

Algebraic General Topology. Volume 1

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generalizations of uniform spaces

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TODD TRIMBLE, ANDREAS BLASS, ROBERT MARTIN SOLOVAY, NIELS
DIEPEVEEN, and others (mentioned below) have proved some theorems which are
now in this book.

ABSTRACT. In this work I introduce and study in details the concepts of fun-
coids which generalize proximity spaces and reloids which generalize uniform
spaces, and generalizations thereof. The concept of funoid is generalized con-
cept of proximity, the concept of reloid is cleared from superfluous details
(generalized) concept of uniformity.

Also funoids and reloids are generalizations of binary relations whose
domains and ranges are filters (instead of sets). Also funoids and reloids can
be considered as a generalization of (oriented) graphs, this provides us with a
common generalization of calculus and discrete mathematics.

It is defined a generalization of limit for arbitrary (including discontinuous
and multivalued) functions, what allows to define for example derivative of an
arbitrary real function.

The concept of continuity is defined by an algebraic formula (instead of old
messy epsilon-delta notation) for arbitrary morphisms (including funoids and
reloids) of a partially ordered category. In one formula continuity, proximity
continuity, and uniform continuity are generalized.

Also I define connectedness for funoids and reloids.

Then I consider generalizations of funoids: pointfree funoids and gen-
eralization of pointfree funoids: staroids and multifunoids. Also I define
several kinds of products of funoids and other morphisms.

Before going to topology, this book studies properties of co-brouwerian
lattices and filters.

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Part 1

Introductory chapters

CHAPTER 1

Introduction

The main purpose of this book is to record the current state of my research. The book is however written in such a way that it can be used as a textbook for studying my research.

For the latest version of this file, related materials, articles, research questions, and erratum consult the Web page of the author of the book:

<http://www.mathematics21.org/algebraic-general-topology.html>

1.1. License and editing

This work is licensed under the Creative Commons Attribution 4.0 International License. To view a copy of the license, visit

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You can create your own copy of L^AT_EX source of the book and edit it (to correct errors, add new results, generalize existing results, enhance readability). The editable source of the book is presented at

<https://bitbucket.org/portonv/algebraic-general-topology>

Please consider reviewing this book at

<http://www.euro-math-soc.eu/node/add/book-review>

If you find any error (or some improvement idea), please report in our bug tracker:

<https://bitbucket.org/portonv/algebraic-general-topology/issues>

1.2. Intended audience

This book is suitable for any math student as well as for researchers.

To make this book be understandable even for first grade students, I made a chapter about basic concepts (posets, lattices, topological spaces, etc.), which an already knowledgeable person may skip reading. It is assumed that the reader knows basic set theory.

But it is also valuable for mature researchers, as it contains much original research which you could not find in any other source except of my work.

Knowledge of the basic set theory is expected from the reader.

Despite that this book presents new research, it is well structured and is suitable to be used as a textbook for a college course.

Your comments about this book are welcome to the email porton@narod.ru.

1.3. Reading Order

If you know basic order and lattice theory (including Galois connections and brouwerian lattices) and basics of category theory, you may skip reading the chapter “[Common knowledge, part 1](#)”.

You are recommended to read the rest of this book by the order.

1.4. Our topic and rationale

From [42]: *Point-set topology, also called set-theoretic topology or general topology, is the study of the general abstract nature of continuity or “closeness” on spaces. Basic point-set topological notions are ones like continuity, dimension, compactness, and connectedness.*

In this work we study a new approach to point-set topology (and pointfree topology).

Traditionally general topology is studied using topological spaces (defined below in the section “**Topological spaces**”). I however argue that the theory of topological spaces is not the best method of studying general topology and introduce an alternative theory, the theory of *funcoids*. Despite of popularity of the theory of topological spaces it has some drawbacks and is in my opinion not the most appropriate formalism to study most of general topology. Because topological spaces are tailored for study of special sets, so called open and closed sets, studying general topology with topological spaces is a little anti-natural and ugly. In my opinion the theory of funcoids is more elegant than the theory of topological spaces, and it is better to study funcoids than topological spaces. One of the main purposes of this work is to present an alternative General Topology based on funcoids instead of being based on topological spaces as it is customary. In order to study funcoids the prior knowledge of topological spaces is not necessary. Nevertheless in this work I will consider topological spaces and the topic of interrelation of funcoids with topological spaces.

In fact funcoids are a generalization of topological spaces, so the well known theory of topological spaces is a special case of the below presented theory of funcoids.

But probably the most important reason to study funcoids is that funcoids are a generalization of proximity spaces (see section “**Proximity spaces**” for the definition of proximity spaces). Before this work it was written that the theory of proximity spaces was an example of a stalled research, almost nothing interesting was discovered about this theory. It was so because the proper way to research proximity spaces is to research their generalization, funcoids. And so it was stalled until discovery of funcoids. That generalized theory of proximity spaces will bring us yet many interesting results.

In addition to *funcoids* I research *reloids*. Using below defined terminology it may be said that reloids are (basically) filters on Cartesian product of sets, and this is a special case of uniform spaces.

Afterward we study some generalizations.

Somebody might ask, why to study it? My approach relates to traditional general topology like complex numbers to real numbers theory. Be sure this will find applications.

This book has a deficiency: It does not properly relate my theory with previous research in general topology and does not consider deeper category theory properties. It is however OK for now, as I am going to do this study in later volumes (continuation of this book).

Many proofs in this book may seem too easy and thus this theory not sophisticated enough. But it is largely a result of a well structured digraph of proofs, where more difficult results are made easy by reducing them to easier lemmas and propositions.

1.5. Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization

which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [11].

Some references to predecessors:

- In [15, 16, 25, 2, 36] generalized uniformities and proximities are studied.
- Proximities and uniformities are also studied in [22, 23, 35, 37, 38].
- [20, 21] contains recent progress in quasi-uniform spaces. [21] has a very long list of related literature.

Some works ([34]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work the attempt to define or research their generalization, compactness of funcoids or reloids is not done. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate volume.

[10] studies mappings between proximity structures. (In this volume no attempt to research mappings between funcoids is done.) [26] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

This book is based partially on my articles [30, 28, 29].

1.6. Kinds of continuity

A research result based on this book but not fully included in this book (and not yet published) is that the following kinds of continuity are described by the same algebraic (or rather categorical) formulas for different kinds of continuity and have common properties:

- discrete continuity (between digraphs);
- (pre)topological continuity;
- proximal continuity;
- uniform continuity;
- Cauchy continuity;
- (probably other kinds of continuity).

Thus my research justifies using the same word “continuity” for these diverse kinds of continuity.

See <http://www.mathematics21.org/algebraic-general-topology.html>

1.7. Responses to some accusations against style of my exposition

The proofs are generally hard to follow and unpleasant to read as they are just a bunch of equations thrown at you, without motivation or underlying reasoning, etc.

I don't think this is essential. The proofs are not the most important thing in my book. The most essential thing are definitions. The proofs are just to fill the gaps. So I deem it not important whether proofs are motivated.

Also, note “algebraic” in the title of my book. The proofs are meant to be just sequences of formulas for as much as possible :-). It is to be thought algebraically. The meaning are the formulas themselves.

Maybe it makes sense to read my book skipping all the proofs? Some proofs contain important ideas, but most don't. The important thing are definitions.

1.8. Structure of this book

In the chapter “**Common knowledge, part 1**” some well known definitions and theories are considered. You may skip its reading if you already know it. That chapter contains info about:

- posets;
- lattices and complete lattices;
- Galois connections;
- co-brouwerian lattices;
- a very short intro into category theory;
- a very short introduction to group theory.

Afterward there are my little additions to poset/lattice and category theory.

Afterward there is the theory of filters and filtrators.

Then there is “**Common knowledge, part 2 (topology)**”, which considers briefly:

- metric spaces;
- topological spaces;
- pretopological spaces;
- proximity spaces.

Despite of the name “Common knowledge” this second common knowledge chapter is recommended to be read completely even if you know topology well, because it contains some rare theorems not known to most mathematicians and hard to find in literature.

Then the most interesting thing in this book, the theory of functors, starts.

Afterwards there is the theory of reloids.

Then I show relationships between functors and reloids.

The last I research generalizations of functors, *pointfree functors*, *staroids*, and *multifunctors* and some different kinds of products of morphisms.

1.9. Basic notation

I will denote a set definition like $\left\{ \frac{x \in A}{P(x)} \right\}$ instead of customary $\{x \in A \mid P(x)\}$. I do this because otherwise formulas don't fit horizontally into the available space.

1.9.1. Grothendieck universes. We will work in ZFC with an infinite and uncountable Grothendieck universe.

A Grothendieck universe is just a set big enough to make all usual set theory inside it. For example if \mathcal{U} is a Grothendieck universe, and sets $X, Y \in \mathcal{U}$, then also $X \cup Y \in \mathcal{U}$, $X \cap Y \in \mathcal{U}$, $X \times Y \in \mathcal{U}$, etc.

A set which is a member of a Grothendieck universe is called a *small set* (regarding this Grothendieck universe). We can restrict our consideration to small sets in order to get rid troubles with proper classes.

DEFINITION 1. Grothendieck universe is a set \mathcal{U} such that:

- 1°. If $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$.
- 2°. If $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$.
- 3°. If $x \in \mathcal{U}$ then $\mathcal{P}x \in \mathcal{U}$.
- 4°. If $\left\{ \frac{x_i}{i \in I \in \mathcal{U}} \right\}$ is a family of elements of \mathcal{U} , then $\bigcup_{i \in I} x_i \in \mathcal{U}$.

One can deduce from this also:

- 1°. If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$.
- 2°. If x is a subset of $y \in \mathcal{U}$, then $x \in \mathcal{U}$.
- 3°. If $x, y \in \mathcal{U}$ then the ordered pair $(x, y) = \{\{x, y\}, x\} \in \mathcal{U}$.
- 4°. If $x, y \in \mathcal{U}$ then $x \cup y$ and $x \times y$ are in \mathcal{U} .
- 5°. If $\left\{ \frac{x_i}{i \in I \in \mathcal{U}} \right\}$ is a family of elements of \mathcal{U} , then the product $\prod_{i \in I} x_i \in \mathcal{U}$.

6°. If $x \in \mathcal{U}$, then the cardinality of x is strictly less than the cardinality of \mathcal{U} .

1.9.2. Misc. In this book quantifiers bind tightly. That is $\forall x \in A : P(x) \wedge Q$ and $\forall x \in A : P(x) \Rightarrow Q$ should be read $(\forall x \in A : P(x)) \wedge Q$ and $(\forall x \in A : P(x)) \Rightarrow Q$ not $\forall x \in A : (P(x) \wedge Q)$ and $\forall x \in A : (P(x) \Rightarrow Q)$.

The set of functions from a set A to a set B is denoted as B^A .

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x .

I will denote $\langle f \rangle^* X = \left\{ \frac{\beta \in \text{im } f}{\exists \alpha \in X : \alpha f \beta} \right\}$ (in other words $\langle f \rangle^* X$ is the image of a set X under a function or binary relation f) and $X [f]^* Y \Leftrightarrow \exists x \in X, y \in Y : x f y$ for sets X, Y and a binary relation f . (Note that functions are a special case of binary relations.)

By just $\langle f \rangle^*$ and $[f]^*$ I will denote the corresponding function and relation on small sets.

OBVIOUS 2. For a function f we have $\langle f \rangle^* X = \left\{ \frac{f(x)}{x \in X} \right\}$.

DEFINITION 3. $\langle f^{-1} \rangle^* X$ is called the *preimage* of a set X by a function (or, more generally, a binary relation) f .

OBVIOUS 4. $\{\alpha\} [f]^* \{\beta\} \Leftrightarrow \alpha f \beta$ for every α and β .

$\lambda x \in D : f(x) = \left\{ \frac{(x, f(x))}{x \in D} \right\}$ for a set D and a form f depending on the variable x . In other words, $\lambda x \in D : f(x)$ is the function which maps elements x of a set D into $f(x)$.

I will denote source and destination of a morphism f of any category (See chapter 2 for a definition of a category.) as $\text{Src } f$ and $\text{Dst } f$ correspondingly. Note that below defined domain and image of a funcoïd are not the same as its source and destination.

I will denote $\text{GR}(A, B, f) = f$ for any morphism (A, B, f) of either **Set** or **Rel**. (See definitions of **Set** and **Rel** below.)

1.10. Implicit arguments

Some notation such that $\perp^{\mathfrak{A}}, \top^{\mathfrak{A}}, \sqcup^{\mathfrak{A}}, \sqcap^{\mathfrak{A}}$ have indexes (in these examples \mathfrak{A}).

We will omit these indexes when they can be restored from the context. For example, having a function $f : \mathfrak{A} \rightarrow \mathfrak{B}$ where $\mathfrak{A}, \mathfrak{B}$ are posets with least elements, we will concisely denote $f \perp = \perp$ for $f \perp^{\mathfrak{A}} = \perp^{\mathfrak{B}}$. (See below for definitions of these operations.)

NOTE 5. In the above formula $f \perp = \perp$ we have the first \perp and the second \perp denoting different objects.

We will assume (skipping this in actual proofs) that all omitted indexes can be restored from context. (Note that so called dependent type theory computer proof assistants do this like we implicitly.)

1.11. Unusual notation

In the chapter “**Common knowledge, part 1**” (which you may skip reading if you are already knowledgeable) some non-standard notation is defined. I summarize here this notation for the case if you choose to skip reading that chapter:

Partial order is denoted as \sqsubseteq .

Meets and joins are denoted as $\sqcap, \sqcup, \sqcap, \sqcup$.

I call element b *subtractive* from an element a (of a distributive lattice \mathfrak{A}) when the difference $a \setminus b$ exists. I call b *complementive* to a when there exists $c \in \mathfrak{A}$ such

that $b \sqcap c = \perp$ and $b \sqcup c = a$. We will prove that b is complementive to a iff b is subtractive from a and $b \sqsubseteq a$.

DEFINITION 6. Call a and b of a poset \mathfrak{A} *intersecting*, denoted $a \not\asymp b$, when there exists a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$.

DEFINITION 7. $a \asymp b \stackrel{\text{def}}{=} \neg(a \not\asymp b)$.

DEFINITION 8. I call elements a and b of a poset \mathfrak{A} *joining* and denote $a \equiv b$ when there are no non-greatest element c such that $c \sqsupseteq a \wedge c \sqsupseteq b$.

DEFINITION 9. $a \not\equiv b \stackrel{\text{def}}{=} \neg(a \equiv b)$.

OBVIOUS 10. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

OBVIOUS 11. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

I extend the definitions of pseudocomplement and dual pseudocomplement to arbitrary posets (not just lattices as it is customary):

DEFINITION 12. Let \mathfrak{A} be a poset. *Pseudocomplement* of a is

$$\max \left\{ \frac{c \in \mathfrak{A}}{c \asymp a} \right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

DEFINITION 13. Let \mathfrak{A} be a poset. *Dual pseudocomplement* of a is

$$\min \left\{ \frac{c \in \mathfrak{A}}{c \equiv a} \right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

CHAPTER 2

Common knowledge, part 1

In this chapter we will consider some well known mathematical theories. If you already know them you may skip reading this chapter (or its parts).

2.1. Order theory

2.1.1. Posets.

DEFINITION 14. The *identity relation* on a set A is $\text{id}_A = \left\{ \frac{(a,a)}{a \in A} \right\}$.

DEFINITION 15. A *preorder* on a set A is a binary relation \sqsubseteq on A which is:

- *reflexive* on A that is $(\sqsubseteq) \supseteq \text{id}_A$ or what is the same $\forall x \in A : x \sqsubseteq x$;
- *transitive* that is $(\sqsubseteq) \circ (\sqsubseteq) \subseteq (\sqsubseteq)$ or what is the same

$$\forall x, y, z : (x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z).$$

DEFINITION 16. A *partial order* on a set A is a preorder on A which is *anti-symmetric* that is $(\sqsubseteq) \cap (\sqsubseteq) \subseteq \text{id}_A$ or what is the same

$$\forall x, y \in A : (x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y).$$

The reverse relation is denoted \supseteq .

DEFINITION 17. a is a subelement of b (or what is the same a is *contained in* b or b *contains* a) iff $a \sqsubseteq b$.

OBVIOUS 18. The reverse of a partial order is also a partial order.

DEFINITION 19. A set A together with a partial order on it is called a *partially ordered set* (*poset* for short).

An example of a poset is the set \mathbb{R} of real numbers with $\sqsubseteq = \leq$.

Another example is the set $\mathcal{P}A$ of all subsets of an arbitrary fixed set A with $\sqsubseteq = \subseteq$. Note that this poset is (in general) not linear (see definition of *linear* poset below.)

DEFINITION 20. Strict partial order \sqsubset corresponding to the partial order \sqsubseteq on a set A is defined by the formula $(\sqsubset) = (\sqsubseteq) \setminus \text{id}_A$. In other words,

$$a \sqsubset b \Leftrightarrow a \sqsubseteq b \wedge a \neq b.$$

An example of strict partial order is $<$ on the set \mathbb{R} of real numbers.

DEFINITION 21. A partial order on a set A *restricted* to a set $B \subseteq A$ is $(\sqsubseteq) \cap (B \times B)$.

OBVIOUS 22. A partial order on a set A restricted to a set $B \subseteq A$ is a partial order on B .

DEFINITION 23.

- The *least* element \perp of a poset \mathfrak{A} is defined by the formula $\forall a \in \mathfrak{A} : \perp \sqsubseteq a$.
- The *greatest* element \top of a poset \mathfrak{A} is defined by the formula $\forall a \in \mathfrak{A} : \top \supseteq a$.

PROPOSITION 24. There exist no more than one least element and no more than one greatest element (for a given poset).

PROOF. By antisymmetry. \square

DEFINITION 25. The *dual* order for \sqsubseteq is \supseteq .

OBVIOUS 26. Dual of a partial order is a partial order.

DEFINITION 27. The *dual* poset for a poset (A, \sqsubseteq) is the poset (A, \supseteq) .

I will denote dual of a poset \mathfrak{A} as $(\text{dual}\mathfrak{A})$ and dual of an element $a \in \mathfrak{A}$ (that is the same element in the dual poset) as $(\text{dual } a)$.

Below we will sometimes use *duality* that is replacement of the partial order and all related operations and relations with their duals. In other words, it is enough to prove a theorem for an order \sqsubseteq and the similar theorem for \supseteq follows by duality.

DEFINITION 28. A subset P of a poset \mathfrak{A} is called *bounded above* if there exists $t \in \mathfrak{A}$ such that $\forall x \in P : t \supseteq x$. *Bounded below* is defined dually.

2.1.1.1. *Intersecting and joining elements.* Let \mathfrak{A} be a poset.

DEFINITION 29. Call elements a and b of \mathfrak{A} *intersecting*, denoted $a \not\sqsupseteq b$, when there exists a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$.

DEFINITION 30. $a \succ b \stackrel{\text{def}}{=} \neg(a \not\sqsupseteq b)$.

OBVIOUS 31. $a_0 \not\sqsupseteq b_0 \wedge a_1 \supseteq a_0 \wedge b_1 \supseteq b_0 \Rightarrow a_1 \not\sqsupseteq b_1$.

DEFINITION 32. I call elements a and b of \mathfrak{A} *joining* and denote $a \equiv b$ when there is no a non-greatest element c such that $c \supseteq a \wedge c \supseteq b$.

DEFINITION 33. $a \not\equiv b \stackrel{\text{def}}{=} \neg(a \equiv b)$.

OBVIOUS 34. Intersecting is the dual of non-joining.

OBVIOUS 35. $a_0 \equiv b_0 \wedge a_1 \supseteq a_0 \wedge b_1 \supseteq b_0 \Rightarrow a_1 \equiv b_1$.

2.1.2. Linear order.

DEFINITION 36. A poset \mathfrak{A} is called *linearly ordered set* (or what is the same, *totally ordered set*) if $a \supseteq b \vee b \supseteq a$ for every $a, b \in \mathfrak{A}$.

EXAMPLE 37. The set of real numbers with the customary order is a linearly ordered set.

DEFINITION 38. A set $X \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a poset is called *chain* if \mathfrak{A} restricted to X is a total order.

2.1.3. *Meets and joins.* Let \mathfrak{A} be a poset.

DEFINITION 39. Given a set $X \in \mathcal{P}\mathfrak{A}$ the *least element* (also called *minimum* and denoted $\min X$) of X is such $a \in X$ that $\forall x \in X : a \sqsubseteq x$.

Least element does not necessarily exist. But if it exists:

PROPOSITION 40. For a given $X \in \mathcal{P}\mathfrak{A}$ there exist no more than one least element.

PROOF. It follows from anti-symmetry. \square

Greatest element is the dual of least element:

DEFINITION 41. Given a set $X \in \mathcal{P}\mathfrak{A}$ the *greatest element* (also called *maximum* and denoted $\max X$) of X is such $a \in X$ that $\forall x \in X : a \supseteq x$.

REMARK 42. Least and greatest elements of a set X is a trivial generalization of the above defined least and greatest element for the entire poset.

DEFINITION 43.

- A *minimal* element of a set $X \in \mathcal{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsubset x$.
- A *maximal* element of a set $X \in \mathcal{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsupset x$.

REMARK 44. Minimal element is not the same as minimum, and maximal element is not the same as maximum.

OBVIOUS 45.

- 1°. The least element (if it exists) is a minimal element.
- 2°. The greatest element (if it exists) is a maximal element.

EXERCISE 46. Show that there may be more than one minimal and more than one maximal element for some poset.

DEFINITION 47. *Upper bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsupseteq x} \right\}$.

The dual notion:

DEFINITION 48. *Lower bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsubseteq x} \right\}$.

DEFINITION 49. *Join* $\bigsqcup X$ (also called *supremum* and denoted “sup X ”) of a set X is the least element of its upper bounds (if it exists).

DEFINITION 50. *Meet* $\bigsqcap X$ (also called *infimum* and denoted “inf X ”) of a set X is the greatest element of its lower bounds (if it exists).

We will also denote $\bigsqcup_{i \in X} f(i) = \bigsqcup \left\{ \frac{f(i)}{x \in X} \right\}$ and $\bigsqcap_{i \in X} f(i) = \bigsqcap \left\{ \frac{f(i)}{x \in X} \right\}$.

We will write $b = \bigsqcup X$ when $b \in \mathfrak{A}$ is the join of X or say that $\bigsqcup X$ does not exist if there are no such $b \in \mathfrak{A}$. (And dually for meets.)

EXERCISE 51. Provide an example of $\bigsqcup X \notin X$ for some set X on some poset.

PROPOSITION 52.

- 1°. If b is the greatest element of X then $\bigsqcup X = b$.
- 2°. If b is the least element of X then $\bigsqcap X = b$.

PROOF. We will prove only the first as the second is dual.

Let b be the greatest element of X . Then upper bounds of X are $\left\{ \frac{y \in \mathfrak{A}}{y \sqsupseteq b} \right\}$. Obviously b is the least element of this set, that is the join. \square

DEFINITION 53. *Binary joins and meets* are defined by the formulas

$$x \sqcup y = \bigsqcup \{x, y\} \quad \text{and} \quad x \sqcap y = \bigsqcap \{x, y\}.$$

OBVIOUS 54. \sqcup and \sqcap are symmetric operations (whenever these are defined for given x and y).

THEOREM 55.

- 1°. If $\bigsqcup X$ exists then $y \sqsupseteq \bigsqcup X \Leftrightarrow \forall x \in X : y \sqsupseteq x$.
- 2°. If $\bigsqcap X$ exists then $y \sqsubseteq \bigsqcap X \Leftrightarrow \forall x \in X : y \sqsubseteq x$.

PROOF. I will prove only the first as the second follows by duality.

$y \sqsupseteq \bigsqcup X \Leftrightarrow y$ is an upper bound for $X \Leftrightarrow \forall x \in X : y \sqsupseteq x$. \square

COROLLARY 56.

- 1°. If $a \sqcup b$ exists then $y \sqsupseteq a \sqcup b \Leftrightarrow y \sqsupseteq a \wedge y \sqsupseteq b$.
- 2°. If $a \sqcap b$ exists then $y \sqsubseteq a \sqcap b \Leftrightarrow y \sqsubseteq a \wedge y \sqsubseteq b$.

I will denote meets and joins for a specific poset \mathfrak{A} as $\prod^{\mathfrak{A}}$, $\bigsqcup^{\mathfrak{A}}$, $\cap^{\mathfrak{A}}$, $\sqcup^{\mathfrak{A}}$.

2.1.4. Semilattices.

DEFINITION 57.

- 1°. A *join-semilattice* is a poset \mathfrak{A} such that $a \sqcup b$ is defined for every $a, b \in \mathfrak{A}$.
- 2°. A *meet-semilattice* is a poset \mathfrak{A} such that $a \sqcap b$ is defined for every $a, b \in \mathfrak{A}$.

THEOREM 58.

- 1°. The operation \sqcup is associative for any join-semilattice.
- 2°. The operation \sqcap is associative for any meet-semilattice.

PROOF. I will prove only the first as the second follows by duality.

We need to prove $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$ for every $a, b, c \in \mathfrak{A}$.

Taking into account the definition of join, it is enough to prove that

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c)$$

for every $x \in \mathfrak{A}$. Really, this follows from the chain of equivalences:

$$\begin{aligned} x \sqsupseteq (a \sqcup b) \sqcup c &\Leftrightarrow \\ x \sqsupseteq a \sqcup b \wedge x \sqsupseteq c &\Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \wedge x \sqsupseteq c &\Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \sqcup c &\Leftrightarrow \\ x \sqsupseteq a \sqcup (b \sqcup c). & \end{aligned}$$

□

OBVIOUS 59. $a \not\leq b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

OBVIOUS 60. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

2.1.5. Lattices and complete lattices.

DEFINITION 61. A *bounded* poset is a poset having both least and greatest elements.

DEFINITION 62. *Lattice* is a poset which is both join-semilattice and meet-semilattice.

DEFINITION 63. A *complete lattice* is a poset \mathfrak{A} such that for every $X \in \mathcal{P}\mathfrak{A}$ both $\bigsqcup X$ and $\bigsqcap X$ exist.

OBVIOUS 64. Every complete lattice is a lattice.

PROPOSITION 65. Every complete lattice is a bounded poset.

PROOF. $\bigsqcup \emptyset$ is the least and $\bigsqcap \emptyset$ is the greatest element. □

THEOREM 66. Let \mathfrak{A} be a poset.

- 1°. If $\bigsqcup X$ is defined for every $X \in \mathcal{P}\mathfrak{A}$, then \mathfrak{A} is a complete lattice.
- 2°. If $\bigsqcap X$ is defined for every $X \in \mathcal{P}\mathfrak{A}$, then \mathfrak{A} is a complete lattice.

PROOF. See [27] or any lattice theory reference. □

OBVIOUS 67. If $X \subseteq Y$ for some $X, Y \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a complete lattice, then

- 1°. $\bigsqcup X \subseteq \bigsqcup Y$;
- 2°. $\bigsqcap X \supseteq \bigsqcap Y$.

PROPOSITION 68. If $S \in \mathcal{P}\mathcal{P}\mathfrak{A}$ then for every complete lattice \mathfrak{A}

$$\begin{aligned} 1^\circ. \bigsqcup \bigcup S &= \bigsqcup_{X \in S} \bigsqcup X; \\ 2^\circ. \prod \bigcup S &= \prod_{X \in S} \prod X. \end{aligned}$$

PROOF. We will prove only the first as the second is dual.

By definition of joins, it is enough to prove $y \sqsupseteq \bigsqcup \bigcup S \Leftrightarrow y \sqsupseteq \bigsqcup_{X \in S} \bigsqcup X$.

Really,

$$\begin{aligned} y \sqsupseteq \bigsqcup \bigcup S &\Leftrightarrow \\ \forall x \in \bigcup S : y \sqsupseteq x &\Leftrightarrow \\ \forall X \in S \forall x \in X : y \sqsupseteq x &\Leftrightarrow \\ \forall X \in S : y \sqsupseteq \bigsqcup X &\Leftrightarrow \\ y \sqsupseteq \bigsqcup_{X \in S} \bigsqcup X. & \end{aligned}$$

□

DEFINITION 69. A *sublattice* of a lattice is its subset closed regarding \sqcup and \sqcap .

OBVIOUS 70. Sublattice with induced order is also a lattice.

2.1.6. Distributivity of lattices.

DEFINITION 71. A *distributive* lattice is such lattice \mathfrak{A} that for every $x, y, z \in \mathfrak{A}$

$$\begin{aligned} 1^\circ. x \sqcap (y \sqcup z) &= (x \sqcap y) \sqcup (x \sqcap z); \\ 2^\circ. x \sqcup (y \sqcap z) &= (x \sqcup y) \sqcap (x \sqcup z). \end{aligned}$$

THEOREM 72. For a lattice to be distributive it is enough just one of the conditions:

$$\begin{aligned} 1^\circ. x \sqcap (y \sqcup z) &= (x \sqcap y) \sqcup (x \sqcap z); \\ 2^\circ. x \sqcup (y \sqcap z) &= (x \sqcup y) \sqcap (x \sqcup z). \end{aligned}$$

PROOF.

$$\begin{aligned} (x \sqcup y) \sqcap (x \sqcup z) &= \\ ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z) &= \\ x \sqcup ((x \sqcap z) \sqcup (y \sqcap z)) &= \\ (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) &= \\ x \sqcup (y \sqcap z) & \end{aligned}$$

(applied $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ twice). □

2.1.7. Difference and complement.

DEFINITION 73. Let \mathfrak{A} be a distributive lattice with least element \perp . The *difference* (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \sqcap c = \perp$ and $a \sqcup b = b \sqcup c$. I will call b *subtractive* from a when $a \setminus b$ exists.

THEOREM 74. If \mathfrak{A} is a distributive lattice with least element \perp , there exists no more than one difference of elements a, b .

PROOF. Let c and d be both differences $a \setminus b$. Then $b \sqcap c = b \sqcap d = \perp$ and $a \sqcup b = b \sqcup c = b \sqcup d$. So

$$c = c \sqcap (b \sqcup c) = c \sqcap (b \sqcup d) = (c \sqcap b) \sqcup (c \sqcap d) = \perp \sqcup (c \sqcap d) = c \sqcap d.$$

Similarly $d = d \sqcap c$. Consequently $c = c \sqcap d = d \sqcap c = d$. □

DEFINITION 75. I will call b *complementive* to a iff there exists $c \in \mathfrak{A}$ such that $b \sqcap c = \perp$ and $b \sqcup c = a$.

PROPOSITION 76. b is complementive to a iff b is subtractive from a and $b \sqsubseteq a$.

PROOF.

\Leftarrow . Obvious.

\Rightarrow . We deduce $b \sqsubseteq a$ from $b \sqcup c = a$. Thus $a \sqcup b = a = b \sqcup c$.

□

PROPOSITION 77. If b is complementive to a then $(a \setminus b) \sqcup b = a$.

PROOF. Because $b \sqsubseteq a$ by the previous proposition. □

DEFINITION 78. Let \mathfrak{A} be a bounded distributive lattice. The *complement* (denoted \bar{a}) of an element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \sqcap b = \perp$ and $a \sqcup b = \top$.

PROPOSITION 79. If \mathfrak{A} is a bounded distributive lattice then $\bar{a} = \top \setminus a$.

PROOF. $b = \bar{a} \Leftrightarrow b \sqcap a = \perp \wedge b \sqcup a = \top \Leftrightarrow b \sqcap a = \perp \wedge \top \sqcup a = a \sqcup b \Leftrightarrow b = \top \setminus a$. □

COROLLARY 80. If \mathfrak{A} is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.

DEFINITION 81. An element of bounded distributive lattice is called *complemented* when its complement exists.

DEFINITION 82. A distributive lattice is a *complemented lattice* iff every its element is complemented.

PROPOSITION 83. For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \sqcup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.

PROOF. $((a \setminus b) \setminus c) \sqcap c = \perp$; $((a \setminus b) \setminus c) \sqcup c = (a \setminus b) \sqcup c$; $(a \setminus b) \sqcap b = \perp$; $(a \setminus b) \sqcup b = a \sqcup b$.

We need to prove $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \perp$ and $((a \setminus b) \setminus c) \sqcup (b \sqcup c) = a \sqcup (b \sqcup c)$.
In fact,

$$\begin{aligned} & ((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \\ & (((a \setminus b) \setminus c) \sqcap b) \sqcup (((a \setminus b) \setminus c) \sqcap c) = \\ & (((a \setminus b) \setminus c) \sqcap b) \sqcup \perp = \\ & ((a \setminus b) \setminus c) \sqcap b \sqsubseteq \\ & (a \setminus b) \sqcap b = \perp, \end{aligned}$$

so $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \perp$;

$$\begin{aligned} & ((a \setminus b) \setminus c) \sqcup (b \sqcup c) = \\ & (((a \setminus b) \setminus c) \sqcup c) \sqcup b = \\ & (a \setminus b) \sqcup c \sqcup b = \\ & ((a \setminus b) \sqcup b) \sqcup c = \\ & a \sqcup b \sqcup c. \end{aligned}$$

□

2.1.8. Boolean lattices.

DEFINITION 84. A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is $\mathcal{P}A$ where A is a set, ordered by set inclusion.

THEOREM 85. (DE MORGAN'S laws) For every elements a, b of a boolean lattice

- 1°. $\overline{a \sqcup b} = \bar{a} \sqcap \bar{b}$;
- 2°. $\overline{a \sqcap b} = \bar{a} \sqcup \bar{b}$.

PROOF. We will prove only the first as the second is dual.

It is enough to prove that $a \sqcup b$ is a complement of $\bar{a} \sqcap \bar{b}$. Really:

$$(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) \sqsubseteq a \sqcap (\bar{a} \sqcap \bar{b}) = (a \sqcap \bar{a}) \sqcap \bar{b} = \perp \sqcap \bar{b} = \perp;$$

$$(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = ((a \sqcup b) \sqcup \bar{a}) \sqcap ((a \sqcup b) \sqcup \bar{b}) \sqsupseteq (a \sqcup \bar{a}) \sqcap (b \sqcup \bar{b}) = \top \sqcap \top = \top.$$

Thus $(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) = \perp$ and $(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = \top$. \square

DEFINITION 86. A complete lattice \mathfrak{A} is *join infinite distributive* when $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle^* S$; a complete lattice \mathfrak{A} is *meet infinite distributive* when $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle^* S$ for all $x \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$.

DEFINITION 87. *Infinite distributive complete lattice* is a complete lattice which is both join infinite distributive and meet infinite distributive.

THEOREM 88. For every boolean lattice \mathfrak{A} , $x \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$ we have:

- 1°. $\bigsqcup \langle x \sqcap \rangle^* S$ is defined and $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle^* S$ whenever $\bigsqcup S$ is defined.
- 2°. $\bigsqcap \langle x \sqcup \rangle^* S$ is defined and $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle^* S$ whenever $\bigsqcap S$ is defined.

PROOF. We will prove only the first, as the other is dual.

We need to prove that $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle^* S$.

That $x \sqcap \bigsqcup S$ is an upper bound of $\langle x \sqcap \rangle^* S$ is obvious.

Now let u be any upper bound of $\langle x \sqcap \rangle^* S$, that is $x \sqcap y \sqsubseteq u$ for all $y \in S$. Then

$$y = y \sqcap (x \sqcup \bar{x}) = (y \sqcap x) \sqcup (y \sqcap \bar{x}) \sqsubseteq u \sqcup \bar{x},$$

and so $\bigsqcup S \sqsubseteq u \sqcup \bar{x}$. Thus

$$x \sqcap \bigsqcup S \sqsubseteq x \sqcap (u \sqcup \bar{x}) = (x \sqcap u) \sqcup (x \sqcap \bar{x}) = (x \sqcap u) \sqcup \perp = x \sqcap u \sqsubseteq u,$$

that is $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle^* S$. \square

COROLLARY 89. Every complete boolean lattice is both join infinite distributive and meet infinite distributive.

THEOREM 90. (infinite DE MORGAN'S laws) For every subset S of a complete boolean lattice

- 1°. $\overline{\bigsqcup S} = \bigsqcap_{x \in S} \bar{x}$;
- 2°. $\overline{\bigsqcap S} = \bigsqcup_{x \in S} \bar{x}$.

PROOF. It's enough to prove that $\bigsqcup S$ is a complement of $\bigsqcap_{x \in S} \bar{x}$ (the second follows from duality). Really, using the previous theorem:

$$\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \bigsqcap_{x \in S} \langle \bigsqcup S \sqcup \rangle^* \bar{x} = \bigsqcap_{x \in S} \left\{ \frac{\bigsqcup S \sqcup \bar{x}}{x \in S} \right\} \sqsupseteq \bigsqcap_{x \in S} \left\{ \frac{x \sqcup \bar{x}}{x \in S} \right\} = \top;$$

$$\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bigsqcup_{y \in S} \left\langle \bigsqcap_{x \in S} \bar{x} \sqcap \right\rangle^* y = \bigsqcup_{y \in S} \left\{ \frac{\bigsqcap_{x \in S} \bar{x} \sqcap y}{y \in S} \right\} \sqsubseteq \bigsqcup_{y \in S} \left\{ \frac{\bar{y} \sqcap y}{y \in S} \right\} = \perp.$$

So $\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \top$ and $\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \perp$. \square

2.1.9. Center of a lattice.

DEFINITION 91. The *center* $Z(\mathfrak{A})$ of a bounded distributive lattice \mathfrak{A} is the set of its complemented elements.

REMARK 92. For a definition of center of non-distributive lattices see [5].

REMARK 93. In [24] the word center and the notation $Z(\mathfrak{A})$ are used in a different sense.

DEFINITION 94. A sublattice K of a complete lattice L is a *closed sublattice* of L if K contains the meet and the join of any its nonempty subset.

THEOREM 95. Center of an infinitely distributive lattice is its closed sublattice.

PROOF. See [17]. \square

REMARK 96. See [18] for a more strong result.

THEOREM 97. The center of a bounded distributive lattice constitutes its sublattice.

PROOF. Let \mathfrak{A} be a bounded distributive lattice and $Z(\mathfrak{A})$ be its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \sqcup \bar{b}$ is the complement of $a \sqcap b$ because

$$\begin{aligned} (a \sqcap b) \sqcap (\bar{a} \sqcup \bar{b}) &= (a \sqcap b \sqcap \bar{a}) \sqcup (a \sqcap b \sqcap \bar{b}) = \perp \sqcup \perp = \perp \quad \text{and} \\ (a \sqcap b) \sqcup (\bar{a} \sqcup \bar{b}) &= (a \sqcup \bar{a} \sqcup \bar{b}) \sqcap (b \sqcup \bar{a} \sqcup \bar{b}) = \top \sqcap \top = \top. \end{aligned}$$

So $a \sqcap b$ is complemented. Similarly $a \sqcup b$ is complemented. \square

THEOREM 98. The center of a bounded distributive lattice constitutes a boolean lattice.

PROOF. Because it is a distributive complemented lattice. \square

2.1.10. Atoms of posets.

DEFINITION 99. An atom of a poset is an element a such that (for every its element x) $x \sqsubset a$ if and only if x is the least element.

REMARK 100. This definition is valid even for posets without least element.

PROPOSITION 101. Element a is an atom iff both:

- 1°. $x \sqsubset a$ implies x is the least element;
- 2°. a is non-least.

PROOF.

\Rightarrow . Let a be an atom. 1° is obvious. If a is least then $a \sqsubset a$ what is impossible, so 2°.

\Leftarrow . Let 1° and 2° hold. We need to prove only that x is least implies that $x \sqsubset a$ but this follows from a being non-least. \square

EXAMPLE 102. Atoms of the boolean algebra $\mathscr{P}A$ (ordered by set inclusion) are one-element sets.

I will denote atoms ^{\mathfrak{A}} a or just (atoms a) the set of atoms contained in an element a of a poset \mathfrak{A} . I will denote atoms ^{\mathfrak{A}} the set of all atoms of a poset \mathfrak{A} .

DEFINITION 103. A poset \mathfrak{A} is called *atomic* iff atoms $a \neq \emptyset$ for every non-least element a of the poset \mathfrak{A} .

DEFINITION 104. *Atomistic poset* is such a poset that $a = \bigsqcup \text{atoms } a$ for every element a of this poset.

OBVIOUS 105. Every atomistic poset is atomic.

PROPOSITION 106. Let \mathfrak{A} be a poset. If a is an atom of \mathfrak{A} and $B \in \mathfrak{A}$ then

$$a \in \text{atoms } B \Leftrightarrow a \sqsubseteq B \Leftrightarrow a \not\prec B.$$

PROOF.

$a \in \text{atoms } B \Leftrightarrow a \sqsubseteq B$. Obvious.

$a \sqsubseteq B \Rightarrow a \not\prec B$. $a \sqsubseteq B \Rightarrow a \sqsubseteq a \wedge a \sqsubseteq B$, thus $a \not\prec B$ because a is not least.

$a \sqsubseteq B \Leftarrow a \not\prec B$. $a \not\prec B$ implies existence of non-least element x such that $x \sqsubseteq B$ and $x \sqsubseteq a$. Because a is an atom, we have $x = a$. So $a \sqsubseteq B$. □

THEOREM 107. A poset is atomistic iff every its element can be represented as join of atoms.

PROOF.

\Rightarrow . Obvious.

\Leftarrow . Let $a = \bigsqcup S$ where S is a set of atoms. We will prove that a is the least upper bound of atoms a .

That a is an upper bound of atoms a is obvious. Let x is an upper bound of atoms a . Then $x \supseteq \bigsqcup S$ because $S \subseteq \text{atoms } a$. Thus $x \supseteq a$. □

THEOREM 108. $\text{atoms } \prod S = \bigcap \langle \text{atoms} \rangle^* S$ whenever $\prod S$ is defined for every $S \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a poset.

PROOF. For any atom

$$\begin{aligned} c \in \text{atoms } \prod S &\Leftrightarrow \\ c \sqsubseteq \prod S &\Leftrightarrow \\ \forall a \in S : c \sqsubseteq a &\Leftrightarrow \\ \forall a \in S : c \in \text{atoms } a &\Leftrightarrow \\ c \in \bigcap \langle \text{atoms} \rangle^* S. & \end{aligned}$$

□

COROLLARY 109. $\text{atoms}(a \sqcap b) = \text{atoms } a \cap \text{atoms } b$ for an arbitrary meet-semilattice.

THEOREM 110. A complete boolean lattice is atomic iff it is atomistic.

PROOF.

\Leftarrow . Obvious.

\Rightarrow . Let \mathfrak{A} be an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigsqcup \text{atoms } a \sqsubset a$. If $x \in \text{atoms}(a \setminus b)$ then $x \sqsubseteq a \setminus b$ and so $x \sqsubseteq a$ and hence $x \sqsubseteq b$. But we have $x = x \sqcap b \sqsubseteq (a \setminus b) \sqcap b = \perp$ what contradicts to our supposition. □

2.1.11. Kuratowski's lemma.

THEOREM 111. (KURATOWSKI'S lemma) Any chain in a poset is contained in a maximal chain (if we order chains by inclusion).

I will skip the proof of KURATOWSKI'S lemma as this proof can be found in any set theory or order theory reference.

2.1.12. Homomorphisms of posets and lattices.

DEFINITION 112. A *monotone* function (also called *order homomorphism*) from a poset \mathfrak{A} to a poset \mathfrak{B} is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$ for every $x, y \in \mathfrak{A}$.

DEFINITION 113. A *antitone* function (also called *antitone order homomorphism*) from a poset \mathfrak{A} to a poset \mathfrak{B} is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsupseteq fy$ for every $x, y \in \mathfrak{A}$.

DEFINITION 114. *Order embedding* is a function f from poset \mathfrak{A} to a poset \mathfrak{B} such that $x \sqsubseteq y \Leftrightarrow fx \sqsubseteq fy$ for every $x, y \in \mathfrak{A}$.

PROPOSITION 115. Every order embedding is injective.

PROOF. $fx = fy$ implies $x \sqsubseteq y$ and $y \sqsubseteq x$. □

OBVIOUS 116. Every order embedding is an order homomorphism.

DEFINITION 117. *Antitone order embedding* is a function f from poset \mathfrak{A} to a poset \mathfrak{B} such that $x \sqsubseteq y \Leftrightarrow fx \sqsupseteq fy$ for every $x, y \in \mathfrak{A}$.

OBVIOUS 118. Antitone order embedding is an order embedding between a poset and a dual of (another) poset.

DEFINITION 119. *Order isomorphism* is a surjective order embedding.

Order isomorphism preserves properties of posets, such as order, joins and meets, etc.

DEFINITION 120. *Antitone order isomorphism* is a surjective antitone order embedding.

DEFINITION 121.

- 1°. *Join semilattice homomorphism* is a function f from a join semilattice \mathfrak{A} to a join semilattice \mathfrak{B} , such that $f(x \sqcup y) = fx \sqcup fy$ for every $x, y \in \mathfrak{A}$.
- 2°. *Meet semilattice homomorphism* is a function f from a meet semilattice \mathfrak{A} to a meet semilattice \mathfrak{B} , such that $f(x \sqcap y) = fx \sqcap fy$ for every $x, y \in \mathfrak{A}$.

OBVIOUS 122.

- 1°. Join semilattice homomorphisms are monotone.
- 2°. Meet semilattice homomorphisms are monotone.

DEFINITION 123. A *lattice homomorphism* is a function from a lattice to a lattice, which is both join semilattice homomorphism and meet semilattice homomorphism.

DEFINITION 124. *Complete lattice homomorphism* from a complete lattice \mathfrak{A} to a complete lattice \mathfrak{B} is a function f from \mathfrak{A} to \mathfrak{B} which preserves all meets and joins, that is $f \sqcup S = \sqcup \langle f \rangle^* S$ and $f \sqcap S = \sqcap \langle f \rangle^* S$ for every $S \in \mathcal{P}\mathfrak{A}$.

2.1.13. Galois connections. See [3, 12] for more detailed treatment of Galois connections.

DEFINITION 125. Let \mathfrak{A} and \mathfrak{B} be two posets. A *Galois connection* between \mathfrak{A} and \mathfrak{B} is a pair of functions $f = (f^*, f_*)$ with $f^* : \mathfrak{A} \rightarrow \mathfrak{B}$ and $f_* : \mathfrak{B} \rightarrow \mathfrak{A}$ such that:

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (f^* x \sqsubseteq y \Leftrightarrow x \sqsubseteq f_* y).$$

f_* is called *the upper adjoint* of f^* and f^* is called *the lower adjoint* of f_* .

THEOREM 126. A pair (f^*, f_*) of functions $f^* : \mathfrak{A} \rightarrow \mathfrak{B}$ and $f_* : \mathfrak{B} \rightarrow \mathfrak{A}$ is a Galois connection iff both of the following:

- 1°. f^* and f_* are monotone.
 2°. $x \sqsubseteq f_* f^* x$ and $f^* f_* y \sqsubseteq y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.

PROOF.

\Rightarrow .

- 2°. $x \sqsubseteq f_* f^* x$ since $f^* x \sqsubseteq f^* x$; $f^* f_* y \sqsubseteq y$ since $f_* y \sqsubseteq f_* y$.
 1°. Let $a, b \in \mathfrak{A}$ and $a \sqsubseteq b$. Then $a \sqsubseteq b \sqsubseteq f_* f^* b$. So by definition $f^* a \sqsubseteq f^* b$ that is f^* is monotone. Analogously f_* is monotone.

\Leftarrow . $f^* x \sqsubseteq y \Rightarrow f_* f^* x \sqsubseteq f_* y \Rightarrow x \sqsubseteq f_* y$. The other direction is analogous. □

THEOREM 127.

- 1°. $f^* \circ f_* \circ f^* = f^*$.
 2°. $f_* \circ f^* \circ f_* = f_*$.

PROOF.

- 1°. Let $x \in \mathfrak{A}$. We have $x \sqsubseteq f_* f^* x$; consequently $f^* x \sqsubseteq f^* f_* f^* x$. On the other hand, $f^* f_* f^* x \sqsubseteq f^* x$. So $f^* f_* f^* x = f^* x$.
 2°. Similar. □

DEFINITION 128. A function f is called *idempotent* iff $f(f(X)) = f(X)$ for every argument X .

PROPOSITION 129. $f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.

PROOF. $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

THEOREM 130. Each of two adjoints is uniquely determined by the other.

PROOF. Let p and q be both upper adjoints of f . We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \sqsubseteq p(y) \Leftrightarrow f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \sqsubseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \sqsubseteq p(y)$. So $q(y) = p(y)$. □

THEOREM 131. Let f be a function from a poset \mathfrak{A} to a poset \mathfrak{B} .

1°. Both:

- (a) If f is monotone and $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{f x \sqsubseteq b}\right\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f .
 (b) If $g : \mathfrak{B} \rightarrow \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{f x \sqsubseteq b}\right\}$ for every $b \in \mathfrak{B}$.

2°. Both:

- (a) If f is monotone and $g(b) = \min\left\{\frac{x \in \mathfrak{A}}{f x \sqsupseteq b}\right\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f .
 (b) If $g : \mathfrak{B} \rightarrow \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min\left\{\frac{x \in \mathfrak{A}}{f x \sqsupseteq b}\right\}$ for every $b \in \mathfrak{B}$.

PROOF. We will prove only the first as the second is its dual.

1°a. Let $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{f x \sqsubseteq b}\right\}$ for every $b \in \mathfrak{B}$. Then

$$x \sqsubseteq g y \Leftrightarrow x \sqsubseteq \max\left\{\frac{x \in \mathfrak{A}}{f x \sqsubseteq y}\right\} \Rightarrow f x \sqsubseteq y$$

(because f is monotone) and

$$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max \left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq y} \right\} \Leftarrow fx \sqsubseteq y.$$

So $fx \sqsubseteq y \Leftrightarrow x \sqsubseteq gy$ that is f is the lower adjoint of g .

1°b. We have

$$g(b) = \max \left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq b} \right\} \Leftrightarrow fgb \sqsubseteq b \wedge \forall x \in \mathfrak{A} : (fx \sqsubseteq b \Rightarrow x \sqsubseteq gb).$$

what is true by properties of adjoints. □

THEOREM 132. Let f be a function from a poset \mathfrak{A} to a poset \mathfrak{B} .

- 1°. If f is an upper adjoint, f preserves all existing infima in \mathfrak{A} .
- 2°. If \mathfrak{A} is a complete lattice and f preserves all infima, then f is an upper adjoint of a function $\mathfrak{B} \rightarrow \mathfrak{A}$.
- 3°. If f is a lower adjoint, f preserves all existing suprema in \mathfrak{A} .
- 4°. If \mathfrak{A} is a complete lattice and f preserves all suprema, then f is a lower adjoint of a function $\mathfrak{B} \rightarrow \mathfrak{A}$.

PROOF. We will prove only first two items because the rest items are similar.

1°. Let $S \in \mathcal{P}\mathfrak{A}$ and $\prod S$ exists. $f \prod S$ is a lower bound for $\langle f \rangle^* S$ because f is order-preserving. If a is a lower bound for $\langle f \rangle^* S$ then $\forall x \in S : a \sqsubseteq fx$ that is $\forall x \in S : ga \sqsubseteq x$ where g is the lower adjoint of f . Thus $ga \sqsubseteq \prod S$ and hence $f \prod S \sqsupseteq a$. So $f \prod S$ is the greatest lower bound for $\langle f \rangle^* S$.

2°. Let \mathfrak{A} be a complete lattice and f preserves all infima. Let

$$g(a) = \prod \left\{ \frac{x \in \mathfrak{A}}{fx \sqsupseteq a} \right\}.$$

Since f preserves infima, we have

$$f(g(a)) = \prod \left\{ \frac{f(x)}{x \in \mathfrak{A}, fx \sqsupseteq a} \right\} \sqsupseteq a.$$

$$g(f(b)) = \prod \left\{ \frac{x \in \mathfrak{A}}{fx \sqsupseteq fb} \right\} \sqsubseteq b.$$

Obviously f is monotone and thus g is also monotone.

So f is the upper adjoint of g . □

COROLLARY 133. Let f be a function from a complete lattice \mathfrak{A} to a poset \mathfrak{B} . Then:

- 1°. f is an upper adjoint of a function $\mathfrak{B} \rightarrow \mathfrak{A}$ iff f preserves all infima in \mathfrak{A} .
- 2°. f is a lower adjoint of a function $\mathfrak{B} \rightarrow \mathfrak{A}$ iff f preserves all suprema in \mathfrak{A} .

2.1.13.1. *Order and composition of Galois connections.* Following [32] we will denote the set of Galois connection between posets \mathfrak{A} and \mathfrak{B} as $\mathfrak{A} \otimes \mathfrak{B}$.

DEFINITION 134. I will order Galois connections by the formula: $f \sqsubseteq g \Leftrightarrow f^* \sqsubseteq g^*$ (where $f^* \sqsubseteq g^* \Leftrightarrow \forall x \in \mathfrak{A} : f^*x \sqsubseteq g^*x$).

OBVIOUS 135. Galois connections $\mathfrak{A} \otimes \mathfrak{B}$ between two given posets form a poset.

PROPOSITION 136. $f \sqsubseteq g \Leftrightarrow f_* \sqsupseteq g_*$.

PROOF. It is enough to prove $f \sqsubseteq g \Rightarrow f_* \sqsupseteq g_*$ (the rest follows from the fact that a Galois connection is determined by one adjoint).

Really, let $f \sqsubseteq g$. Then $f_0^* \sqsubseteq f_1^*$ and thus:

$$f_{0*}(b) = \max \left\{ \frac{x \in \mathfrak{A}}{f_0^* x \sqsubseteq b} \right\}, f_{1*}(b) = \max \left\{ \frac{x \in \mathfrak{A}}{f_1^* x \sqsubseteq b} \right\}.$$

Thus $f_{0*}(b) \sqsupseteq f_{1*}(b)$ for every $b \in \mathfrak{B}$ and so $f_{0*} \sqsupseteq f_{1*}$. \square

DEFINITION 137. Composition of Galois connections is defined by the formula: $g \circ f = (g^* \circ f^*, f_* \circ g_*)$.

PROPOSITION 138. Composition of Galois connections is a Galois connection.

PROOF. $g^* \circ f^*$ and $f_* \circ g_*$ are monotone as composition of monotone functions;

$$(g^* \circ f^*)x \sqsubseteq z \Leftrightarrow g^* f^* x \sqsubseteq z \Leftrightarrow f^* x \sqsubseteq g_* z \Leftrightarrow x \sqsubseteq f_* g_* z \Leftrightarrow x \sqsubseteq (f_* \circ g_*)z.$$

\square

OBVIOUS 139. Composition of Galois connections preserves order.

2.1.13.2. *Antitone Galois connections.*

DEFINITION 140. An *antitone Galois connection* between posets \mathfrak{A} and \mathfrak{B} is a Galois connection between \mathfrak{A} and dual \mathfrak{B} .

OBVIOUS 141. An antitone Galois connection is a pair of antitone functions $f : \mathfrak{A} \rightarrow \mathfrak{B}, g : \mathfrak{B} \rightarrow \mathfrak{A}$ such that $b \sqsubseteq fa \Leftrightarrow a \sqsubseteq gb$ for every $a \in \mathfrak{A}, b \in \mathfrak{B}$.

Such f and g are called *polarities* (between \mathfrak{A} and \mathfrak{B}).

OBVIOUS 142. $f \sqcup S = \bigcap \langle f \rangle^* S$ if f is a polarity between \mathfrak{A} and \mathfrak{B} and $S \in \mathcal{P}\mathfrak{A}$.

Galois connections (particularly between boolean lattices) are studied in [32] and [33].

2.1.14. Co-Brouwerian lattices.

DEFINITION 143. Let \mathfrak{A} be a poset. *Pseudocomplement* of $a \in \mathfrak{A}$ is

$$\max \left\{ \frac{c \in \mathfrak{A}}{c \succ a} \right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

DEFINITION 144. Let \mathfrak{A} be a poset. *Dual pseudocomplement* of $a \in \mathfrak{A}$ is

$$\min \left\{ \frac{c \in \mathfrak{A}}{c \equiv a} \right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

PROPOSITION 145. If a is a complemented element of a bounded distributive lattice, then \bar{a} is both pseudocomplement and dual pseudocomplement of a .

PROOF. Because of duality it is enough to prove that \bar{a} is pseudocomplement of a .

We need to prove $c \succ a \Rightarrow c \sqsubseteq \bar{a}$ for every element c of our poset, and $\bar{a} \succ a$. The second is obvious. Let's prove $c \succ a \Rightarrow c \sqsubseteq \bar{a}$.

Really, let $c \succ a$. Then $c \sqcap a = \perp$; $\bar{a} \sqcup (c \sqcap a) = \bar{a}$; $(\bar{a} \sqcup c) \sqcap (\bar{a} \sqcup a) = \bar{a}$; $\bar{a} \sqcup c = \bar{a}$; $c \sqsubseteq \bar{a}$. \square

DEFINITION 146. Let \mathfrak{A} be a join-semilattice. Let $a, b \in \mathfrak{A}$. *Pseudodifference* of a and b is

$$\min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.

REMARK 147. I do not require that a^* is undefined if there are no pseudocomplement of a and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplement, and quasidifference which generalize pseudo-* counterparts. I will denote a^* the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

OBVIOUS 148. Dual pseudocomplement is the dual of pseudocomplement.

THEOREM 149. Let \mathfrak{A} be a distributive lattice with least element. Let $a, b \in \mathfrak{A}$. If $a \setminus b$ exists, then $a \setminus^* b$ also exists and $a \setminus^* b = a \setminus b$.

PROOF. Because \mathfrak{A} be a distributive lattice with least element, the definition of $a \setminus b$ is correct.

Let $x = a \setminus b$ and let $S = \left\{ \frac{y \in \mathfrak{A}}{a \sqsubseteq b \sqcup y} \right\}$.

We need to show

- 1°. $x \in S$;
- 2°. $y \in S \Rightarrow x \sqsubseteq y$ (for every $y \in \mathfrak{A}$).

Really,

- 1°. Because $b \sqcup x = a \sqcup b$.
- 2°.

$$\begin{aligned}
& y \in S \\
\Rightarrow & a \sqsubseteq b \sqcup y && \text{(by definition of } S) \\
\Rightarrow & a \sqcup b \sqsubseteq b \sqcup y \\
\Rightarrow & x \sqcup b \sqsubseteq b \sqcup y && \text{(since } x \sqcup b = a \sqcup b) \\
\Rightarrow & x \sqcap (x \sqcup b) \sqsubseteq x \sqcap (b \sqcup y) \\
\Rightarrow & (x \sqcap x) \sqcup (x \sqcap b) \sqsubseteq (x \sqcap b) \sqcup (x \sqcap y) && \text{(by distributive law)} \\
\Rightarrow & x \sqcup \perp \sqsubseteq \perp \sqcup (x \sqcap y) && \text{(since } x \sqcap b = \perp) \\
\Rightarrow & x \sqsubseteq x \sqcap y \\
\Rightarrow & x \sqsubseteq y.
\end{aligned}$$

□

DEFINITION 150. *Co-brouwerian lattice* is a lattice for which pseudodifference of any two its elements is defined.

PROPOSITION 151. Every non-empty co-brouwerian lattice \mathfrak{A} has least element.

PROOF. Let a be an arbitrary lattice element. Then

$$a \setminus^* a = \min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq a \sqcup z} \right\} = \min \mathfrak{A}.$$

So $\min \mathfrak{A}$ exists. □

DEFINITION 152. *Co-Heyting lattice* is co-brouwerian lattice with greatest element.

DEFINITION 153. A *co-frame* is the same as a complete co-brouwerian lattice.

THEOREM 154. For a co-brouwerian lattice $a \sqcup -$ is an upper adjoint of $- \setminus^* a$ for every $a \in \mathfrak{A}$.

PROOF. $g(b) = \min \left\{ \frac{x \in \mathfrak{A}}{a \sqcup x \sqsupseteq b} \right\} = b \setminus^* a$ exists for every $b \in \mathfrak{A}$ and thus is the lower adjoint of $a \sqcup -$. □

COROLLARY 155. $\forall a, x, y \in \mathfrak{A} : (x \setminus^* a \sqsubseteq y \Leftrightarrow x \sqsubseteq a \sqcup y)$ for a co-brouwerian lattice.

COROLLARY 156. For a co-brouwerian lattice $a \sqcup \sqcap S = \sqcap \langle a \sqcup \rangle^* S$ whenever $\sqcap S$ exists (for a being a lattice element and S being a set of lattice elements).

DEFINITION 157. Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a complete lattice. *Quasidifference* $a \setminus^* b$ is defined by the formula:

$$a \setminus^* b = \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

REMARK 158. A more detailed theory of quasidifference (as well as quasicomplement and dual quasicomplement) will be considered below.

LEMMA 159. $(a \setminus^* b) \sqcup b = a \sqcup b$ for elements a, b of a meet infinite distributive complete lattice.

PROOF.

$$\begin{aligned} (a \setminus^* b) \sqcup b &= \\ \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\} \sqcup b &= \\ \sqcap \left\{ \frac{z \sqcup b}{z \in \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\} &= \\ \sqcap \left\{ \frac{t \in \mathfrak{A}}{t \sqsupseteq b, a \sqsubseteq t} \right\} &= \\ a \sqcup b. & \end{aligned}$$

□

THEOREM 160. The following are equivalent for a complete lattice \mathfrak{A} :

- 1°. \mathfrak{A} is a co-frame.
- 2°. \mathfrak{A} is meet infinite distributive.
- 3°. \mathfrak{A} is a co-brouwerian lattice.
- 4°. \mathfrak{A} is a co-Heyting lattice.
- 5°. $a \sqcup -$ has lower adjoint for every $a \in \mathfrak{A}$.

PROOF.

□

1° \Leftrightarrow 3°. Because it is complete.

3° \Leftrightarrow 4°. Obvious (taking into account completeness of \mathfrak{A}).

5° \Rightarrow 2°. Let $- \setminus^* a$ be the lower adjoint of $a \sqcup -$. Let $S \in \mathcal{P}\mathfrak{A}$. For every $y \in S$ we have $y \sqsupseteq (a \sqcup y) \setminus^* a$ by properties of Galois connections; consequently $y \sqsupseteq (\sqcap \langle a \sqcup \rangle^* S) \setminus^* a$; $\sqcap S \sqsupseteq (\sqcap \langle a \sqcup \rangle^* S) \setminus^* a$. So

$$a \sqcup \sqcap S \sqsupseteq \left((\sqcap \langle a \sqcup \rangle^* S) \setminus^* a \right) \sqcup a \sqsupseteq \sqcap \langle a \sqcup \rangle^* S.$$

But $a \sqcup \sqcap S \sqsubseteq \sqcap \langle a \sqcup \rangle^* S$ is obvious.

2° \Rightarrow 3°. Let $a \setminus^* b = \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. To prove that \mathfrak{A} is a co-brouwerian lattice it is enough to prove $a \sqsubseteq b \sqcup (a \setminus^* b)$. But it follows from the lemma.

3° \Rightarrow 5°. $a \setminus^* b = \min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. So $a \sqcup -$ is the upper adjoint of $- \setminus^* a$.

2° \Rightarrow 5°. Because $a \sqcup -$ preserves all meets.

COROLLARY 161. Co-brouwerian lattices are distributive.

The following theorem is essentially borrowed from [19]:

THEOREM 162. A lattice \mathfrak{A} with least element \perp is co-brouwerian with pseudodifference \setminus^* iff \setminus^* is a binary operation on \mathfrak{A} satisfying the following identities:

- 1°. $a \setminus^* a = \perp$;

- 2°. $a \sqcup (b \setminus^* a) = a \sqcup b$;
 3°. $b \sqcup (b \setminus^* a) = b$;
 4°. $(b \sqcup c) \setminus^* a = (b \setminus^* a) \sqcup (c \setminus^* a)$.

PROOF.

\Leftarrow . We have

$$c \sqsupseteq b \setminus^* a \Rightarrow c \sqcup a \sqsupseteq a \sqcup (b \setminus^* a) = a \sqcup b \sqsupseteq b;$$

$$c \sqcup a \sqsupseteq b \Rightarrow c = c \sqcup (c \setminus^* a) \sqsupseteq (a \setminus^* a) \sqcup (c \setminus^* a) = (a \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a.$$

So $c \sqsupseteq b \setminus^* a \Leftrightarrow c \sqcup a \sqsupseteq b$ that is $a \sqcup -$ is an upper adjoint of $-\setminus^* a$. By a theorem above our lattice is co-brouwerian. By another theorem above \setminus^* is a pseudodifference.

\Rightarrow .

1°. Obvious.

2°.

$$\begin{aligned} a \sqcup (b \setminus^* a) &= \\ a \sqcup \prod \left\{ \frac{z \in \mathfrak{A}}{b \sqsubseteq a \sqcup z} \right\} &= \\ \prod \left\{ \frac{a \sqcup z}{z \in \mathfrak{A}, b \sqsubseteq a \sqcup z} \right\} &= \\ a \sqcup b. & \end{aligned}$$

$$3°. \quad b \sqcup (b \setminus^* a) = b \sqcup \prod \left\{ \frac{z \in \mathfrak{A}}{b \sqsubseteq a \sqcup z} \right\} = \prod \left\{ \frac{b \sqcup z}{z \in \mathfrak{A}, b \sqsubseteq a \sqcup z} \right\} = b.$$

4°. Obviously $(b \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a$ and $(b \sqcup c) \setminus^* a \sqsupseteq c \setminus^* a$. Thus $(b \sqcup c) \setminus^* a \sqsupseteq (b \setminus^* a) \sqcup (c \setminus^* a)$. We have

$$\begin{aligned} (b \setminus^* a) \sqcup (c \setminus^* a) \sqcup a &= \\ ((b \setminus^* a) \sqcup a) \sqcup ((c \setminus^* a) \sqcup a) &= \\ (b \sqcup a) \sqcup (c \sqcup a) &= \\ a \sqcup b \sqcup c \sqsupseteq & \\ b \sqcup c. & \end{aligned}$$

From this by definition of adjoints: $(b \setminus^* a) \sqcup (c \setminus^* a) \sqsupseteq (b \sqcup c) \setminus^* a$. \square

THEOREM 163. $(\sqcup S) \setminus^* a = \sqcup_{x \in S} (x \setminus^* a)$ for all $a \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a co-brouwerian lattice and $\sqcup S$ is defined.

PROOF. Because lower adjoint preserves all suprema. \square

THEOREM 164. $(a \setminus^* b) \setminus^* c = a \setminus^* (b \sqcup c)$ for elements a, b, c of a co-frame.

$$\text{PROOF. } a \setminus^* b = \prod \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

$$(a \setminus^* b) \setminus^* c = \prod \left\{ \frac{z \in \mathfrak{A}}{a \setminus^* b \sqsubseteq c \sqcup z} \right\}.$$

$$a \setminus^* (b \sqcup c) = \prod \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup c \sqcup z} \right\}.$$

It is left to prove $a \setminus^* b \sqsubseteq c \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup c \sqcup z$. But this follows from corollary 155. \square

COROLLARY 165. $((a_0 \setminus^* a_1) \setminus^* \dots) \setminus^* a_n = a_0 \setminus^* (a_1 \sqcup \dots \sqcup a_n)$.

PROOF. By math induction. \square

2.1.15. Dual pseudocomplement on co-Heyting lattices.

THEOREM 166. For co-Heyting algebras $\top \setminus^* b = b^+$.

PROOF.

$$\top \setminus^* b = \min \left\{ \frac{z \in \mathfrak{A}}{\top \sqsubseteq b \sqcup z} \right\} = \min \left\{ \frac{z \in \mathfrak{A}}{\top = b \sqcup z} \right\} = \min \left\{ \frac{z \in \mathfrak{A}}{b \equiv z} \right\} = b^+.$$

□

THEOREM 167. $(a \sqcap b)^+ = a^+ \sqcup b^+$ for every elements a, b of a co-Heyting algebra.

PROOF. $a \sqcup (a \sqcap b)^+ \sqsupseteq (a \sqcap b) \sqcup (a \sqcap b)^+ \sqsupseteq \top$. So $a \sqcup (a \sqcap b)^+ \sqsupseteq \top$; $(a \sqcap b)^+ \sqsupseteq \top \setminus^* a = a^+$.

We have $(a \sqcap b)^+ \sqsupseteq a^+$. Similarly $(a \sqcap b)^+ \sqsupseteq b^+$. Thus $(a \sqcap b)^+ \sqsupseteq a^+ \sqcup b^+$.

On the other hand, $a^+ \sqcup b^+ \sqcup (a \sqcap b) = (a^+ \sqcup b^+ \sqcup a) \sqcap (a^+ \sqcup b^+ \sqcup b)$. Obviously $a^+ \sqcup b^+ \sqcup a = a^+ \sqcup b^+ \sqcup b = \top$. So $a^+ \sqcup b^+ \sqcup (a \sqcap b) \sqsupseteq \top$ and thus $a^+ \sqcup b^+ \sqsupseteq \top \setminus^* (a \sqcap b) = (a \sqcap b)^+$.

So $(a \sqcap b)^+ = a^+ \sqcup b^+$. □

2.2. Intro to category theory

This is a *very* basic introduction to category theory.

DEFINITION 168. A *directed multigraph* (also known as *quiver*) is:

- 1°. a set \mathcal{O} (*vertices*);
- 2°. a set \mathcal{M} (*edges*);
- 3°. functions Src and Dst (*source* and *destination*) from \mathcal{M} to \mathcal{O} .

Note that in category theory vertices are called *objects* and edges are called *morphisms*.

DEFINITION 169. A *precategory* is a directed multigraph together with a partial binary operation \circ on the set \mathcal{M} such that $g \circ f$ is defined iff $\text{Dst } f = \text{Src } g$ (for every morphisms f and g) such that

- 1°. $\text{Src}(g \circ f) = \text{Src } f$ and $\text{Dst}(g \circ f) = \text{Dst } g$ whenever the composition $g \circ f$ of morphisms f and g is defined.
- 2°. $(h \circ g) \circ f = h \circ (g \circ f)$ whenever compositions in this equation are defined.

DEFINITION 170. The set $\text{Hom}(A, B)$ (also denoted as $\text{Hom}_C(A, B)$ or just $C(A, B)$, where C is our category) (morphisms from an object A to an object B) is exactly morphisms which have A as the source and B as the destination.

DEFINITION 171. *Identity morphism* is such a morphism e that $e \circ f = f$ and $g \circ e = g$ whenever compositions in these formulas are defined.

DEFINITION 172. A *category* is a precategory with additional requirement that for every object X there exists identity morphism 1_X .

PROPOSITION 173. For every object X there exist no more than one identity morphism.

PROOF. Let p and q be both identity morphisms for a object X . Then $p = p \circ q = q$. □

DEFINITION 174. An *isomorphism* is such a morphism f of a category that there exists a morphism f^{-1} (*inverse* of f) such that $f \circ f^{-1} = 1_{\text{Dst } f}$ and $f^{-1} \circ f = 1_{\text{Src } f}$.

PROPOSITION 175. An isomorphism has exactly one inverse.

PROOF. Let g and h be both inverses of f . Then $h = h \circ 1_{\text{Dst } f} = h \circ f \circ g = 1_{\text{Src } f} \circ g = g$. \square

DEFINITION 176. A *groupoid* is a category all of whose morphisms are isomorphisms.

DEFINITION 177. A morphism whose source is the same as destination is called *endomorphism*.

DEFINITION 178. An *involution* or *involution morphism* is an endomorphism f that $f \circ f = 1_{\text{Obj } f}$. In other words, an involution is such a self-inverse (that is conforming to the formula $f = f^{-1}$) isomorphism.

DEFINITION 179. *Functor* from category C to category D is a mapping F which associates every object X of C with an object $F(X)$ of D and every morphism $f : X \rightarrow Y$ of C with morphism $F(f) : F(X) \rightarrow F(Y)$ of D , such that:

- 1°. $F(g \circ f) = F(g) \circ F(f)$ for every composable morphisms f, g of C ;
- 2°. $F(1_X^C) = 1_{F(X)}^D$ for every object X of C .

DEFINITION 180. *Opposite category* C^{op} of category C is the category where “all arrows are reversed” that is every morphism f is replaced with so called *opposite morphism* f^{op} such that $\text{Src } f^{\text{op}} = \text{Dst } f$, $\text{Dst } f^{\text{op}} = \text{Src } f$ and $g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}$ (composition taken in the “opposite order”).

DEFINITION 181. A *monomorphism* (also called a *monic* morphism or a *mono*) is a left-cancellative morphism. That is, an arrow $f : X \rightarrow Y$ such that for all objects Z and all morphisms $g_1, g_2 : Z \rightarrow X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2.$$

Monomorphisms are a categorical generalization of injective functions (also called “one-to-one functions”); in some categories the notions coincide, but monomorphisms are more general.

DEFINITION 182. The categorical dual of a monomorphism is an *epimorphism*, i.e. a monomorphism in a category C is an epimorphism in the dual category C^{op} that is it conforms to the formula

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2.$$

2.2.1. Some important examples of categories.

EXERCISE 183. Prove that the below examples of categories are really categories.

DEFINITION 184. The category **Set** is:

- Objects are small sets.
- Morphisms from an object A to an object B are triples (A, B, f) where f is a function from A to B .
- Composition of morphisms is defined by the formula: $(B, C, g) \circ (A, B, f) = (A, C, g \circ f)$ where $g \circ f$ is function composition.

DEFINITION 185. The category **Rel** is:

- Objects are small sets.
- Morphisms from an object A to an object B are triples (A, B, f) where f is a binary relation between A and B .
- Composition of morphisms is defined by the formula: $(B, C, g) \circ (A, B, f) = (A, C, g \circ f)$ where $g \circ f$ is relation composition.

I will denote $\text{GR}(A, B, f) = f$ for any morphism (A, B, f) of either **Set** or **Rel**.

DEFINITION 186. A *subcategory* of a category C is a category whose set of objects is a subset of the set of objects of C and whose set of morphisms is a subset of the set of morphisms of C .

DEFINITION 187. *Wide subcategory* of a category $(\mathcal{O}, \mathcal{M})$ is a category $(\mathcal{O}, \mathcal{M}')$ where $\mathcal{M} \subseteq \mathcal{M}'$ and the composition on $(\mathcal{O}, \mathcal{M}')$ is a restriction of composition of $(\mathcal{O}, \mathcal{M})$. (Similarly *wide sub-precategory* can be defined.)

2.2.2. Commutative diagrams.

DEFINITION 188. A *finite path in directed multigraph* is a tuple $[[e_0, \dots, e_n]]$ of edges (where $i \in \mathbb{N}$) such that $\text{Dst } e_i = \text{Src } e_{i+1}$ for every $i = 0, \dots, n-1$.

DEFINITION 189. The vertices of a finite path are $\text{Src } e_0$, $\text{Dst } e_0 = \text{Src } e_1$, $\text{Dst } e_1 = \text{Src } e_2$, \dots , $\text{Dst } e_n$.

DEFINITION 190. Composition of finite paths $[[e_0, \dots, e_n]]$ and $[[e_k, \dots, e_m]]$ (where $\text{Dst } e_n = \text{Src } e_k$) is the path $[[e_0, \dots, e_n, e_k, \dots, e_m]]$. (It is a path because $\text{Dst } e_n = \text{Src } e_k$.)

DEFINITION 191. A *cycle* is a finite path whose first vertex is the same as the last vertex (in other words $\text{Dst } e_n = \text{Src } e_0$).

DEFINITION 192. A *diagram* in C is a directed multigraph, whose vertices are labeled with objects of C and whose edges are labeled with morphisms of C .

I will denote the morphism corresponding to a edge e as $D(e)$.

DEFINITION 193. A diagram in C is *commutative* when the composition of morphisms corresponding to a finite path is always the same for finite paths from a fixed vertex A to a fixed vertex B independently of the path choice.

We will say “*commutative diagram*” when commutativity of a diagram is implied by the context.

REMARK 194. See [Wikipedia](#) for more on definition and examples of commutative diagrams.

The following is an example of a commutative diagram in **Set** (because $x + 5 - 3 = x + 4 - 2$):

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{+5} & \mathbb{N} \\ \downarrow +4 & & \downarrow -3 \\ \mathbb{N} & \xrightarrow{-2} & \mathbb{N} \end{array}$$

We are especially interested in the special case of commutative diagrams every morphism of which is an isomorphism. So, the below theorem.

THEOREM 195. If morphisms corresponding to every edge e_i of a cycle $[[e_0, \dots, e_n]]$ are isomorphisms then the following are equivalent:

- The morphism induced by $[[e_0, \dots, e_n]]$ is identity.
- The morphism induced by $[[e_n, e_0, \dots, e_{n-1}]]$ is identity.
- The morphism induced by $[[e_{n-1}, e_n, e_0, \dots, e_{n-2}]]$ is identity.
- ...
- The morphism induced by $[[e_1, e_2, \dots, e_n, e_0]]$ is identity.

In other words, the cycle being an identity does not depend on the choice of the start edge in the cycle.

PROOF. Each step in the proof is like:

$$\begin{aligned} D(n) \circ \dots \circ D(e_0) &= 1_{\text{Src } D(e_0)} \Leftrightarrow \\ D(n)^{-1} \circ D(n) \circ \dots \circ D(e_0) \circ D(n) &= D(n)^{-1} \circ 1_{\text{Src } D(e_0)} \circ D(n) \Leftrightarrow \\ D(n-1) \circ \dots \circ D(e_0) \circ D(n) &= 1_{\text{Src } D(e_n)}. \end{aligned}$$

□

LEMMA 196. Let f, g, h be isomorphisms. Let $g \circ f = h^{-1}$. The diagram at the figure 1 is commutative, every cycle in the diagram is an identity.

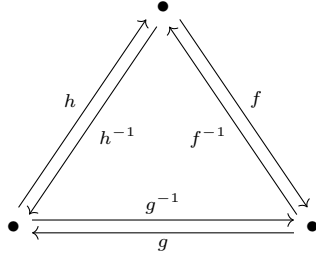


FIGURE 1

PROOF. We will prove by induction that every cycle of the length N in the diagram is an identity.

For cycles of length 2 it holds by definition of isomorphism.

For cycles of length 3 it holds by theorem 195.

Consider a cycle of length above 3. It is easy to show that this cycle contains a sub-cycle of length 3 or below. (Consider three first edges $a \xrightarrow{e_0} b \xrightarrow{e_1} c \xrightarrow{e_2} d$ of the path, by pigeonhole principle we have that there are equal elements among a, b, c, d .) We can exclude the sub-cycle because it is identity. Thus we reduce to cycles of lesser length. Applying math induction, we get that every cycle in the diagram is an identity.

That the diagram is commutative follows from it (because for paths σ, τ we have the paths $\sigma \circ \tau^{-1}$ and $\tau \circ \sigma^{-1}$ being identities). □

LEMMA 197. Let f, g, h, t be isomorphisms. Let $t \circ h \circ g \circ f = 1_{\text{Src } f}$. The diagram at the figure 2 is commutative, every cycle in the diagram is an identity.

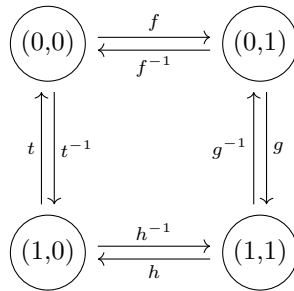


FIGURE 2

PROOF. Assign to every vertex (i, j) of the diagram morphism $W(i, j)$ defined by the table 1.

It is easy to verify by induction that the morphism corresponding to every path in the diagram starting at the vertex $(0, 0)$ and ending with a vertex (x, y) is $W(x, y)$.

Thus the morphism corresponding to every cycle starting at the vertex $(0, 0)$ is identity.

By symmetry, the morphism corresponding to every cycle is identity.

TABLE 1

i	j	$W(i, j)$
0	0	$1_{\text{Src } f}$
0	1	f
1	0	t^{-1}
1	1	$g \circ f$

That the diagram is commutative follows from it (because for paths σ, τ we have the paths $\sigma \circ \tau^{-1}$ and $\tau \circ \sigma^{-1}$ being identities). \square

2.3. Intro to group theory

DEFINITION 198. A *semigroup* is a pair of a set G and an associative binary operation on G .

DEFINITION 199. A *group* is a pair of a set G and a binary operation \cdot on G such that:

- 1°. $(h \cdot g) \cdot f = h \cdot (g \cdot f)$ for every $f, g, h \in G$.
- 2°. There exists an element e (*identity*) of G such that $f \cdot e = e \cdot f = f$ for every $f \in G$.
- 3°. For every element f there exists an element f^{-1} (*inverse of f*) such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

OBVIOUS 200. Every group is a semigroup.

PROPOSITION 201. In every group there exists exactly one identity element.

PROOF. If p and q are both identities, then $p = p \cdot q = q$. \square

PROPOSITION 202. Every group element has exactly one inverse.

PROOF. Let p and q be both inverses of $f \in G$. Then $f \cdot p = p \cdot f = e$ and $f \cdot q = q \cdot f = e$. Then $p = p \cdot e = p \cdot f \cdot q = e \cdot q = q$. \square

PROPOSITION 203. $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$ for every group elements f and g .

PROOF. $(f^{-1} \cdot g^{-1}) \cdot (g \cdot f) = f^{-1} \cdot g^{-1} \cdot g \cdot f = f^{-1} \cdot e \cdot f = f^{-1} \cdot f = e$. Similarly $(g \cdot f) \cdot (f^{-1} \cdot g^{-1}) = e$. So $f^{-1} \cdot g^{-1}$ is the inverse of $g \cdot f$. \square

DEFINITION 204. A *permutation group* on a set D is a group whose elements are functions on D and whose composition is function composition.

OBVIOUS 205. Elements of a permutation group are bijections.

DEFINITION 206. A *transitive* permutation group on a set D is such a permutation group G on D that for every $x, y \in D$ there exists $r \in G$ such that $y = r(x)$.

A groupoid with single (arbitrarily chosen) object corresponds to every group. The morphisms of this category are elements of the group and the composition of morphisms is the group operation.

More on order theory

3.1. Straight maps and separation subsets

3.1.1. Straight maps.

DEFINITION 207. An *order reflecting* map from a poset \mathfrak{A} to a poset \mathfrak{B} is such a function f that (for every $x, y \in \mathfrak{A}$)

$$fx \sqsubseteq fy \Rightarrow x \sqsubseteq y.$$

OBVIOUS 208. Order embeddings are exactly the same as monotone and order reflecting maps.

DEFINITION 209. Let f be a monotone map from a meet-semilattice \mathfrak{A} to some poset \mathfrak{B} . I call f a *straight* map when

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa = f(a \sqcap b)).$$

PROPOSITION 210. The following statements are equivalent for a monotone map f :

- 1°. f is a straight map.
- 2°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \sqsubseteq f(a \sqcap b))$.
- 3°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \not\sqsupseteq f(a \sqcap b))$.
- 4°. $\forall a, b \in \mathfrak{A} : (fa \sqsupseteq f(a \sqcap b) \Rightarrow fa \not\sqsubseteq fb)$.

PROOF.

1° \Leftrightarrow 2° \Leftrightarrow 3°. Due $fa \sqsupseteq f(a \sqcap b)$.

3° \Leftrightarrow 4°. Obvious. □

REMARK 211. The definition of straight map can be generalized for any poset \mathfrak{A} by the formula

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow \exists c \in \mathfrak{A} : (c \sqsubseteq a \wedge c \sqsubseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

PROPOSITION 212. Let f be a monotone map from a meet-semilattice \mathfrak{A} to a meet-semilattice \mathfrak{B} . If

$$\forall a, b \in \mathfrak{A} : f(a \sqcap b) = fa \sqcap fb$$

then f is a straight map.

PROOF. Let $fa \sqsubseteq fb$. Then $f(a \sqcap b) = fa \sqcap fb = fa$. □

PROPOSITION 213. Let f be a monotone map from a meet-semilattice \mathfrak{A} to some poset \mathfrak{B} . If f is order reflecting, then f is a straight map.

PROOF. $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b \Rightarrow a = a \sqcap b \Rightarrow fa = f(a \sqcap b)$. □

The following theorem is the main reason of why we are interested in straight maps:

THEOREM 214. If f is a straight monotone map from a meet-semilattice \mathfrak{A} then the following statements are equivalent:

- 1°. f is an injection.
- 2°. f is order reflecting.
- 3°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \sqsubset fb)$.
- 4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \neq fb)$.
- 5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \not\sqsupseteq fb)$.
- 6°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow a \not\sqsupset b)$.

PROOF.

- 1° \Rightarrow 3°. Let $a, b \in \mathfrak{A}$. Let $fa = fb \Rightarrow a = b$. Let $a \sqsubset b$. $fa \neq fb$ because $a \neq b$.
 $fa \sqsubseteq fb$ because $a \sqsubseteq b$. So $fa \sqsubset fb$.
- 2° \Rightarrow 1°. Let $a, b \in \mathfrak{A}$. Let $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b$. Let $fa = fb$. Then $a \sqsubseteq b$ and $b \sqsubseteq a$
and consequently $a = b$.
- 3° \Rightarrow 2°. Let $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \sqsubset fb)$. Let $a \not\sqsubseteq b$. Then $a \sqsupset a \sqcap b$. So
 $fa \sqsupset f(a \sqcap b)$. If $fa \sqsubseteq fb$ then $fa \sqsubseteq f(a \sqcap b)$ what is a contradiction.
- 3° \Rightarrow 5° \Rightarrow 4°. Obvious.
- 4° \Rightarrow 3°. Because $a \sqsubset b \Rightarrow a \sqsubseteq b \Rightarrow fa \sqsubseteq fb$.
- 5° \Leftrightarrow 6°. Obvious.

□

3.1.2. Separation subsets and full stars.

DEFINITION 215. $\partial_Y a = \left\{ \begin{array}{l} x \in Y \\ x \neq a \end{array} \right\}$ for an element a of a poset \mathfrak{A} and $Y \in \mathcal{P}\mathfrak{A}$.

DEFINITION 216. Full star of $a \in \mathfrak{A}$ is $\star a = \partial_{\mathfrak{A}} a$.

PROPOSITION 217. If \mathfrak{A} is a meet-semilattice, then \star is a straight monotone map.

PROOF. Monotonicity is obvious. Let $\star a \not\sqsubseteq \star(a \sqcap b)$. Then it exists $x \in \star a$ such
that $x \notin \star(a \sqcap b)$. So $x \sqcap a \notin \star b$ but $x \sqcap a \in \star a$ and consequently $\star a \not\sqsubseteq \star b$. □

DEFINITION 218. A *separation subset* of a poset \mathfrak{A} is such its subset Y that

$$\forall a, b \in \mathfrak{A} : (\partial_Y a = \partial_Y b \Rightarrow a = b).$$

DEFINITION 219. I call *separable* such poset that \star is an injection.

DEFINITION 220. I call *strongly separable* such poset that \star is order reflecting.

OBVIOUS 221. A poset is separable iff it has a separation subset.

OBVIOUS 222. A poset is strongly separable iff \star is order embedding.

OBVIOUS 223. Strong separability implies separability.

DEFINITION 224. A poset \mathfrak{A} has *disjunction property of Wallman* iff for any
 $a, b \in \mathfrak{A}$ either $b \sqsubseteq a$ or there exists a non-least element $c \sqsubseteq b$ such that $a \succ c$.

THEOREM 225. For a meet-semilattice with least element the following statements are equivalent:

- 1°. \mathfrak{A} is separable.
- 2°. \mathfrak{A} is strongly separable.
- 3°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \sqsubset \star b)$.
- 4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \neq \star b)$.
- 5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \not\sqsupseteq \star b)$.
- 6°. $\forall a, b \in \mathfrak{A} : (\star a \sqsubseteq \star b \Rightarrow a \not\sqsupset b)$.
- 7°. \mathfrak{A} conforms to Wallman's disjunction property.

8°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\perp\} : (c \succ a \wedge c \sqsubseteq b))$.

PROOF.

1° \Leftrightarrow 2° \Leftrightarrow 3° \Leftrightarrow 4° \Leftrightarrow 5° \Leftrightarrow 6°. By the above theorem.

8° \Rightarrow 4°. Let property 8° hold. Let $a \sqsubset b$. Then it exists element $c \sqsubseteq b$ such that $c \neq \perp$ and $c \sqcap a = \perp$. But $c \sqcap b \neq \perp$. So $\star a \neq \star b$.

2° \Rightarrow 7°. Let property 2° hold. Let $a \not\sqsubseteq b$. Then $\star a \not\sqsubseteq \star b$ that is it there exists $c \in \star a$ such that $c \notin \star b$, in other words $c \sqcap a \neq \perp$ and $c \sqcap b = \perp$. Let $d = c \sqcap a$. Then $d \sqsubseteq a$ and $d \neq \perp$ and $d \sqcap b = \perp$. So disjunction property of Wallman holds.

7° \Rightarrow 8°. Obvious.

8° \Rightarrow 7°. Let $b \not\sqsubseteq a$. Then $a \sqcap b \sqsubset b$ that is $a' \sqsubset b$ where $a' = a \sqcap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{\perp\} : (c \succ a' \wedge c \sqsubseteq b)$. We have $c \sqcap a = c \sqcap b \sqcap a = c \sqcap a' = \perp$. So $c \sqsubseteq b$ and $c \sqcap a = \perp$. Thus Wallman's disjunction property holds. □

PROPOSITION 226. Every boolean lattice is strongly separable.

PROOF. Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a boolean lattice and $a \neq b$. Then $a \sqcap \bar{b} \neq \perp$ or $\bar{a} \sqcap b \neq \perp$ because otherwise $a \sqcap \bar{b} = \perp$ and $a \sqcup \bar{b} = \top$ and thus $a = b$. Without loss of generality assume $a \sqcap \bar{b} \neq \perp$. Then $a \sqcap c \neq \perp$ and $b \sqcap c = \perp$ for $c = a \sqcap \bar{b} \neq \perp$, that is our lattice is separable.

It is strongly separable by theorem 225. □

3.1.3. Atomically Separable Lattices.

PROPOSITION 227. “atoms” is a straight monotone map (for any meet-semilattice).

PROOF. Monotonicity is obvious. The rest follows from the formula

$$\text{atoms}(a \sqcap b) = \text{atoms } a \cap \text{atoms } b$$

(corollary 109). □

DEFINITION 228. I will call *atomically separable* such a poset that “atoms” is an injection.

PROPOSITION 229. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \text{atoms } a \subset \text{atoms } b)$ iff \mathfrak{A} is atomically separable for a poset \mathfrak{A} .

PROOF.

\Leftarrow . Obvious.

\Rightarrow . Let $a \neq b$ for example $a \not\sqsubseteq b$. Then $a \sqcap b \sqsubset a$; $\text{atoms } a \supset \text{atoms}(a \sqcap b) = \text{atoms } a \cap \text{atoms } b$ and thus $\text{atoms } a \neq \text{atoms } b$. □

PROPOSITION 230. Any atomistic poset is atomically separable.

PROOF. We need to prove that $\text{atoms } a = \text{atoms } b \Rightarrow a = b$. But it is obvious because

$$a = \bigsqcup \text{atoms } a \quad \text{and} \quad b = \bigsqcup \text{atoms } b.$$

□

THEOREM 231. A complete lattice is atomistic iff it is atomically separable.

PROOF. Direct implication is the above proposition. Let's prove the reverse implication.

Let "atoms" be injective. Consider an element a of our poset. Let $b = \bigsqcup \text{atoms } a$. Obviously $b \sqsubseteq a$ and thus $\text{atoms } b \subseteq \text{atoms } a$. But if $x \in \text{atoms } a$ then $x \sqsubseteq b$ and thus $x \in \text{atoms } b$. So $\text{atoms } a = \text{atoms } b$. By injectivity $a = b$ that is $a = \bigsqcup \text{atoms } a$. \square

THEOREM 232. If a lattice with least element is atomic and separable then it is atomistic.

PROOF. Suppose the contrary that is $a \sqsupset \bigsqcup \text{atoms } a$. Then, because our lattice is separable, there exists $c \in \mathfrak{A}$ such that $c \sqcap a \neq \perp$ and $c \sqcap \bigsqcup \text{atoms } a = \perp$. There exists atom $d \sqsubseteq c$ such that $d \sqsubseteq c \sqcap a$. $d \sqcap \bigsqcup \text{atoms } a \sqsubseteq c \sqcap \bigsqcup \text{atoms } a = \perp$. But $d \in \text{atoms } a$. Contradiction. \square

THEOREM 233. Let \mathfrak{A} be an atomic meet-semilattice with least element. Then the following statements are equivalent:

- 1°. \mathfrak{A} is separable.
- 2°. \mathfrak{A} is strongly separable.
- 3°. \mathfrak{A} is atomically separable.
- 4°. \mathfrak{A} conforms to Wallman's disjunction property.
- 5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\perp\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1° \Leftrightarrow 2° \Leftrightarrow 4° \Leftrightarrow 5°. Proved above.

3° \Rightarrow 5°. Let our semilattice be atomically separable. Let $a \sqsubset b$. Then $\text{atoms } a \subset \text{atoms } b$ and there exists $c \in \text{atoms } b$ such that $c \notin \text{atoms } a$. $c \neq \perp$ and $c \sqsubseteq b$, from which (taking into account that c is an atom) $c \sqsubseteq b$ and $c \sqcap a = \perp$. So our semilattice conforms to the formula 5°.

5° \Rightarrow 3°. Let formula 5° hold. Then for any elements $a \sqsubset b$ there exists $c \neq \perp$ such that $c \sqsubseteq b$ and $c \sqcap a = \perp$. Because \mathfrak{A} is atomic there exists atom $d \sqsubseteq c$. $d \in \text{atoms } b$ and $d \notin \text{atoms } a$. So $\text{atoms } a \neq \text{atoms } b$ and $\text{atoms } a \subset \text{atoms } b$. Consequently $\text{atoms } a \subset \text{atoms } b$. \square

THEOREM 234. Any atomistic poset is strongly separable.

PROOF. $\star x \sqsubseteq \star y \Rightarrow \text{atoms } x \sqsubseteq \text{atoms } y \Rightarrow x \sqsubseteq y$ because $\text{atoms } x = \star x \cap \text{atoms}^{\mathfrak{A}}$. \square

3.2. Quasidifference and Quasicomplement

I've got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with \bigsqcup and \sqcap . Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo- counterparts.

REMARK 235. *Pseudocomplements* and *pseudodifferences* are standard terminology. *Quasi-* counterparts are my neologisms.

DEFINITION 236. Let \mathfrak{A} be a poset, $a \in \mathfrak{A}$. *Quasicomplement* of a is

$$a^* = \bigsqcup \left\{ \frac{c \in \mathfrak{A}}{c \asymp a} \right\}.$$

DEFINITION 237. Let \mathfrak{A} be a poset, $a \in \mathfrak{A}$. *Dual quasicomplement* of a is

$$a^+ = \sqcap \left\{ \frac{c \in \mathfrak{A}}{c \equiv a} \right\}.$$

I will denote quasicomplement and dual quasicomplement for a specific poset \mathfrak{A} as $a^*(\mathfrak{A})$ and $a^+(\mathfrak{A})$.

DEFINITION 238. Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a distributive lattice. *Quasidifference* of a and b is

$$a \setminus^* b = \prod \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

DEFINITION 239. Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a distributive lattice. *Second quasidifference* of a and b is

$$a \# b = \sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \asymp b} \right\}.$$

THEOREM 240. $a \setminus^* b = \prod \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\}$ where \mathfrak{A} is a distributive lattice and $a, b \in \mathfrak{A}$.

PROOF. Obviously $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \subseteq \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. Thus $\prod \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \supseteq a \setminus^* b$.

Let $z \in \mathfrak{A}$ and $z' = z \sqcap a$.

$a \sqsubseteq b \sqcup z \Rightarrow a \sqsubseteq (b \sqcup z) \sqcap a \Leftrightarrow a \sqsubseteq (b \sqcap a) \sqcup (z \sqcap a) \Leftrightarrow a \sqsubseteq (b \sqcap a) \sqcup z' \Rightarrow a \sqsubseteq b \sqcup z'$
and $a \sqsubseteq b \sqcup z \Leftarrow a \sqsubseteq b \sqcup z'$. Thus $a \sqsubseteq b \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup z'$.

If $z \in \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ then $a \sqsubseteq b \sqcup z$ and thus

$$z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\}.$$

But $z' \sqsubseteq z$ thus having $\prod \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \subseteq \prod \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. \square

REMARK 241. If we drop the requirement that \mathfrak{A} is distributive, two formulas for quasidifference (the definition and the last theorem) fork.

OBVIOUS 242. Dual quasicomplement is the dual of quasicomplement.

OBVIOUS 243.

- Every pseudocomplement is quasicomplement.
- Every dual pseudocomplement is dual quasicomplement.
- Every pseudodifference is quasidifference.

Below we will stick to the more general quasies than pseudos. If needed, one can check that a quasicomplement a^* is a pseudocomplement by the equation $a^* \asymp a$ (and analogously with other quasies).

Next we will express quasidifference through quasicomplement.

PROPOSITION 244.

- 1°. $a \setminus^* b = a \setminus^* (a \sqcap b)$ for any distributive lattice;
- 2°. $a \# b = a \# (a \sqcap b)$ for any distributive lattice with least element.

PROOF.

1°. $a \sqsubseteq (a \sqcap b) \sqcup z \Leftrightarrow a \sqsubseteq (a \sqcup z) \sqcap (b \sqcup z) \Leftrightarrow a \sqsubseteq a \sqcup z \wedge a \sqsubseteq b \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup z$.

Thus $a \setminus^* (a \sqcap b) = \prod \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq (a \sqcap b) \sqcup z} \right\} = \prod \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\} = a \setminus^* b$.

2°.

$$\begin{aligned}
a \# (a \sqcap b) &= \\
\sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap a \sqcap b = \perp} \right\} &= \\
\sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge (z \sqcap a) \sqcap a \sqcap b = \perp} \right\} &= \\
\sqcup \left\{ \frac{z \sqcap a}{z \in \mathfrak{A}, z \sqcap a \sqcap b = \perp} \right\} &= \\
\sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a, z \sqcap b = \perp} \right\} &= \\
a \# b. &
\end{aligned}$$

□

I will denote Da the lattice $\left\{ \frac{x \in \mathfrak{A}}{x \sqsubseteq a} \right\}$.

THEOREM 245. For $a, b \in \mathfrak{A}$ where \mathfrak{A} is a distributive lattice

- 1°. $a \setminus^* b = (a \sqcap b)^{+(Da)}$;
- 2°. $a \# b = (a \sqcap b)^{*(Da)}$ if \mathfrak{A} has least element.

PROOF.

1°.

$$\begin{aligned}
(a \sqcap b)^{+(Da)} &= \\
\sqcap \left\{ \frac{c \in Da}{c \sqcup (a \sqcap b) = a} \right\} &= \\
\sqcap \left\{ \frac{c \in Da}{c \sqcup (a \sqcap b) \sqsupseteq a} \right\} &= \\
\sqcap \left\{ \frac{c \in Da}{(c \sqcup a) \sqcap (c \sqcup b) \sqsupseteq a} \right\} &= \\
\sqcap \left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcup b \sqsupseteq a} \right\} &= \\
a \setminus^* b. &
\end{aligned}$$

2°.

$$\begin{aligned}
(a \sqcap b)^{*(Da)} &= \\
\sqcup \left\{ \frac{c \in Da}{c \sqcap a \sqcap b = \perp} \right\} &= \\
\sqcup \left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcap a \sqcap b = \perp} \right\} &= \\
\sqcup \left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcap b = \perp} \right\} &= \\
a \# b. &
\end{aligned}$$

□

PROPOSITION 246. $(a \sqcup b) \setminus^* b \sqsubseteq a$ for an arbitrary complete lattice.

PROOF. $(a \sqcup b) \setminus^* b = \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqcup b \sqsubseteq b \sqcup z} \right\}$.

But $a \sqsubseteq z \Rightarrow a \sqcup b \sqsubseteq b \sqcup z$. So $\left\{ \frac{z \in \mathfrak{A}}{a \sqcup b \sqsubseteq b \sqcup z} \right\} \supseteq \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\}$.

Consequently, $(a \sqcup b) \setminus^* b \sqsubseteq \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\} = a$.

□

3.3. Several equal ways to express pseudodifference

THEOREM 247. For an atomistic co-brouwerian lattice \mathfrak{A} and $a, b \in \mathfrak{A}$ the following expressions are always equal:

- 1°. $a \setminus^* b = \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);
- 2°. $a \# b = \sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$ (second quasidifference of a and b);
- 3°. $\sqcup(\text{atoms } a \setminus \text{atoms } b)$.

PROOF.

Proof of $1^\circ = 3^\circ$.

$$\begin{aligned}
 a \setminus^* b &= \\
 \left(\sqcup \text{atoms } a \right) \setminus^* b &= \text{(theorem 163)} \\
 \sqcup_{A \in \text{atoms } a} (A \setminus^* b) &= \\
 \sqcup_{A \in \text{atoms } a} \left(\begin{cases} A & \text{if } A \notin \text{atoms } b \\ \perp & \text{if } A \in \text{atoms } b \end{cases} \right) &= \\
 \sqcup \left\{ \frac{A}{A \in \text{atoms } a, A \notin \text{atoms } b} \right\} &= \\
 \sqcup(\text{atoms } a \setminus \text{atoms } b). &
 \end{aligned}$$

Proof of $2^\circ = 3^\circ$. $a \setminus^* b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$\begin{aligned}
 a \setminus^* b &= \\
 \sqcup(\text{atoms } a \setminus \text{atoms } b) &= \\
 \sqcup \left\{ \frac{z \in \text{atoms } a}{z \sqcap b = \perp} \right\}. &
 \end{aligned}$$

So $\sqcup \left\{ \frac{z \in \text{atoms } a}{z \sqcap b = \perp} \right\}$ is defined.

If $z \sqsubseteq a \wedge z \sqcap b = \perp$ then $z' = \sqcup \left\{ \frac{x \in \text{atoms } z}{x \sqcap b = \perp} \right\}$ is defined because $z' = z \setminus^* b$ (atomisticity taken into account). z' is a lower bound for $\left\{ \frac{z \in \text{atoms } a}{x \sqcap b = \perp} \right\}$.

Thus $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$ and so $\sqcup \left\{ \frac{z \in \text{atoms } a}{z \sqcap b = \perp} \right\}$ is an upper bound of $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$.

If y is above every $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$ then y is above every $z \in \text{atoms } a$ such that $z \sqcap b = \perp$ and thus y is above $\sqcup \left\{ \frac{z \in \text{atoms } a}{z \sqcap b = \perp} \right\}$.

Thus $\sqcup \left\{ \frac{z \in \text{atoms } a}{z \sqcap b = \perp} \right\}$ is least upper bound of

$$\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\},$$

that is

$$\begin{aligned}
 \sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\} &= \\
 \sqcup \left\{ \frac{z \in \text{atoms } a}{z \sqcap b = \perp} \right\} &= \\
 \sqcup(\text{atoms } a \setminus \text{atoms } b). &
 \end{aligned}$$

□

3.4. Partially ordered categories

3.4.1. Definition.

DEFINITION 248. I will call a partially ordered (pre)category a (pre)category together with partial order \sqsubseteq on each of its Mor-sets with the additional requirement that

$$f_1 \sqsubseteq f_2 \wedge g_1 \sqsubseteq g_2 \Rightarrow g_1 \circ f_1 \sqsubseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\text{Src } f_1 = \text{Src } f_2$ and $\text{Dst } f_1 = \text{Dst } f_2 = \text{Src } g_1 = \text{Src } g_2$ and $\text{Dst } g_1 = \text{Dst } g_2$.

I will denote lattice operations on a Hom-set $C(A, B)$ of a category (or any directed multigraph) like \sqcup^C instead of writing $\sqcup^{C(A, B)}$ explicitly.

3.4.2. Dagger categories.

DEFINITION 249. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a dagger precategory is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g :

- 1°. $f^{\dagger\dagger} = f$;
- 2°. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

DEFINITION 250. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a dagger category is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A :

- 1°. $f^{\dagger\dagger} = f$;
- 2°. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
- 3°. $(1_A)^\dagger = 1_A$.

THEOREM 251. If a category is a dagger precategory then it is a dagger category.

PROOF. We need to prove only that $(1_A)^\dagger = 1_A$. Really,

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A.$$

□

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g with the same source and destination)

$$f^\dagger \sqsubseteq g^\dagger \Leftrightarrow f \sqsubseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

DEFINITION 252. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

DEFINITION 253. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.

DEFINITION 254. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

THEOREM 255. The following conditions are equivalent for a morphism f of a dagger precategory:

- 1°. f is symmetric and transitive.
- 2°. $f = f^\dagger \circ f$.

PROOF.

1° \Rightarrow 2°. If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

2° \Rightarrow 1°. $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive. □

3.4.2.1. Some special classes of morphisms.

DEFINITION 256. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \sqsubseteq 1_{\text{Dst } f}$.

DEFINITION 257. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \supseteq 1_{\text{Src } f}$.

DEFINITION 258. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^\dagger \circ f \sqsubseteq 1_{\text{Src } f}$.

DEFINITION 259. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^\dagger \supseteq 1_{\text{Dst } f}$.

REMARK 260. It is easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective functions as morphisms of the category **Rel**.

OBVIOUS 261. “Injective morphism” is a dual of “monovalued morphism” and “surjective morphism” is a dual of “entirely defined morphism”.

DEFINITION 262. For a given partially ordered dagger category C the *category of monovalued (entirely defined, injective, surjective) morphisms of C* is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C .

We need to prove that these are really categories, that is that composition of monovalued (entirely defined, injective, surjective) morphisms is monovalued (entirely defined, injective, surjective) and that identity morphisms are monovalued, entirely defined, injective, and surjective.

PROOF. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let f and g be monovalued morphisms, $\text{Dst } f = \text{Src } g$. Then

$$\begin{aligned} (g \circ f) \circ (g \circ f)^\dagger &= \\ g \circ f \circ f^\dagger \circ g^\dagger &\sqsubseteq \\ g \circ 1_{\text{Src } g} \circ g^\dagger &= \\ g \circ g^\dagger &\sqsubseteq \\ 1_{\text{Dst } g} &= 1_{\text{Dst}(g \circ f)}. \end{aligned}$$

So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following:

$$1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\text{Dst } 1_A} \sqsubseteq 1_{\text{Dst } 1_A}.$$

Entirely defined. Let f and g be entirely defined morphisms, $\text{Dst } f = \text{Src } g$. Then

$$\begin{aligned} (g \circ f)^\dagger \circ (g \circ f) &= \\ f^\dagger \circ g^\dagger \circ g \circ f &\sqsupseteq \\ f^\dagger \circ 1_{\text{Src } g} \circ f &= \\ f^\dagger \circ 1_{\text{Dst } f} \circ f &= \\ f^\dagger \circ f &\sqsupseteq \\ 1_{\text{Src } f} &= 1_{\text{Src}(g \circ f)}. \end{aligned}$$

So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following:

$$(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\text{Src } 1_A} \sqsupseteq 1_{\text{Src } 1_A}.$$

□

DEFINITION 263. I will call a *bijective* morphism a morphism which is entirely defined, monovalued, injective, and surjective.

PROPOSITION 264. If a morphism is bijective then it is an isomorphism.

PROOF. Let f be bijective. Then $f \circ f^\dagger \sqsubseteq 1_{\text{Dst } f}$, $f^\dagger \circ f \sqsupseteq 1_{\text{Src } f}$, $f^\dagger \circ f \sqsubseteq 1_{\text{Src } f}$, $f \circ f^\dagger \sqsupseteq 1_{\text{Dst } f}$. Thus $f \circ f^\dagger = 1_{\text{Dst } f}$ and $f^\dagger \circ f = 1_{\text{Src } f}$ that is f^\dagger is an inverse of f . □

Let Hom-sets be complete lattices.

DEFINITION 265. A morphism f of a partially ordered category is *metamonovalued* when $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$ whenever G is a set of morphisms with a suitable source and destination.

DEFINITION 266. A morphism f of a partially ordered category is *metainjective* when $f \circ (\prod G) = \prod_{g \in G} (f \circ g)$ whenever G is a set of morphisms with a suitable source and destination.

OBVIOUS 267. Metamonovaluedness and metainjectivity are dual to each other.

DEFINITION 268. A morphism f of a partially ordered category is *metacomplete* when $f \circ (\sqcup G) = \sqcup_{g \in G} (f \circ g)$ whenever G is a set of morphisms with a suitable source and destination.

DEFINITION 269. A morphism f of a partially ordered category is *co-metacomplete* when $(\sqcup G) \circ f = \sqcup_{g \in G} (g \circ f)$ whenever G is a set of morphisms with a suitable source and destination.

Let now Hom-sets be meet-semilattices.

DEFINITION 270. A morphism f of a partially ordered category is *weakly metamonovalued* when $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ whenever g and h are morphisms with a suitable source and destination.

DEFINITION 271. A morphism f of a partially ordered category is *weakly metainjective* when $f \circ (g \sqcap h) = (f \circ g) \sqcap (f \circ h)$ whenever g and h are morphisms with a suitable source and destination.

Let now Hom-sets be join-semilattices.

DEFINITION 272. A morphism f of a partially ordered category is *weakly metacomplete* when $f \circ (g \sqcup h) = (f \circ g) \sqcup (f \circ h)$ whenever g and h are morphisms with a suitable source and destination.

DEFINITION 273. A morphism f of a partially ordered category is *weakly co-metacomplete* when $(g \sqcup h) \circ f = (g \circ f) \sqcup (h \circ f)$ whenever g and h are morphisms with a suitable source and destination.

OBVIOUS 274.

- 1°. Metamonovalued morphisms are weakly metamonovalued.
- 2°. Metainjective morphisms are weakly metainjective.
- 3°. Metacomplete morphisms are weakly metacomplete.
- 4°. Co-metacomplete morphisms are weakly co-metacomplete.

3.5. Partitioning

DEFINITION 275. Let \mathfrak{A} be a complete lattice. *Torning* of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{\perp\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall x, y \in S : (x \neq y \Rightarrow x \asymp y).$$

DEFINITION 276. Let \mathfrak{A} be a complete lattice. *Weak partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{\perp\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall x \in S : x \asymp \bigsqcup (S \setminus \{x\}).$$

DEFINITION 277. Let \mathfrak{A} be a complete lattice. *Strong partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{\perp\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall A, B \in \mathcal{P}S : (A \asymp B \Rightarrow \bigsqcup A \asymp \bigsqcup B).$$

OBVIOUS 278.

- 1°. Every strong partition is a weak partition.
- 2°. Every weak partition is a torning.

DEFINITION 279. *Complete lattice generated by* a set P (on a complete lattice) is the set (obviously having the structure of complete lattice) $P_0 \cup P_1 \cup \dots$ where $P_0 = P$ and $P_{i+1} = \left\{ \bigsqcup_{K \in \mathcal{P}P_i} K \right\}$.

OBVIOUS 280. Complete lattice generated by a set is indeed a complete lattice.

EXAMPLE 281. $[S] \neq \left\{ \bigsqcup_{X \in \mathcal{P}S} X \right\}$, where $[S]$ is the complete lattice generated by a strong partition S of a filter on a set.

PROOF. Consider any infinite set U and its strong partition $S = \left\{ \uparrow_{x \in U} \{x\} \right\}$. The set S consists only of principal filters. But $[S]$ contains (exercise!) some nonprincipal filters. \square

By the way:

PROPOSITION 282. $\left\{ \bigsqcup_{X \in \mathcal{P}S} X \right\}$ is closed under binary meets, if S is a strong partition of an element of a complete lattice.

PROOF. Let $R = \left\{ \bigsqcup_{X \in \mathcal{P}S} X \right\}$. Then for every $X, Y \in \mathcal{P}S$

$$\begin{aligned}
& \bigsqcup X \sqcap \bigsqcup Y = \\
& \bigsqcup ((X \cap Y) \cup (X \setminus Y)) \sqcap \bigsqcup Y = \\
& \left(\bigsqcup (X \cap Y) \sqcup \bigsqcup (X \setminus Y) \right) \sqcap \bigsqcup Y = \\
& \left(\bigsqcup (X \cap Y) \sqcap \bigsqcup Y \right) \sqcup \left(\bigsqcup (X \setminus Y) \sqcap \bigsqcup Y \right) = \\
& \left(\bigsqcup (X \cap Y) \sqcap \bigsqcup Y \right) \sqcup \perp = \\
& \bigsqcup (X \cap Y) \sqcap \bigsqcup Y.
\end{aligned}$$

Applying the formula $\bigsqcup X \sqcap \bigsqcup Y = \bigsqcup (X \cap Y) \sqcap \bigsqcup Y$ twice we get

$$\begin{aligned}
& \bigsqcup X \sqcap \bigsqcup Y = \\
& \bigsqcup (X \cap Y) \sqcap \bigsqcup (Y \cap (X \cap Y)) = \\
& \bigsqcup (X \cap Y) \sqcap \bigsqcup (X \cap Y) = \\
& \bigsqcup (X \cap Y).
\end{aligned}$$

But for any $A, B \in R$ there exist $X, Y \in \mathcal{P}S$ such that $A = \bigsqcup X$, $B = \bigsqcup Y$. So $A \sqcap B = \bigsqcup X \sqcap \bigsqcup Y = \bigsqcup (X \cap Y) \in R$. \square

3.6. A proposition about binary relations

PROPOSITION 283. Let f, g, h be binary relations. Then $g \circ f \not\leq h \Leftrightarrow g \not\leq h \circ f^{-1}$.

PROOF.

$$\begin{aligned}
& g \circ f \not\leq h \Leftrightarrow \\
& \exists a, c : a ((g \circ f) \cap h) c \Leftrightarrow \\
& \exists a, c : (a (g \circ f) c \wedge a h c) \Leftrightarrow \\
& \exists a, b, c : (a f b \wedge b g c \wedge a h c) \Leftrightarrow \\
& \exists b, c : (b g c \wedge b (h \circ f^{-1}) c) \Leftrightarrow \\
& \exists b, c : b (g \cap (h \circ f^{-1})) c \Leftrightarrow \\
& g \not\leq h \circ f^{-1}.
\end{aligned}$$

\square

3.7. Infinite associativity and ordinated product

3.7.1. Introduction. We will consider some function f which takes an arbitrary ordinal number of arguments. That is f can be taken for arbitrary (small, if to be precise) ordinal number of arguments. More formally: Let $x = x_{i \in n}$ be a

family indexed by an ordinal n . Then $f(x)$ can be taken. The same function f can take different number of arguments. (See below for the exact definition.)

Some of such functions f are associative in the sense defined below. If a function is associative in the below defined sense, then the binary operation induced by this function is associative in the usual meaning of the word “associativity” as defined in basic algebra.

I also introduce and research an important example of infinitely associative function, which I call *ordinated product*.

Note that my searching about infinite associativity and ordinals in Internet has provided no useful results. As such there is a reason to assume that my research of generalized associativity in terms of ordinals is novel.

3.7.2. Used notation. We identify natural numbers with finite Von Neumann’s ordinals (further just *ordinals* or *ordinal numbers*).

For simplicity we will deal with small sets (members of a Grothendieck universe). We will denote the Grothendieck universe (aka *universal set*) as \mathcal{U} .

I will denote a tuple of n elements like $\llbracket a_0, \dots, a_{n-1} \rrbracket$. By definition

$$\llbracket a_0, \dots, a_{n-1} \rrbracket = \{(0, a_0), \dots, (n-1, a_{n-1})\}.$$

Note that an ordered pair (a, b) is not the same as the tuple $\llbracket a, b \rrbracket$ of two elements. (However, we will use them interchangeably.)

DEFINITION 284. An *anchored relation* is a tuple $\llbracket n, r \rrbracket$ where n is an index set and r is an n -ary relation.

For an anchored relation $\text{arity}\llbracket n, r \rrbracket = n$. The graph¹ of $\llbracket n, r \rrbracket$ is defined as follows: $\text{GR}\llbracket n, r \rrbracket = r$.

DEFINITION 285. $\text{Pr}_i f$ is a function defined by the formula

$$\text{Pr}_i f = \left\{ \frac{x_i}{x \in f} \right\}$$

for every small n -ary relation f where n is an ordinal number and $i \in n$. Particularly for every n -ary relation f and $i \in n$ where $n \in \mathbb{N}$

$$\text{Pr}_i f = \left\{ \frac{x_i}{\llbracket x_0, \dots, x_{n-1} \rrbracket \in f} \right\}.$$

Recall that Cartesian product is defined as follows:

$$\prod a = \left\{ \frac{z \in (\bigcup \text{im } a)^{\text{dom } a}}{\forall i \in \text{dom } a : z(i) \in a_i} \right\}.$$

OBVIOUS 286. If a is a small function, then $\prod a = \left\{ \frac{z \in \mathcal{U}^{\text{dom } a}}{\forall i \in \text{dom } a : z(i) \in a_i} \right\}$.

3.7.2.1. *Currying and uncurrying.*

The customary definition. Let X, Y, Z be sets.

We will consider variables $x \in X$ and $y \in Y$.

Let a function $f \in Z^{X \times Y}$. Then $\text{curry}(f) \in (Z^Y)^X$ is the function defined by the formula $(\text{curry}(f)x)y = f(x, y)$.

Let now $f \in (Z^Y)^X$. Then $\text{uncurry}(f) \in Z^{X \times Y}$ is the function defined by the formula $\text{uncurry}(f)(x, y) = (fx)y$.

OBVIOUS 287.

1°. $\text{uncurry}(\text{curry}(f)) = f$ for every $f \in Z^{X \times Y}$.

2°. $\text{curry}(\text{uncurry}(f)) = f$ for every $f \in (Z^Y)^X$.

¹It is unrelated with graph theory.

Currying and uncurrying with a dependent variable. Let X, Z be sets and Y be a function with the domain X . (Vaguely saying, Y is a variable dependent on X .)

The disjoint union $\coprod Y = \bigcup_{i \in \text{dom } Y} (\{i\} \times Y_i) = \left\{ \frac{(i,x)}{i \in \text{dom } Y, x \in Y_i} \right\}$.

We will consider variables $x \in X$ and $y \in Y_x$.

Let a function $f \in Z^{\coprod_{i \in X} Y_i}$ (or equivalently $f \in Z^{\coprod Y}$). Then $\text{curry}(f) \in \prod_{i \in X} Z^{Y_i}$ is the function defined by the formula $(\text{curry}(f)x)y = f(x, y)$.

Let now $f \in \prod_{i \in X} Z^{Y_i}$. Then $\text{uncurry}(f) \in Z^{\coprod_{i \in X} Y_i}$ is the function defined by the formula $\text{uncurry}(f)(x, y) = (fx)y$.

OBVIOUS 288.

- 1°. $\text{uncurry}(\text{curry}(f)) = f$ for every $f \in Z^{\coprod_{i \in X} Y_i}$.
- 2°. $\text{curry}(\text{uncurry}(f)) = f$ for every $f \in \prod_{i \in X} Z^{Y_i}$.

3.7.2.2. *Functions with ordinal numbers of arguments.* Let Ord be the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n} .

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n \in \text{Ord}} X^n}$.

I will denote $\text{OrdVar}(X) = \bigcup_{n \in \text{Ord}} X^n$ and call it *ordinal variadic*. (“Var” in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3.7.3. On sums of ordinals. Let a be an ordinal-indexed family of ordinals.

PROPOSITION 289. $\coprod a$ with lexicographic order is a well-ordered set.

PROOF. Let S be non-empty subset of $\coprod a$.

Take $i_0 = \min \text{Pr}_0 S$ and $x_0 = \min \left\{ \frac{\text{Pr}_1 y}{y \in S, y(0) = i_0} \right\}$ (these exist by properties of ordinals). Then (i_0, x_0) is the least element of S . \square

DEFINITION 290. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$.

EXERCISE 291. Prove that for finite ordinals it is just a sum of natural numbers.

This ordinal exists and is unique because our set is well-ordered.

REMARK 292. An infinite sum of ordinals is not customary defined.

The *structured sum* $\oplus a$ of a is an order isomorphism from lexicographically ordered set $\coprod a$ into $\sum a$.

There exists (for a given a) exactly one structured sum, by properties of well-ordered sets.

OBVIOUS 293. $\sum a = \text{im } \oplus a$.

THEOREM 294. $(\oplus a)(n, x) = \sum_{i \in n} a_i + x$.

PROOF. We need to prove that it is an order isomorphism. Let’s prove it is an injection that is $m > n \Rightarrow \sum_{i \in m} a_i + x > \sum_{i \in n} a_i + x$ and $y > x \Rightarrow \sum_{i \in n} a_i + y > \sum_{i \in n} a_i + x$.

Really, if $m > n$ then $\sum_{i \in m} a_i + x \geq \sum_{i \in n+1} a_i + x > \sum_{i \in n} a_i + x$. The second formula is true by properties of ordinals.

Let’s prove that it is a surjection. Let $r \in \sum a$. There exist $n \in \text{dom } a$ and $x \in a_n$ such that $r = (\oplus a)(n, x)$. Thus $r = (\oplus a)(n, 0) + x = \sum_{i \in n} a_i + x$ because $(\oplus a)(n, 0) = \sum_{i \in n} a_i$ since $(n, 0)$ has $\sum_{i \in n} a_i$ predecessors. \square

3.7.4. Ordinated product.

3.7.4.1. *Introduction.* *Ordinated product* defined below is a variation of Cartesian product, but is associative unlike Cartesian product. However, ordinated product unlike Cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let F indexed by an ordinal number be a small family of anchored relations.

3.7.4.2. Concatenation.

DEFINITION 295. Let z be an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The *concatenation* of z is

$$\text{concat } z = \text{uncurry}(z) \circ \left(\bigoplus (\text{dom } \circ z) \right)^{-1}.$$

EXERCISE 296. Prove, that if z is a finite family of finitary tuples, it is concatenation of $\text{dom } z$ tuples in the usual sense (as it is commonly used in computer science).

PROPOSITION 297. If $z \in \prod(\text{GR} \circ F)$ then $\text{concat } z = \text{uncurry}(z) \circ \left(\bigoplus (\text{arity} \circ F) \right)^{-1}$.

PROOF. If $z \in \prod(\text{GR} \circ F)$ then $\text{dom } z(i) = \text{dom}(\text{GR} \circ F)_i = \text{arity } F_i$ for every $i \in \text{dom } F$. Thus $\text{dom } \circ z = \text{arity} \circ F$. \square

PROPOSITION 298. $\text{dom } \text{concat } z = \sum_{i \in \text{dom } z} \text{dom } z_i$.

PROOF. Because $\text{dom}(\bigoplus (\text{dom } \circ z))^{-1} = \sum_{i \in \text{dom } f} (\text{dom } \circ z)$, it is enough to prove that

$$\text{dom } \text{uncurry}(z) = \text{dom } \bigoplus (\text{dom } \circ z).$$

Really,

$$\begin{aligned} \sum_{i \in \text{dom } f} (\text{dom } \circ z) &= \\ \left\{ \frac{(i, x)}{i \in \text{dom}(\text{dom } \circ z), x \in \text{dom } z_i} \right\} &= \\ \left\{ \frac{(i, x)}{i \in \text{dom } z, x \in \text{dom } z_i} \right\} &= \\ \prod z & \end{aligned}$$

and $\text{dom } \text{uncurry}(z) = \prod_{i \in X} z_i = \prod z$. \square

3.7.4.3. *Finite example.* If F is a finite family (indexed by a natural number $\text{dom } F$) of anchored finitary relations, then by definition

$$\text{GR } \prod^{(\text{ord})} = \left\{ \frac{\llbracket a_{0,0}, \dots, a_{0,\text{arity } F_0-1}, \dots, a_{\text{dom } F-1,0}, \dots, a_{\text{dom } F-1,\text{arity } F_{\text{dom } F-1}-1} \rrbracket}{\begin{array}{l} \llbracket a_{0,0}, \dots, a_{0,\text{arity } F_0-1} \rrbracket \in \text{GR } F_0 \wedge \dots \wedge \\ \llbracket a_{\text{dom } F-1,\text{arity } F_{\text{dom } F-1}-1} \rrbracket \in \text{GR } F_{\text{dom } F-1} \end{array}} \right\}$$

and

$$\text{arity } \prod^{(\text{ord})} F = \text{arity } F_0 + \dots + \text{arity } F_{\text{dom } F-1}.$$

The above formula can be shortened to

$$\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{\text{concat } z}{z \in \prod(\text{GR} \circ F)} \right\}.$$

3.7.4.4. *The definition.*

DEFINITION 299. The anchored relation (which I call *ordinated product*) $\prod^{(\text{ord})} F$ is defined by the formulas:

$$\begin{aligned} \text{arity } \prod^{(\text{ord})} F &= \sum (\text{arity } \circ f); \\ \text{GR } \prod^{(\text{ord})} F &= \left\{ \frac{\text{concat } z}{z \in \prod (\text{GR } \circ F)} \right\}. \end{aligned}$$

PROPOSITION 300. $\prod^{(\text{ord})} F$ is a properly defined anchored relation.

PROOF. $\text{dom concat } z = \sum_{i \in \text{dom } F} \text{dom } z_i = \sum_{i \in \text{dom } F} \text{arity } f_i = \sum (\text{arity } \circ F)$. \square

3.7.4.5. *Definition with composition for every multiplier.*

$$q(F)_i \stackrel{\text{def}}{=} \left(\text{curry} \left(\bigoplus (\text{arity } \circ F) \right) \right) i.$$

PROPOSITION 301. $\prod^{(\text{ord})} F = \left\{ \frac{L \in \bigcup \sum^{(\text{arity } \circ F)}}{\forall i \in \text{dom } F: L \circ q(F)_i \in \text{GR } F_i} \right\}$.

PROOF. $\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{\text{concat } z}{z \in \prod (\text{GR } \circ F)} \right\}$;

$$\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{\text{uncurry}(z) \circ \left(\bigoplus (\text{arity } \circ f) \right)^{-1}}{z \in \prod_{i \in \text{dom } F} \mathcal{U}^{\text{arity } F_i}, \forall i \in \text{dom } F: z(i) \in \text{GR } F_i} \right\}.$$

Let $L = \text{uncurry}(z)$. Then $z = \text{curry}(L)$.

$$\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{L \circ \left(\bigoplus (\text{arity } \circ f) \right)^{-1}}{\text{curry}(L) \in \prod_{i \in \text{dom } F} \mathcal{U}^{\text{arity } F_i}, \forall i \in \text{dom } F: \text{curry}(L)_i \in \text{GR } F_i} \right\};$$

$$\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{L \circ \left(\bigoplus (\text{arity } \circ f) \right)^{-1}}{L \in \bigcup \prod_{i \in \text{dom } F} \text{arity } F_i, \forall i \in \text{dom } F: \text{curry}(L)_i \in \text{GR } F_i} \right\};$$

$$\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{L \in \bigcup \sum^{(\text{arity } \circ f)}}{\forall i \in \text{dom } F: \text{curry}(L \circ \bigoplus (\text{arity } \circ F))_i \in \text{GR } F_i} \right\};$$

$(\text{curry}(L \circ \bigoplus (\text{arity } \circ F)))_i x = L((\text{curry}(\bigoplus (\text{arity } \circ F)))_i x) = L(q(F)_i x) = (L \circ q(F)_i) x$;

$$\text{curry}(L \circ \bigoplus (\text{arity } \circ F))_i = L \circ q(F)_i;$$

$$\prod^{(\text{ord})} F = \left\{ \frac{L \in \bigcup \sum^{(\text{arity } \circ F)}}{\forall i \in \text{dom } F: L \circ q(F)_i \in \text{GR } F_i} \right\}. \quad \square$$

COROLLARY 302. $\prod^{(\text{ord})} F = \left\{ \frac{L \in \left(\bigcup \text{im}(\text{GR } \circ F) \right) \sum^{(\text{arity } \circ F)}}{\forall i \in \text{dom } F: L \circ q(F)_i \in \text{GR } F_i} \right\}$.

COROLLARY 303. $\prod^{(\text{ord})} F$ is small if F is small.

3.7.4.6. *Definition with shifting arguments.* Let $F'_i = \left\{ \frac{L \circ \text{Pr}_1 |_{\{i\} \times \text{arity } F_i}}{L \in \text{GR } F_i} \right\}$.

PROPOSITION 304. $F'_i = \left\{ \frac{L \circ \text{Pr}_1 |_{\{i\} \times \mathcal{U}}}{L \in \text{GR } F_i} \right\}$.

PROOF. If $L \in \text{GR } F_i$ then $\text{dom } L = \text{arity } F_i$. Thus

$$L \circ \text{Pr}_1 |_{\{i\} \times \text{arity } F_i} = L \circ \text{Pr}_1 |_{\{i\} \times \text{dom } L} = L \circ \text{Pr}_1 |_{\{i\} \times \mathcal{U}}.$$

\square

PROPOSITION 305. F'_i is an $(\{i\} \times \text{arity } F_i)$ -ary relation.

PROOF. We need to prove that $\text{dom}(L \circ \text{Pr}_1 |_{\{i\} \times \text{arity } F_i}) = \{i\} \times \text{arity } F_i$ for $L \in \text{GR } F_i$, but that's obvious. \square

OBVIOUS 306. $\prod(\text{arity} \circ F) = \bigcup_{i \in \text{dom } F} (\{i\} \times \text{arity } F_i) = \bigcup_{i \in \text{dom } F} \text{dom } F'_i$.

LEMMA 307. $P \in \prod_{i \in \text{dom } F} F'_i \Leftrightarrow \text{curry}(\bigcup \text{im } P) \in \prod(\text{GR} \circ F)$ for a $(\text{dom } F)$ -indexed family P where $P_i \in \mathcal{U}^{\{i\} \times \text{arity } F_i}$ for every $i \in \text{dom } F$, that is for $P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i}$.

PROOF. For every $P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i}$ we have:

$$\begin{aligned}
P &\in \prod_{i \in \text{dom } F} F'_i \Leftrightarrow \\
P &\in \left\{ \frac{z \in \mathcal{U}^{\text{dom } F}}{\forall i \in \text{dom } F : z(i) \in F'_i} \right\} \Leftrightarrow \\
P &\in \mathcal{U}^{\text{dom } F} \wedge \forall i \in \text{dom } F : P(i) \in F'_i \Leftrightarrow \\
P &\in \mathcal{U}^{\text{dom } F} \wedge \forall i \in \text{dom } F \exists L \in \text{GR } F_i : P_i = L \circ (\text{Pr}_1 |_{\{i\} \times \mathcal{U}}) \Leftrightarrow \\
P &\in \mathcal{U}^{\text{dom } F} \wedge \forall i \in \text{dom } F \exists L \in \text{GR } F_i : (P_i \in \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \forall x \in \text{arity } F_i : P_i(i, x) = Lx) \Leftrightarrow \\
P &\in \mathcal{U}^{\text{dom } F} \wedge \forall i \in \text{dom } F \exists L \in \text{GR } F_i : (P_i \in \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \text{curry}(P_i)i = L) \Leftrightarrow \\
P &\in \mathcal{U}^{\text{dom } F} \wedge \forall i \in \text{dom } F : (P_i \in \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \text{curry}(P_i)i \in \text{GR } F_i) \Leftrightarrow \\
\forall i \in \text{dom } F \exists Q_i \in (\mathcal{U}^{\text{arity } F_i})^{\{i\}} : (P_i = \text{uncurry}(Q_i) \wedge (Q_i)i \in \mathcal{U}^{\text{arity } F_i} \wedge Q_i i \in \text{GR } F_i) &\Leftrightarrow \\
\forall i \in \text{dom } F \exists Q_i \in (\mathcal{U}^{\text{arity } F_i})^{\{i\}} : \left(P_i = \text{uncurry}(Q_i) \wedge \left(\bigcup_{i \in \text{dom } F} Q_i \right) i \in \text{GR } F_i \right) &\Leftrightarrow \\
\forall i \in \text{dom } F \exists Q_i \in (\mathcal{U}^{\text{arity } F_i})^{\{i\}} : \left(P_i = \text{uncurry}(Q_i) \wedge \bigcup_{i \in \text{dom } F} Q_i \in \prod(\text{GR} \circ F) \right) &\Leftrightarrow \\
\forall i \in \text{dom } F : \bigcup_{i \in \text{dom } F} \text{curry}(P_i) \in \prod(\text{GR} \circ F) &\Leftrightarrow \\
\text{curry} \left(\bigcup_{i \in \text{dom } F} P_i \right) \in \prod(\text{GR} \circ F) &\Leftrightarrow \\
\text{curry}(\bigcup \text{im } P) \in \prod(\text{GR} \circ F). &
\end{aligned}$$

\square

LEMMA 308. $\left\{ \frac{\text{curry}(f) \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} = \prod(\text{GR} \circ F)$.

PROOF. First $\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{\text{uncurry}(z) \circ (\bigoplus(\text{dom} \circ z))^{-1}}{z \in \prod(\text{GR} \circ F)} \right\}$, that is

$$\left\{ \frac{f}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} = \left\{ \frac{\text{uncurry}(z) \circ (\bigoplus(\text{arity} \circ F))^{-1}}{z \in \prod(\text{GR} \circ F)} \right\}.$$

Since $\bigoplus(\text{arity} \circ F)$ is a bijection, we have

$$\left\{ \frac{f \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} = \left\{ \frac{\text{uncurry}(z)}{z \in \prod(\text{GR} \circ F)} \right\} \text{ what is equivalent to}$$

$$\left\{ \frac{\text{curry}(f) \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} = \left\{ \frac{z}{z \in \prod(\text{GR} \circ F)} \right\} \text{ that is } \left\{ \frac{\text{curry}(f) \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} = \prod(\text{GR} \circ F). \quad \square$$

LEMMA 309. $\left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \text{curry}(\bigcup \text{im } P) \in \prod(\text{GR} \circ F)} \right\} = \left\{ \frac{L \in \mathcal{U}^{\prod_{i \in \text{dom } F} \text{arity } F_i}}{\text{curry}(L) \in \prod(\text{GR} \circ F)} \right\}$.

PROOF. Let $L' \in \left\{ \frac{L \in \mathcal{U} \prod_{i \in \text{dom } F} \text{arity } F_i}{\text{curry}(L) \in \prod(\text{GR} \circ F)} \right\}$. Then $L' \in \mathcal{U} \prod_{i \in \text{dom } F} \text{arity } F_i$ and $\text{curry}(L') \in \prod(\text{GR} \circ F)$.

Let $P = \lambda i \in \text{dom } F : L'|_{\{i\} \times \text{arity } F_i}$. Then $P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i}$ and $\bigcup \text{im } P = L'$. So $L' \in \left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \text{curry}(\bigcup \text{im } P) \in \prod(\text{GR} \circ F)} \right\}$.

Let now $L' \in \left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \text{curry}(\bigcup \text{im } P) \in \prod(\text{GR} \circ F)} \right\}$. Then there exists $P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i}$ such that $L' = \bigcup \text{im } P$ and $\text{curry}(L') \in \prod(\text{GR} \circ F)$.

Evidently $L' \in \mathcal{U} \prod_{i \in \text{dom } F} \text{arity } F_i$. So $L' \in \left\{ \frac{L \in \mathcal{U} \prod_{i \in \text{dom } F} \text{arity } F_i}{\text{curry}(L) \in \prod(\text{GR} \circ F)} \right\}$. \square

$$\text{LEMMA 310. } \left\{ \frac{f \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} = \left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} F'_i} \right\}.$$

PROOF.

$$\begin{aligned} L &\in \left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} F'_i} \right\} \Leftrightarrow \\ L &\in \left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} \mathcal{U}^{\{i\} \times \text{arity } F_i} \wedge \text{curry}(\bigcup \text{im } P) \in \prod(\text{GR} \circ F)} \right\} \Leftrightarrow \\ &L \in \mathcal{U} \prod_{i \in \text{dom } F} \text{arity } F_i \wedge \text{curry}(L) \in \prod(\text{GR} \circ F) \Leftrightarrow \\ L &\in \mathcal{U} \prod_{i \in \text{dom } F} \text{arity } F_i \wedge \text{curry}(L) \in \left\{ \frac{\text{curry}(f) \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} \Leftrightarrow \\ &\text{(because } \bigoplus(\text{arity} \circ F) \text{ is a bijection)} \\ \text{curry}(L) \circ \left(\bigoplus(\text{arity} \circ F) \right)^{-1} &\in \left\{ \frac{\text{curry}(f)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} \Leftrightarrow \\ L \circ \left(\bigoplus(\text{arity} \circ F) \right)^{-1} &\in \left\{ \frac{f}{f \in \text{GR } \prod^{(\text{ord})} F} \right\} \Leftrightarrow \\ &\text{(because } \bigoplus(\text{arity} \circ F) \text{ is a bijection)} \\ L &\in \left\{ \frac{f \circ \bigoplus(\text{arity} \circ F)}{f \in \text{GR } \prod^{(\text{ord})} F} \right\}. \end{aligned}$$

\square

$$\text{THEOREM 311. } \text{GR } \prod^{(\text{ord})} F = \left\{ \frac{(\bigcup \text{im } P) \circ \left(\bigoplus(\text{arity} \circ F) \right)^{-1}}{P \in \prod_{i \in \text{dom } F} F'_i} \right\}.$$

PROOF. From the lemma, because $\bigoplus(\text{arity} \circ F)$ is a bijection. \square

$$\text{THEOREM 312. } \text{GR } \prod^{(\text{ord})} F = \left\{ \frac{\bigcup_{i \in \text{dom } F} (P_i \circ \left(\bigoplus(\text{arity} \circ f) \right)^{-1})}{P \in \prod_{i \in \text{dom } F} F'_i} \right\}.$$

PROOF. From the previous theorem. \square

$$\text{THEOREM 313. } \text{GR } \prod^{(\text{ord})} F = \left\{ \frac{\bigcup \text{im } P}{P \in \prod_{i \in \text{dom } F} \left\{ \frac{f \circ \left(\bigoplus(\text{arity} \circ f) \right)^{-1}}{f \in F'_i} \right\}} \right\}.$$

PROOF. From the previous. \square

REMARK 314. Note that the above formulas contain both $\bigcup_{i \in \text{dom } F} \text{dom } F'_i$ and $\bigcup_{i \in \text{dom } F} F'_i$. These forms are similar but different.

3.7.4.7. *Associativity of ordinated product.* Let f be an ordinal variadic function.

Let S be an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set X .

I call f *infinite associative* when

- 1°. $f(f \circ S) = f(\text{concat } S)$ for every S ;
- 2°. $f(\llbracket x \rrbracket) = x$ for $x \in X$.

Infinite associativity implies associativity.

PROPOSITION 315. Let f be an infinitely associative function taking an ordinal number of arguments in a set X . Define $x \star y = f\llbracket x, y \rrbracket$ for $x, y \in X$. Then the binary operation \star is associative.

PROOF. Let $x, y, z \in X$. Then $(x \star y) \star z = f\llbracket f\llbracket x, y \rrbracket, z \rrbracket = f(f\llbracket x, y \rrbracket, f\llbracket z \rrbracket) = f\llbracket x, y, z \rrbracket$. Similarly $x \star (y \star z) = f\llbracket x, y, z \rrbracket$. So $(x \star y) \star z = x \star (y \star z)$. \square

Concatenation is associative. First we will prove some lemmas.

Let a and b be functions on a poset. Let $a \sim b$ iff there exist an order isomorphism f such that $a = b \circ f$. Evidently \sim is an equivalence relation.

OBVIOUS 316. $\text{concat } a = \text{concat } b \Leftrightarrow \text{uncurry}(a) \sim \text{uncurry}(b)$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry .

LEMMA 317. $a \sim b \Rightarrow \text{uncurry } a \sim \text{uncurry } b$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

PROOF. There exists an order isomorphism f such that $a = b \circ f$.

$$\begin{aligned} \text{uncurry}(a)(x, y) &= (ax)y = (bfxy)y = \\ & \text{uncurry}(b)(fx, y) = \text{uncurry}(b)g(x, y) \end{aligned}$$

where $g(x, y) = (fx, y)$.

g is an order isomorphism because $g(x_0, y_0) \geq g(x_1, y_1) \Leftrightarrow (x_0, y_0) \geq (x_1, y_1)$. (Injectivity and surjectivity are obvious.) \square

LEMMA 318. Let $a_i \sim b_i$ for every i . Then $\text{uncurry } a \sim \text{uncurry } b$ for every ordinal indexed families a and b of ordinal indexed families of functions taking an ordinal number of arguments.

PROOF. Let $a_i = b_i \circ f_i$ where f_i is an order isomorphism for every i .

$$\begin{aligned} \text{uncurry}(a)(i, y) &= a_i y = b_i f_i y = \\ & \text{uncurry}(b)(i, f_i y) = \text{uncurry}(b)g(i, y) = (\text{uncurry}(b) \circ g)(i, y) \end{aligned}$$

where $g(i, y) = (i, f_i y)$.

g is an order isomorphism because $g(i, y_0) \geq g(i, y_1) \Leftrightarrow f_i y_0 \geq f_i y_1 \Leftrightarrow y_0 \geq y_1$ and $i_0 > i_1 \Rightarrow g(i, y_0) > g(i, y_1)$. (Injectivity and surjectivity are obvious.) \square

Let now S be an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

LEMMA 319. $\text{uncurry}(\text{uncurry} \circ S) \sim \text{uncurry}(\text{uncurry } S)$.

PROOF. $\text{uncurry} \circ S = \lambda i \in S : \text{uncurry}(S_i)$;

$(\text{uncurry}(\text{uncurry} \circ S))((i, x), y) = (\text{uncurry } S_i)(x, y) = (S_i x)y$;

$(\text{uncurry}(\text{uncurry } S))((i, x), y) = ((\text{uncurry } S)(i, x))y = (S_i x)y$.

Thus $(\text{uncurry}(\text{uncurry} \circ S))((i, x), y) = (\text{uncurry}(\text{uncurry } S))((i, x), y)$ and thus evidently $\text{uncurry}(\text{uncurry} \circ S) \sim \text{uncurry}(\text{uncurry } S)$. \square

THEOREM 320. concat is an infinitely associative function.

PROOF. $\text{concat}(\llbracket x \rrbracket) = x$ for a function x taking an ordinal number of argument is obvious. It is remained to prove

$$\text{concat}(\text{concat} \circ S) = \text{concat}(\text{concat } S);$$

We have, using the lemmas,

$$\begin{aligned} \text{concat}(\text{concat} \circ S) &\sim \\ \text{uncurry}(\text{concat} \circ S) &\sim \\ &\text{(by lemma 318)} \\ \text{uncurry}(\text{uncurry} \circ S) &\sim \\ \text{uncurry}(\text{uncurry } S) &\sim \\ \text{uncurry}(\text{concat } S) &\sim \\ \text{concat}(\text{concat } S). & \end{aligned}$$

Consequently $\text{concat}(\text{concat} \circ S) = \text{concat}(\text{concat } S)$. \square

COROLLARY 321. Ordinated product is an infinitely associative function.

3.8. Galois surjections

DEFINITION 322. *Galois surjection* is the special case of Galois connection such that $f^* \circ f_*$ is identity.

PROPOSITION 323. For Galois surjection $\mathfrak{A} \rightarrow \mathfrak{B}$ such that \mathfrak{A} is a join-semilattice we have (for every $y \in \mathfrak{B}$)

$$f_* y = \max \left\{ \frac{x \in \mathfrak{A}}{f^* x = y} \right\}.$$

PROOF. We need to prove (theorem 131)

$$\max \left\{ \frac{x \in \mathfrak{A}}{f^* x = y} \right\} = \max \left\{ \frac{x \in \mathfrak{A}}{f^* x \sqsubseteq y} \right\}.$$

To prove it, it's enough to show that for each $f^* x \sqsubseteq y$ there exists an $x' \sqsupseteq x$ such that $f^* x' = y$.

Really, $y = f^* f_* y$. It's enough to prove $f^*(x \sqcup f_* y) = y$.

Indeed (because lower adjoints preserve joins), $f^*(x \sqcup f_* y) = f^* x \sqcup f^* f_* y = f^* x \sqcup y = y$. \square

3.9. Some properties of frames

This section is based on a TODD TRIMBLE's proof. A shorter but less elementary proof (also by TODD TRIMBLE) is available at

<http://ncatlab.org/toddtrimble/published/topogeny>

I will abbreviate *join-semilattice with least element* as JSWLE.

OBVIOUS 324. JSWLEs are the same as finitely join-closed posets (with nullary joins included).

DEFINITION 325. It is said that a function f from a poset \mathfrak{A} to a poset \mathfrak{B} *preserves finite joins*, when for every finite set $S \in \mathscr{P}\mathfrak{A}$ such that $\bigsqcup^{\mathfrak{A}} S$ exists we have $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S = f \bigsqcup^{\mathfrak{A}} S$.

OBVIOUS 326. A function between JSWLEs preserves finite joins iff it preserves binary joins ($f(x \sqcup y) = fx \sqcup fy$) and nullary joins ($f(\perp^{\mathfrak{A}}) = \perp^{\mathfrak{B}}$).

DEFINITION 327. A *fixed point* of a function F is such x that $F(x) = x$. We will denote $\text{Fix}(F)$ the set of all fixed points of a function F .

DEFINITION 328. Let \mathfrak{A} be a JSWLE. A *co-nucleus* is a function $F : \mathfrak{A} \rightarrow \mathfrak{A}$ such that for every $p, q \in \mathfrak{A}$ we have:

- 1°. $F(p) \sqsubseteq p$;
- 2°. $F(F(p)) = F(p)$;
- 3°. $F(p \sqcup q) = F(p) \sqcup F(q)$.

PROPOSITION 329. Every co-nucleus is a monotone function.

PROOF. It follows from $F(p \sqcup q) = F(p) \sqcup F(q)$. □

LEMMA 330. $\bigsqcup^{\text{Fix}(F)} S = \bigsqcup S$ for every $S \in \mathscr{P}\text{Fix}(F)$ for every co-nucleus F on a complete lattice.

PROOF. Obviously $\bigsqcup S \sqsupseteq x$ for every $x \in S$.

Suppose $z \sqsupseteq x$ for every $x \in S$ for a $z \in \text{Fix}(F)$. Then $z \sqsupseteq \bigsqcup S$.

$F(\bigsqcup S) \sqsupseteq F(x)$ for every $x \in S$. Thus $F(\bigsqcup S) \sqsupseteq \bigsqcup_{x \in S} F(x) = \bigsqcup S$. But $F(\bigsqcup S) \sqsubseteq \bigsqcup S$. Thus $F(\bigsqcup S) = \bigsqcup S$ that is $\bigsqcup S \in \text{Fix}(F)$.

So $\bigsqcup^{\text{Fix}(F)} S = \bigsqcup S$ by the definition of join. □

COROLLARY 331. $\bigsqcup^{\text{Fix}(F)} S$ is defined for every $S \in \mathscr{P}\text{Fix}(F)$.

LEMMA 332. $\prod^{\text{Fix}(F)} S = F(\prod S)$ for every $S \in \mathscr{P}\text{Fix}(F)$ for every co-nucleus F on a complete lattice.

PROOF. Obviously $F(\prod S) \sqsubseteq x$ for every $x \in S$.

Suppose $z \sqsubseteq x$ for every $x \in S$ for a $z \in \text{Fix}(F)$. Then $z \sqsubseteq \prod S$ and thus $z \sqsubseteq F(\prod S)$.

So $\prod^{\text{Fix}(F)} S = F(\prod S)$ by the definition of meet. □

COROLLARY 333. $\prod^{\text{Fix}(F)} S$ is defined for every $S \in \mathscr{P}\text{Fix}(F)$.

OBVIOUS 334. $\text{Fix}(F)$ with induced order is a complete lattice.

LEMMA 335. If F is a co-nucleus on a co-frame \mathfrak{A} , then the poset $\text{Fix}(F)$ of fixed points of F , with order inherited from \mathfrak{A} , is also a co-frame.

PROOF. Let $b \in \text{Fix}(F)$, $S \in \mathcal{P} \text{Fix}(F)$. Then

$$\begin{aligned}
& b \sqcup^{\text{Fix}(F)} \prod^{\text{Fix}(F)} S = \\
& b \sqcup^{\text{Fix}(F)} F\left(\prod S\right) = \\
& F(b) \sqcup F\left(\prod S\right) = \\
& F\left(b \sqcup \prod S\right) = \\
& F\left(\prod \langle b \sqcup \rangle^* S\right) = \\
& \prod^{\text{Fix}(F)} \langle b \sqcup \rangle^* S = \\
& \prod^{\text{Fix}(F)} \langle b \sqcup^{\text{Fix}(F)} \rangle^* S.
\end{aligned}$$

□

DEFINITION 336. Denote $\text{Up}(\mathfrak{A})$ the set of upper sets on \mathfrak{A} ordered *reverse* to set theoretic inclusion.

DEFINITION 337. Denote $\uparrow a = \left\{ \frac{x \sqsubseteq \mathfrak{A}}{x \sqsupseteq a} \right\} \in \text{Up}(\mathfrak{A})$.

LEMMA 338. The set $\text{Up}(\mathfrak{A})$ is closed under arbitrary meets and joins.

PROOF. Let $S \in \mathcal{P} \text{Up}(\mathfrak{A})$.

Let $X \in \bigcup S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then there is $P \in S$ such that $X \in P$ and thus $Y \in P$ and so $Y \in \bigcup S$. So $\bigcup S \in \text{Up}(\mathfrak{A})$.

Let now $X \in \bigcap S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then $\forall T \in S : X \in T$ and so $\forall T \in S : Y \in T$, thus $Y \in \bigcap S$. So $\bigcap S \in \text{Up}(\mathfrak{A})$. □

THEOREM 339. A poset \mathfrak{A} is a complete lattice iff there is a antitone map $s : \text{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ such that

- 1°. $s(\uparrow p) = p$ for every $p \in \mathfrak{A}$;
- 2°. $D \sqsubseteq \uparrow s(D)$ for every $D \in \text{Up}(\mathfrak{A})$.

Moreover, in this case $s(D) = \prod D$ for every $D \in \text{Up}(\mathfrak{A})$.

PROOF.

⇒. Take $s(D) = \prod D$.

⇐. $\forall x \in D : x \sqsupseteq s(D)$ from the second formula.

Let $\forall x \in D : y \sqsubseteq x$. Then $x \in \uparrow y$, $D \sqsubseteq \uparrow y$; because s is an antitone map, thus follows $s(D) \sqsupseteq s(\uparrow y) = y$. So $\forall x \in D : y \sqsubseteq s(D)$.

That s is the meet follows from the definition of meets.

It remains to prove that \mathfrak{A} is a complete lattice.

Take any subset S of \mathfrak{A} . Let D be the smallest upper set containing S . (It exists because $\text{Up}(\mathfrak{A})$ is closed under arbitrary joins.) This is

$$D = \left\{ \frac{x \in \mathfrak{A}}{\exists s \in S : x \sqsupseteq s} \right\}.$$

Any lower bound of D is clearly a lower bound of S since $D \supseteq S$. Conversely any lower bound of S is a lower bound of D . Thus S and D have the same set of lower bounds, hence have the same greatest lower bound.

□

PROPOSITION 340. For any poset \mathfrak{A} the following are mutually reverse order isomorphisms between upper sets F (ordered reverse to set-theoretic inclusion) on \mathfrak{A} and order homomorphisms $\varphi : \mathfrak{A}^{\text{op}} \rightarrow 2$ (here 2 is the partially ordered set of two elements: 0 and 1 where $0 \sqsubseteq 1$), defined by the formulas

$$\begin{aligned} 1^\circ. \varphi(a) &= \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases} \text{ for every } a \in \mathfrak{A}; \\ 2^\circ. F &= \varphi^{-1}(1). \end{aligned}$$

PROOF. Let $X \in \varphi^{-1}(1)$ and $Y \sqsupseteq X$. Then $\varphi(X) = 1$ and thus $\varphi(Y) = 1$. Thus $\varphi^{-1}(1)$ is an upper set.

It is easy to show that φ defined by the formula 1° is an order homomorphism $\mathfrak{A}^{\text{op}} \rightarrow 2$ whenever F is an upper set.

Finally we need to prove that they are mutually inverse. Really: Let φ be defined by the formula 1° . Then take $F' = \varphi^{-1}(1)$ and define $\varphi'(a)$ by the formula 1° . We have

$$\varphi'(a) = \begin{cases} 1 & \text{if } a \in \varphi^{-1}(1) \\ 0 & \text{if } a \notin \varphi^{-1}(1) \end{cases} = \begin{cases} 1 & \text{if } \varphi(a) = 1 \\ 0 & \text{if } \varphi(a) \neq 1 \end{cases} = \varphi(a).$$

Let now F be defined by the formula 2° . Then take $\varphi'(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ as defined by the formula 1° and define $F' = \varphi'^{-1}(1)$. Then

$$F' = \varphi'^{-1}(1) = F.$$

□

LEMMA 341. For a complete lattice \mathfrak{A} , the map $\prod : \text{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ preserves arbitrary meets.

PROOF. Let $S \in \mathcal{P} \text{Up}(\mathfrak{A})$. We have $\prod S \in \text{Up}(\mathfrak{A})$.

$\prod \prod S = \prod \prod_{X \in S} X = \prod_{X \in S} \prod X$ is what we needed to prove. □

LEMMA 342. A complete lattice \mathfrak{A} is a co-frame iff $\prod : \text{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ preserves finite joins.

PROOF.

\Rightarrow . Let \mathfrak{A} be a co-frame. Let $D, D' \in \text{Up}(\mathfrak{A})$. Obviously $\prod(D \sqcup D') \sqsupseteq \prod D$ and $\prod(D \sqcup D') \sqsupseteq \prod D'$, so $\prod(D \sqcup D') \sqsupseteq \prod D \sqcup \prod D'$.

Also

$$\begin{aligned} \prod D \sqcup \prod D' &= \bigcup D \sqcup \bigcup D' = (\text{because } \mathfrak{A} \text{ is a co-frame}) = \\ &= \bigcup \left\{ \frac{d \sqcup d'}{d \in D, d' \in D'} \right\}. \end{aligned}$$

Obviously $d \sqcup d' \in D \cap D'$, thus $\prod D \sqcup \prod D' \subseteq \bigcup(D \cap D') = \prod(D \cap D')$ that is $\prod D \sqcup \prod D' \sqsupseteq \prod(D \cap D')$. So $\prod(D \sqcup D') = \prod D \sqcup \prod D'$ that is $\prod : \text{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ preserves binary joins.

It preserves nullary joins since $\prod^{\text{Up}(\mathfrak{A})} \perp_{\text{Up}(\mathfrak{A})} = \prod^{\text{Up}(\mathfrak{A})} \mathfrak{A} = \perp_{\mathfrak{A}}$.

\Leftarrow . Suppose $\sqcap : \text{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ preserves finite joins. Let $b \in \mathfrak{A}$, $S \in \mathscr{P}\mathfrak{A}$. Let D be the smallest upper set containing S (so $D = \bigcup \langle \uparrow \rangle^* S$). Then

$$\begin{aligned}
b \sqcup \sqcap S &= \\
\sqcap \uparrow b \sqcup \bigcup \sqcap \langle \uparrow \rangle^* S &= \\
\sqcap \uparrow b \sqcup \sqcap \bigcup \langle \uparrow \rangle^* S &= (\text{since } \sqcap \text{ preserves finite joins}) \\
\sqcap (\uparrow b \sqcup \bigcup \langle \uparrow \rangle^* S) &= \\
\bigcup (\uparrow b \sqcap \bigcup \langle \uparrow \rangle^* S) &= \\
\sqcap \bigcup_{a \in S} (\uparrow b \sqcap \uparrow a) &= \\
\sqcap \bigcup_{a \in S} \uparrow (b \sqcup a) &= (\text{since } \sqcap \text{ preserves all meets}) \\
\bigcup_{a \in S} \sqcap \uparrow (b \sqcup a) &= \\
\bigcup_{a \in S} (b \sqcup a) &= \\
\sqcap_{a \in S} (b \sqcup a). &
\end{aligned}$$

□

COROLLARY 343. If \mathfrak{A} is a co-frame, then the composition $F = \uparrow \circ \sqcap : \text{Up}(\mathfrak{A}) \rightarrow \text{Up}(\mathfrak{A})$ is a co-nucleus. The embedding $\uparrow : \mathfrak{A} \rightarrow \text{Up}(\mathfrak{A})$ is an isomorphism of \mathfrak{A} onto the co-frame $\text{Fix}(F)$.

PROOF. $D \sqsupseteq F(D)$ follows from theorem 339.

We have $F(F(D)) = F(D)$ for all $D \in \text{Up}(\mathfrak{A})$ since $F(F(D)) = \uparrow \sqcap \uparrow \sqcap D =$ (because $\sqcap \uparrow s = s$ for any s) $= \uparrow \sqcap D = F(D)$.

And since both $\sqcap : \text{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ and \uparrow preserve finite joins, F preserves finite joins. Thus F is a co-nucleus.

Finally, we have $a \sqsupseteq a'$ if and only if $\uparrow a \subseteq \uparrow a'$, so that $\uparrow : \mathfrak{A} \rightarrow \text{Up}(\mathfrak{A})$ maps \mathfrak{A} isomorphically onto its image $\langle \uparrow \rangle^* \mathfrak{A}$. This image is $\text{Fix}(F)$ because if D is any fixed point (i.e. if $D = \uparrow \sqcap D$), then D clearly belongs to $\langle \uparrow \rangle^* \mathfrak{A}$; and conversely $\uparrow a$ is always a fixed point of $F = \uparrow \circ \sqcap$ since $F(\uparrow a) = \uparrow \sqcap \uparrow a = \uparrow a$. □

DEFINITION 344. If $\mathfrak{A}, \mathfrak{B}$ are two JSWLEs, then $\text{Join}(\mathfrak{A}, \mathfrak{B})$ is the (ordered pointwise) set of finite joins preserving maps $\mathfrak{A} \rightarrow \mathfrak{B}$.

OBVIOUS 345. $\text{Join}(\mathfrak{A}, \mathfrak{B})$ is a JSWLE, where $f \sqcup g$ is given by the formula $(f \sqcup g)(p) = f(p) \sqcup g(p)$, $\perp^{\text{Join}(\mathfrak{A}, \mathfrak{B})}$ is given by the formula $\perp^{\text{Join}(\mathfrak{A}, \mathfrak{B})}(p) = \perp^{\mathfrak{B}}$.

DEFINITION 346. Let $h : Q \rightarrow R$ be a finite joins preserving map. Then by definition $\text{Join}(P, h) : \text{Join}(P, Q) \rightarrow \text{Join}(P, R)$ takes $f \in \text{Join}(P, Q)$ into the composition $h \circ f \in \text{Join}(P, R)$.

LEMMA 347. Above defined $\text{Join}(P, h)$ is a finite joins preserving map.

PROOF.

$$\begin{aligned}
(h \circ (f \sqcup f'))x &= h(f \sqcup f')x = h(fx \sqcup f'x) = \\
&= hfx \sqcup hf'x = (h \circ f)x \sqcup (h \circ f')x = ((h \circ f) \sqcup (h \circ f'))x.
\end{aligned}$$

Thus $h \circ (f \sqcup f') = (h \circ f) \sqcup (h \circ f')$.
 $(h \circ \perp^{\text{Join}(P,Q)})x = h \perp^{\text{Join}(P,Q)}x = h \perp^Q = \perp^R$. \square

PROPOSITION 348. If $h, h' : Q \rightarrow R$ are finite join preserving maps and $h \sqsupseteq h'$, then $\text{Join}(P, h) \sqsupseteq \text{Join}(P, h')$.

PROOF. $\text{Join}(P, h)(f)(x) = (h \circ f)(x) = hfx \sqsupseteq h'fx = (h' \circ f)(x) = \text{Join}(P, h')(f)(x)$. \square

LEMMA 349. If $g : Q \rightarrow R$ and $h : R \rightarrow S$ are finite joins preserving, then the composition $\text{Join}(P, h) \circ \text{Join}(P, g)$ is equal to $\text{Join}(P, h \circ g)$. Also $\text{Join}(P, \text{id}_Q)$ for identity map id_Q on Q is the identity map $\text{id}_{\text{Join}(P,Q)}$ on $\text{Join}(P, Q)$.

PROOF. $\text{Join}(P, h) \text{Join}(P, g)f = \text{Join}(P, h)(g \circ f) = h \circ g \circ f = \text{Join}(P, h \circ g)f$.
 $\text{Join}(P, \text{id}_Q)f = \text{id}_Q \circ f = f$. \square

COROLLARY 350. If Q is a JSWLE and $F : Q \rightarrow Q$ is a co-nucleus, then for any JSWLE P we have that

$$\text{Join}(P, F) : \text{Join}(P, Q) \rightarrow \text{Join}(P, Q)$$

is also a co-nucleus.

PROOF. From $\text{id}_Q \sqsupseteq F$ (co-nucleus axiom 1 $^\circ$) we have $\text{Join}(P, \text{id}_Q) \sqsupseteq \text{Join}(P, F)$ and since by the last lemma the left side is the identity on $\text{Join}(P, Q)$, we see that $\text{Join}(P, F)$ also satisfies co-nucleus axiom 1 $^\circ$.

$\text{Join}(P, F) \circ \text{Join}(P, F) = \text{Join}(P, F \circ F)$ by the same lemma and thus $\text{Join}(P, F) \circ \text{Join}(P, F) = \text{Join}(P, F)$ by the second co-nucleus axiom for F , showing that $\text{Join}(P, F)$ satisfies the second co-nucleus axiom.

By an other lemma, we have that $\text{Join}(P, F)$ preserves binary joins, given that F preserves binary joins, which is the third co-nucleus axiom. \square

LEMMA 351. $\text{Fix}(\text{Join}(P, F)) = \text{Join}(P, \text{Fix}(F))$ for every JSWLEs P, Q and a join preserving function $F : Q \rightarrow Q$.

PROOF. $a \in \text{Fix}(\text{Join}(P, F)) \Leftrightarrow a \in F^P \wedge F \circ a = a \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

$$a \in \text{Join}(P, \text{Fix}(F)) \Leftrightarrow a \in \text{Fix}(F)^P \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x).$$

Thus $\text{Fix}(\text{Join}(P, F)) = \text{Join}(P, \text{Fix}(F))$. That the order of the left and right sides of the equality agrees is obvious. \square

DEFINITION 352. $\mathbf{Pos}(\mathfrak{A}, \mathfrak{B})$ is the pointwise ordered poset of monotone maps from a poset \mathfrak{A} to a poset \mathfrak{B} .

LEMMA 353. If Q, R are JSWLEs and P is a poset, then $\mathbf{Pos}(P, R)$ is a JSWLE and $\mathbf{Pos}(P, \text{Join}(Q, R))$ is isomorphic to $\text{Join}(Q, \mathbf{Pos}(P, R))$. If R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

PROOF. Let $f, g \in \mathbf{Pos}(P, R)$. Then $\lambda x \in P : (fx \sqcup gx)$ is obviously monotone and then it is evident that $f \sqcup^{\mathbf{Pos}(P,R)} g = \lambda x \in P : (fx \sqcup gx)$. $\lambda x \in P : \perp^R$ is also obviously monotone and it is evident that $\perp^{\mathbf{Pos}(P,R)} = \lambda x \in P : \perp^R$.

Obviously both $\mathbf{Pos}(P, \text{Join}(Q, R))$ and $\text{Join}(Q, \mathbf{Pos}(P, R))$ are sets of order preserving maps.

Let f be a monotone map.

$f \in \mathbf{Pos}(P, \text{Join}(Q, R))$ iff $f \in \text{Join}(Q, R)^P$ iff $f \in \left\{ \frac{g \in R^Q}{g \text{ preserves finite joins}} \right\}^P$ iff $f \in (R^Q)^P$ and every $g = f(x)$ (for $x \in P$) preserving finite joins. This is bijectively equivalent ($f \mapsto f'$) to $f' \in (R^P)^Q$ preserving finite joins.

$f' \in \text{Join}(Q, \mathbf{Pos}(P, R))$ iff f' preserves finite joins and $f' \in \mathbf{Pos}(P, R)^Q$ iff f' preserves finite joins and $f' \in \left\{ \frac{g \in (R^P)^Q}{g(x) \text{ is monotone}} \right\}$ iff f' preserves finite joins and $f' \in (R^P)^Q$.

So we have proved that $f \mapsto f'$ is a bijection between $\mathbf{Pos}(P, \text{Join}(Q, R))$ and $\text{Join}(Q, \mathbf{Pos}(P, R))$. That it preserves order is obvious.

It remains to prove that if R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

First, we need to prove that $\mathbf{Pos}(P, R)$ is a complete lattice. But it is easy to prove that for every set $S \in \mathcal{P}\mathbf{Pos}(P, R)$ we have $\lambda x \in P : \sqcup_{f \in S} f(x)$ and $\lambda x \in P : \prod_{f \in S} f(x)$ are monotone and thus are the joins and meets on $\mathbf{Pos}(P, R)$.

Next we need to prove that

$$b \sqcup^{\mathbf{Pos}(P, R)} \prod_{f \in S} f = \prod_{f \in S} \left\langle b \sqcup^{\mathbf{Pos}(P, R)} f \right\rangle^* S.$$

Really (for every $x \in P$),

$$\begin{aligned} \left(b \sqcup^{\mathbf{Pos}(P, R)} \prod_{f \in S} f \right) x &= b(x) \sqcup \left(\prod_{f \in S} f(x) \right) = \\ &= b(x) \sqcup \prod_{f \in S} f(x) = \prod_{f \in S} (b(x) \sqcup f(x)) = \prod_{f \in S} \left(b \sqcup^{\mathbf{Pos}(P, R)} f \right) x = \\ &= \left(\prod_{f \in S} \left(b \sqcup^{\mathbf{Pos}(P, R)} f \right) \right) x. \end{aligned}$$

Thus

$$b \sqcup^{\mathbf{Pos}(P, R)} \prod_{f \in S} f = \prod_{f \in S} \left(b \sqcup^{\mathbf{Pos}(P, R)} f \right) = \prod_{f \in S} \left\langle b \sqcup^{\mathbf{Pos}(P, R)} f \right\rangle^* S.$$

□

DEFINITION 354. $P \cong Q$ means that posets P and Q are isomorphic.

Typed sets and category Rel

4.1. Relational structures

DEFINITION 355. A *relational structure* is a pair consisting of a set and a tuple of relations on this set.

A poset $(\mathfrak{A}, \sqsubseteq)$ can be considered as a relational structure: $(\mathfrak{A}, [\sqsubseteq])$.

A set can X be considered as a relational structure with zero relations: $(X, [])$.

This book is not about relational structures. So I will not introduce more examples.

Think about relational structures as a common place for sets or posets, as far as they are considered in this book.

We will denote $x \in (\mathfrak{A}, R)$ iff $x \in \mathfrak{A}$ for a relational structure (\mathfrak{A}, R) .

4.2. Typed elements and typed sets

We sometimes want to differentiate between the same element of two different sets. For example, we may want to consider different the natural number 3 and the rational number 3. In order to describe this in a formal way we consider elements of sets together with sets themselves. For example, we can consider the pairs $(\mathbb{N}, 3)$ and $(\mathbb{Q}, 3)$.

DEFINITION 356. A *typed element* is a pair (\mathfrak{A}, a) where \mathfrak{A} is a relational structure and $a \in \mathfrak{A}$.

I denote $\text{type}(\mathfrak{A}, a) = \mathfrak{A}$ and $\text{GR}(\mathfrak{A}, a) = a$.

DEFINITION 357. I will denote typed element (\mathfrak{A}, a) as $@^{\mathfrak{A}}a$ or just $@a$ when \mathfrak{A} is clear from context.

DEFINITION 358. A *typed set* is a typed element equal to $(\mathcal{P}U, A)$ where U is a set and A is its subset.

REMARK 359. *Typed sets* is an awkward formalization of type theory sets in ZFC (U is meant to express the *type* of the set). This book could be better written using type theory instead of ZFC, but I want my book to be understandable for everyone knowing ZFC. $(\mathcal{P}U, A)$ should be understood as a set A of type U . For an example, consider $(\mathcal{P}\mathbb{R}, [0; 10])$; it is the closed interval $[0; 10]$ whose elements are considered as real numbers.

DEFINITION 360. $\mathfrak{T}\mathfrak{A} = \left\{ \frac{(\mathfrak{A}, a)}{a \in \mathfrak{A}} \right\} = \{\mathfrak{A}\} \times \mathfrak{A}$ for every relational structure \mathfrak{A} .

REMARK 361. $\mathfrak{T}\mathfrak{A}$ is the set of typed elements of \mathfrak{A} .

DEFINITION 362. If \mathfrak{A} is a poset, we introduce order on its typed elements isomorphic to the order of the original poset: $(\mathfrak{A}, a) \sqsubseteq (\mathfrak{A}, b) \Leftrightarrow a \sqsubseteq b$.

DEFINITION 363. I denote $\text{GR}(\mathfrak{A}, a) = a$ for a typed element (\mathfrak{A}, a) .

DEFINITION 364. I will denote *typed subsets* of a typed poset $(\mathcal{P}U, A)$ as $\mathcal{P}(\mathcal{P}U, A) = \left\{ \frac{(\mathcal{P}U, X)}{X \in \mathcal{P}A} \right\} = \{\mathcal{P}U\} \times \mathcal{P}A$.

OBVIOUS 365. $\mathcal{P}(\mathcal{P}U, A)$ is also a set of typed sets.

DEFINITION 366. I will denote $\mathcal{T}U = \mathfrak{T}\mathcal{P}U$.

REMARK 367. This means that $\mathcal{T}U$ is the set of typed subsets of a set U .

OBVIOUS 368. $\mathcal{T}U = \left\{ \frac{(\mathcal{P}U, X)}{X \in \mathcal{P}U} \right\} = \{\mathcal{P}U\} \times \mathcal{P}U = \mathcal{P}(\mathcal{P}U, U)$.

OBVIOUS 369. $\mathcal{T}U$ is a complete atomistic boolean lattice. Particularly:

- 1°. $\perp^{\mathcal{T}U} = (\mathcal{P}U, \emptyset)$;
- 2°. $\top^{\mathcal{T}U} = (\mathcal{P}U, U)$;
- 3°. $(\mathcal{P}U, A) \sqcup (\mathcal{P}U, B) = (\mathcal{P}U, A \cup B)$;
- 4°. $(\mathcal{P}U, A) \sqcap (\mathcal{P}U, B) = (\mathcal{P}U, A \cap B)$;
- 5°. $\bigsqcup_{A \in S} (\mathcal{P}U, A) = (\mathcal{P}U, \bigcup_{A \in S} A)$;
- 6°. $\bigsqcap_{A \in S} (\mathcal{P}U, A) = \left(\mathcal{P}U, \begin{cases} \bigcap_{A \in S} A & \text{if } A \neq \emptyset \\ U & \text{if } A = \emptyset \end{cases} \right)$;
- 7°. $\overline{(\mathcal{P}U, A)} = (\mathcal{P}U, U \setminus A)$;
- 8°. atomic elements are $(\mathcal{P}U, \{x\})$ where $x \in U$.

Typed sets are “better” than regular sets as (for example) for a set U and a typed set X the following are defined by regular order theory:

- atoms X ;
- \overline{X} ;
- $\bigsqcap^{\mathcal{T}U} \emptyset$.

For regular (“non-typed”) sets these are not defined (except of atoms X which however needs a special definition instead of using the standard order-theory definition of atoms).

Typed sets are convenient to be used together with filters on sets (see below), because both typed sets and filters have a set $\mathcal{P}U$ as their type.

Another advantage of typed sets is that their binary product (as defined below) is a **Rel**-morphism. This is especially convenient because below defined products of filters are also morphisms of related categories.

Well, typed sets are also quite awkward, but the proper way of doing modern mathematics is *type theory* not ZFC, what is however outside of the topic of this book.

4.3. Category **Rel**

I remind that **Rel** is the category of (small) binary relations between sets, and **Set** is its subcategory where only monovalued entirely defined morphisms (functions) are considered.

DEFINITION 370. Order on **Rel**(A, B) is defined by the formula $f \sqsubseteq g \Leftrightarrow \text{GR } f \subseteq \text{GR } g$.

OBVIOUS 371. This order is isomorphic to the natural order of subsets of the set $A \times B$.

DEFINITION 372. $X [f]^* Y \Leftrightarrow \text{GR } X [\text{GR } f]^* \text{GR } Y$ and $\langle f \rangle^* X = (\text{Dst } f, \langle \text{GR } f \rangle^* \text{GR } X)$ for a **Rel**-morphism f and typed sets $X \in \mathcal{T} \text{Src } f$, $Y \in \mathcal{T} \text{Dst } f$.

DEFINITION 373. For category **Rel** there is defined reverse morphism: $(A, B, F)^{-1} = (B, A, F^{-1})$.

OBVIOUS 374. $(f^{-1})^{-1} = f$ for every **Rel**-morphism f .

OBVIOUS 375. $[f^{-1}]^* = [f]^*{}^{-1}$ for every **Rel**-morphism f .

OBVIOUS 376. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable **Rel**-morphisms f and g .

PROPOSITION 377. $\langle g \circ f \rangle^* = \langle g \rangle^* \circ \langle f \rangle^*$ for every composable **Rel**-morphisms f and g .

PROOF. Exercise. □

PROPOSITION 378. The above definitions of monovalued morphisms of **Rel** and of injective morphisms of **Set** coincide with how mathematicians usually define monovalued functions (that is morphisms of **Set**) and injective functions.

PROOF. Let f be a **Rel**-morphism $A \rightarrow B$.

The following are equivalent:

- f is a monovalued relation;
- $\forall x \in A, y_0, y_1 \in B : (x f y_0 \wedge x f y_1 \Rightarrow y_0 = y_1)$;
- $\forall x \in A, y_0, y_1 \in B : (y_0 \neq y_1 \Rightarrow \neg(x f y_0) \vee \neg(x f y_1))$;
- $\forall y_0, y_1 \in B \forall x \in A : (y_0 \neq y_1 \Rightarrow \neg(x f y_0) \vee \neg(x f y_1))$;
- $\forall y_0, y_1 \in B : (y_0 \neq y_1 \Rightarrow \forall x \in A : (\neg(x f y_0) \vee \neg(x f y_1)))$;
- $\forall y_0, y_1 \in B : (\exists x \in A : (x f y_0 \wedge x f y_1) \Rightarrow y_0 = y_1)$;
- $\forall y_0, y_1 \in B : y_0 (f \circ f^{-1}) y_1 \Rightarrow y_0 = y_1$;
- $f \circ f^{-1} \sqsubseteq 1_B$.

Let now f be a **Set**-morphism $A \rightarrow B$.

The following are equivalent:

- f is an injective function;
- $\forall y \in B, a, b \in A : (a f y \wedge b f y \Rightarrow a = b)$;
- $\forall y \in B, a, b \in A : (a \neq b \Rightarrow \neg(a f y) \vee \neg(b f y))$;
- $\forall y \in B : (a \neq b \Rightarrow \forall a, b \in A : (\neg(a f y) \vee \neg(b f y)))$;
- $\forall y \in B : (\exists a, b \in A : (a f y \wedge b f y) \Rightarrow a = b)$;
- $f^{-1} \circ f \sqsubseteq 1_A$.

□

PROPOSITION 379. For a binary relation f we have:

- 1°. $\langle f \rangle^* \bigcup S = \bigcup \langle \langle f \rangle^* \rangle^* S$ for a set of sets S ;
- 2°. $\bigcup S [f]^* Y \Leftrightarrow \exists X \in S : X [f]^* Y$ for a set of sets S ;
- 3°. $X [f]^* \bigcup T \Leftrightarrow \exists Y \in T : X [f]^* Y$ for a set of sets T ;
- 4°. $\bigcup S [f]^* \bigcup T \Leftrightarrow \exists X \in S, Y \in T : X [f]^* Y$ for sets of sets S and T ;
- 5°. $X [f]^* Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\} [f]^* \{\beta\}$ for sets X and Y ;
- 6°. $\langle f \rangle^* X = \bigcup \langle \langle f \rangle^* \rangle^* \text{atoms } X$ for a set X (where $\text{atoms } X = \left\{ \frac{\{x\}}{x \in X} \right\}$).

PROOF.

1°.

$$y \in \langle f \rangle^* \bigcup S \Leftrightarrow \exists x \in \bigcup S : x f y \Leftrightarrow \exists P \in S, x \in P : x f y \Leftrightarrow \\ \exists P \in S : y \in \langle f \rangle^* P \Leftrightarrow \exists Q \in \langle \langle f \rangle^* \rangle^* S : y \in Q \Leftrightarrow y \in \bigcup \langle \langle f \rangle^* \rangle^* S.$$

2°.

$$\bigcup S [f]^* Y \Leftrightarrow \exists x \in \bigcup S, y \in Y : x f y \Leftrightarrow \\ \exists X \in S, x \in X, y \in Y : x f y \Leftrightarrow \exists X \in S : X [f]^* Y.$$

3°. By symmetry.

- 4°. From two previous formulas.
 5°. $X [f]^* Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \alpha f \beta \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\} [f]^* \{\beta\}$.
 6°. Obvious.

□

COROLLARY 380. For a **Rel**-morphism f we have:

- 1°. $\langle f \rangle^* \sqcup S = \sqcup \langle \langle f \rangle^* \rangle^* S$ for $S \in \mathcal{PT} \text{Src } f$;
 2°. $\sqcup S [f]^* Y \Leftrightarrow \exists X \in S : X [f]^* Y$ for $S \in \mathcal{PT} \text{Src } f$;
 3°. $X [f]^* \sqcup T \Leftrightarrow \exists Y \in T : X [f]^* Y$ for $T \in \mathcal{PT} \text{Dst } f$;
 4°. $\sqcup S [f]^* \sqcup T \Leftrightarrow \exists X \in S, Y \in T : X [f]^* Y$ for $S \in \mathcal{PT} \text{Src } f, T \in \mathcal{PT} \text{Dst } f$;
 5°. $X [f]^* Y \Leftrightarrow \exists x \in \text{atoms } X, y \in \text{atoms } Y : x [f]^* y$ for $X \in \mathcal{T} \text{Src } f, Y \in \mathcal{T} \text{Dst } f$;
 6°. $\langle f \rangle^* X = \sqcup \langle \langle f \rangle^* \rangle^* \text{atoms } X$ for $X \in \mathcal{T} \text{Src } f$.

COROLLARY 381. A **Rel**-morphism f can be restored knowing either $\langle f \rangle^* x$ for atoms $x \in \mathcal{T} \text{Src } f$ or $x [f]^* y$ for atoms $x \in \mathcal{T} \text{Src } f, y \in \mathcal{T} \text{Dst } f$.

PROPOSITION 382. Let A, B be sets, R be a set of binary relations.

- 1°. $\langle \bigcup R \rangle^* X = \bigcup_{f \in R} \langle f \rangle^* X$ for every set X ;
 2°. $\langle \bigcap R \rangle^* \{\alpha\} = \bigcap_{f \in R} \langle f \rangle^* \{\alpha\}$ for every α , if R is nonempty;
 3°. $X [\bigcup R]^* Y \Leftrightarrow \exists f \in R : X [f]^* Y$ for every sets X, Y ;
 4°. $\alpha [\bigcap R] \beta \Leftrightarrow \forall f \in R : \alpha f \beta$ for every α and β , if R is nonempty.

PROOF.

1°.

$$y \in \langle \bigcup R \rangle^* X \Leftrightarrow \exists x \in X : x \left(\bigcup R \right) y \Leftrightarrow \exists x \in X, f \in R : x f y \Leftrightarrow \\ \exists f \in R : y \in \langle f \rangle^* X \Leftrightarrow y \in \bigcup_{f \in R} \langle f \rangle^* X.$$

2°.

$$y \in \langle \bigcap R \rangle^* \{\alpha\} \Leftrightarrow \forall f \in R : \alpha f y \Leftrightarrow \forall f \in R : y \in \langle f \rangle^* \{\alpha\} \Leftrightarrow y \in \bigcap_{f \in R} \langle f \rangle^* \{\alpha\}.$$

3°.

$$X [\bigcup R]^* Y \Leftrightarrow \exists x \in X, y \in Y : x \left(\bigcup R \right) y \Leftrightarrow \\ \exists x \in X, y \in Y, f \in R : x f y \Leftrightarrow \exists f \in R : X [f]^* Y.$$

4°. Obvious.

□

COROLLARY 383. Let A, B be sets, $R \in \mathcal{P} \text{Rel}(A, B)$.

- 1°. $\langle \sqcup R \rangle^* X = \sqcup_{f \in R} \langle f \rangle^* X$ for $X \in \mathcal{T} A$;
 2°. $\langle \sqcap R \rangle^* x = \sqcap_{f \in R} \langle f \rangle^* x$ for atomic $x \in \mathcal{T} A$;
 3°. $X [\sqcup R]^* Y \Leftrightarrow \exists f \in R : X [f]^* Y$ for $X \in \mathcal{T} A, Y \in \mathcal{T} B$;
 4°. $x [\sqcap R]^* y \Leftrightarrow \forall f \in R : x [f]^* y$ for every atomic $x \in \mathcal{T} A, y \in \mathcal{T} B$.

PROPOSITION 384. $X [g \circ f]^* Z \Leftrightarrow \exists \beta : (X [f]^* \{\beta\} \wedge \{\beta\} [g]^* Z)$ for every binary relation f and sets X and Z .

PROOF.

$$\begin{aligned} X [g \circ f]^* Z &\Leftrightarrow \exists x \in X, z \in Z : x (g \circ f) z \Leftrightarrow \\ &\quad \exists x \in X, z \in Z, \beta : (x f \beta \wedge \beta g z) \Leftrightarrow \\ &\quad \exists \beta : (\exists x \in X : x f \beta \wedge \exists y \in Y : \beta g z) \Leftrightarrow \exists \beta : (X [f]^* \{\beta\} \wedge \{\beta\} [g]^* Z). \end{aligned}$$

□

COROLLARY 385. $X [g \circ f]^* Z \Leftrightarrow \exists y \in \text{atoms}^{\mathcal{T}B} : (X [f]^* y \wedge y [g]^* Z)$ for $f \in \mathbf{Rel}(A, B)$, $g \in \mathbf{Rel}(B, C)$ (for sets A, B, C).

PROPOSITION 386. $f \circ \bigcup G = \bigcup_{g \in G} (f \circ g)$ and $\bigcup G \circ f = \bigcup_{g \in G} (g \circ f)$ for every binary relation f and set G of binary relations.

PROOF. We will prove only $\bigcup G \circ f = \bigcup_{g \in G} (g \circ f)$ as the other formula follows from duality. Really

$$\begin{aligned} (x, z) \in \bigcup G \circ f &\Leftrightarrow \exists y : ((x, y) \in f \wedge (y, z) \in \bigcup G) \Leftrightarrow \\ \exists y, g \in G : ((x, y) \in f \wedge (y, z) \in g) &\Leftrightarrow \exists g \in G : (x, z) \in g \circ f \Leftrightarrow (x, z) \in \bigcup_{g \in G} (g \circ f). \end{aligned}$$

□

COROLLARY 387. Every **Rel**-morphism is metacomplete and co-metacomplete.

PROPOSITION 388. The following are equivalent for a **Rel**-morphism f :

- 1°. f is monovalued.
- 2°. f is metamonovalued.
- 3°. f is weakly metamonovalued.
- 4°. $\langle f \rangle^* a$ is either atomic or least whenever $a \in \text{atoms}^{\mathcal{T} \text{Src } f}$.
- 5°. $\langle f^{-1} \rangle^* (I \sqcap J) = \langle f^{-1} \rangle^* I \sqcap \langle f^{-1} \rangle^* J$ for every $I, J \in \mathcal{T} \text{Src } f$.
- 6°. $\langle f^{-1} \rangle^* \prod S = \prod_{Y \in S} \langle f^{-1} \rangle^* Y$ for every $S \in \mathcal{P} \mathcal{T} \text{Src } f$.

PROOF.

2° ⇒ 3°. Obvious.

1° ⇒ 2°. Take $x \in \text{atoms}^{\mathcal{T} \text{Src } f}$; then $fx \in \text{atoms}^{\mathcal{T} \text{Dst } f} \cup \{\perp^{\mathcal{T} \text{Dst } f}\}$ and thus

$$\begin{aligned} \langle (\prod G) \circ f \rangle^* x &= \langle \prod G \rangle^* \langle f \rangle^* x = \prod_{g \in G} \langle g \rangle^* \langle f \rangle^* x = \\ &= \prod_{g \in G} \langle g \circ f \rangle^* x = \left\langle \prod_{g \in G} (g \circ f) \right\rangle^* x; \end{aligned}$$

so $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$.

3° ⇒ 1°. Take $g = \{(a, y)\}$ and $h = \{(b, y)\}$ for arbitrary $a \neq b$ and arbitrary y . We have $g \cap h = \emptyset$; thus $(g \circ f) \cap (h \circ f) = (g \cap h) \circ f = \perp$ and thus impossible $x f a \wedge x f b$ as otherwise $(x, y) \in (g \circ f) \cap (h \circ f)$. Thus f is monovalued.

$4^\circ \Rightarrow 6^\circ$. Let $a \in \text{atoms}^{\mathcal{T} \text{Src } f}$, $\langle f \rangle^* a = b$. Then because $b \in \text{atoms}^{\mathcal{T} \text{Dst } f} \cup \{\perp\}^{\mathcal{T} \text{Dst } f}$

$$\begin{aligned} \prod S \sqcap b \neq \perp &\Leftrightarrow \forall Y \in S : Y \sqcap b \neq \perp; \\ a [f]^* \prod S &\Leftrightarrow \forall Y \in S : a [f]^* Y; \\ \prod S [f^{-1}]^* a &\Leftrightarrow \forall Y \in S : Y [f^{-1}]^* a; \\ a \neq \langle f^{-1} \rangle^* \prod S &\Leftrightarrow \forall Y \in S : a \neq \langle f^{-1} \rangle^* Y; \\ a \neq \langle f^{-1} \rangle^* \prod S &\Leftrightarrow a \neq \prod_{Y \in S} \langle f^{-1} \rangle^* Y; \\ \langle f^{-1} \rangle^* \prod S &= \prod_{X \in S} \langle f^{-1} \rangle^* X. \end{aligned}$$

$6^\circ \Rightarrow 5^\circ$. Obvious.

$5^\circ \Rightarrow 1^\circ$. $\langle f^{-1} \rangle^* a \sqcap \langle f^{-1} \rangle^* b = \langle f^{-1} \rangle^* (a \sqcap b) = \langle f^{-1} \rangle^* \perp = \perp$ for every two distinct atoms $a = \{\alpha\}, b = \{\beta\} \in \mathcal{T} \text{Dst } f$. From this

$$\begin{aligned} \alpha (f \circ f^{-1}) \beta &\Leftrightarrow \exists y \in \text{Dst } f : (\alpha f^{-1} y \wedge y f \beta) \Leftrightarrow \\ &\exists y \in \text{Dst } f : (y \in \langle f^{-1} \rangle^* a \wedge y \in \langle f^{-1} \rangle^* b) \end{aligned}$$

is impossible. Thus $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\mathbf{Rel}}$.

$\neg 4^\circ \Rightarrow \neg 1^\circ$. Suppose $\langle f \rangle^* a \notin \text{atoms}^{\mathcal{T} \text{Dst } f} \cup \{\perp\}^{\mathcal{T} \text{Dst } f}$ for some $a \in \text{atoms}^{\mathcal{T} \text{Src } f}$. Then there exist distinct points p, q such that $p, q \in \langle f \rangle^* a$. Thus $p (f \circ f^{-1}) q$ and so $f \circ f^{-1} \not\sqsubseteq 1_{\text{Dst } f}^{\mathbf{Rel}}$. \square

4.4. Product of typed sets

DEFINITION 389. Product of typed sets is defined by the formula

$$(\mathcal{P}U, A) \times (\mathcal{P}W, B) = (U, W, A \times B).$$

PROPOSITION 390. Product of typed sets is a **Rel**-morphism.

PROOF. We need to prove $A \times B \subseteq U \times W$, but this is obvious. \square

OBVIOUS 391. Atoms of $\mathbf{Rel}(A, B)$ are exactly products $a \times b$ where a and b are atoms correspondingly of $\mathcal{T}A$ and $\mathcal{T}B$. $\mathbf{Rel}(A, B)$ is an atomistic poset.

PROPOSITION 392. $f \not\neq A \times B \Leftrightarrow A [f]^* B$ for every **Rel**-morphism f and $A \in \mathcal{T} \text{Src } f, B \in \mathcal{T} \text{Dst } f$.

PROOF.

$$\begin{aligned} A [f]^* B &\Leftrightarrow \exists x \in \text{atoms } A, y \in \text{atoms } B : x [f]^* y \Leftrightarrow \\ \exists x \in \text{atoms}^{\mathcal{T} \text{Src } f}, y \in \text{atoms}^{\mathcal{T} \text{Dst } f} & : (x \times y \sqsubseteq f \wedge x \times y \sqsubseteq A \times B) \Leftrightarrow f \not\neq A \times B. \end{aligned}$$

\square

DEFINITION 393. *Image* and *domain* of a **Rel**-morphism f are typed sets defined by the formulas

$$\text{dom}(U, W, f) = (\mathcal{P}U, \text{dom } f) \quad \text{and} \quad \text{im}(U, W, f) = (\mathcal{P}W, \text{im } f).$$

OBVIOUS 394. Image and domain of a **Rel**-morphism are really typed sets.

DEFINITION 395. *Restriction* of a **Rel**-morphism to a typed set is defined by the formula $(U, W, f)|_{(\mathcal{P}U, X)} = (U, W, f|_X)$.

OBVIOUS 396. Restriction of a **Rel**-morphism is **Rel**-morphism.

OBVIOUS 397. $f|_A = f \sqcap (A \times \top^{\mathcal{T} \text{Dst } f})$ for every **Rel**-morphism f and $A \in \mathcal{T} \text{Src } f$.

OBVIOUS 398. $\langle f \rangle^* X = \langle f \rangle^*(X \sqcap \text{dom } f) = \text{im}(f|_X)$ for every **Rel**-morphism f and $X \in \mathcal{T} \text{Src } f$.

OBVIOUS 399. $f \sqsubseteq A \times B \Leftrightarrow \text{dom } f \sqsubseteq A \wedge \text{im } f \sqsubseteq B$ for every **Rel**-morphism f and $A \in \mathcal{T} \text{Src } f, B \in \mathcal{T} \text{Dst } f$.

THEOREM 400. Let A, B be sets. If $S \in \mathcal{P}(\mathcal{T}A \times \mathcal{T}B)$ then

$$\bigsqcap_{(A,B) \in S} (A \times B) = \bigsqcap \text{dom } S \times \bigsqcap \text{im } S.$$

PROOF. For every atomic $x \in \mathcal{T}A, y \in \mathcal{T}B$ we have

$$\begin{aligned} x \times y \sqsubseteq \bigsqcap_{(A,B) \in S} (A \times B) &\Leftrightarrow \forall (A,B) \in S : x \times y \sqsubseteq A \times B \Leftrightarrow \\ \forall (A,B) \in S : (x \sqsubseteq A \wedge y \sqsubseteq B) &\Leftrightarrow \forall A \in \text{dom } S : x \sqsubseteq A \wedge \forall B \in \text{im } S : y \sqsubseteq B \Leftrightarrow \\ x \sqsubseteq \bigsqcap \text{dom } S \wedge y \sqsubseteq \bigsqcap \text{im } S &\Leftrightarrow x \times y \sqsubseteq \bigsqcap \text{dom } S \times \bigsqcap \text{im } S. \end{aligned}$$

□

OBVIOUS 401. If U, W are sets and $A \in \mathcal{T}(U)$ then $A \times$ is a complete homomorphism from the lattice $\mathcal{T}(W)$ to the lattice **Rel**(U, W), if also $A \neq \perp$ then it is an order embedding.

Filters and filtrators

This chapter is based on my article [30].

This chapter is grouped in the following way:

- First it goes a short introduction in pedagogical order (first less general stuff and examples, last the most general stuff):
 - filters on a set;
 - filters on a meet-semilattice;
 - filters on a poset.
- Then it goes the formal part.

5.1. Implication tuples

DEFINITION 402. An *implications tuple* is a tuple (P_1, \dots, P_n) such that $P_1 \Rightarrow \dots \Rightarrow P_n$.

OBVIOUS 403. (P_1, \dots, P_n) is an implications tuple iff $P_i \Rightarrow P_j$ for every $i < j$ (where $i, j \in \{1, \dots, n\}$).

The following is an example of a theorem using an implication tuple:

EXAMPLE 404. The following is an implications tuple:

- 1°. A .
- 2°. B .
- 3°. C .

This example means just that $A \Rightarrow B \Rightarrow C$.

I prefer here a verbal description instead of symbolic implications $A \Rightarrow B \Rightarrow C$, because A, B, C may be long English phrases and they may not fit into the formula layout.

The main (intuitive) idea of the theorem is expressed by the implication $P_1 \Rightarrow P_n$, the rest implications ($P_2 \Rightarrow P_n, P_3 \Rightarrow P_n, \dots$) are purely technical, as they express generalizations of the main idea.

For uniformity theorems in the section about filters and filtrators start with the same P_1 : “ $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.” (defined below) That means that the main idea of the theorem is about powerset filtrators, the rest implications (like $P_2 \Rightarrow P_n, P_3 \Rightarrow P_n, \dots$) are just technical generalizations.

5.2. Introduction to filters and filtrators

5.2.1. Filters on a set. We sometimes want to define something resembling an infinitely small (or infinitely big) set, for example the infinitely small interval near 0 on the real line. Of course there is no such set, just like as there is no natural number which is the difference $2 - 3$. To overcome this shortcoming we introduce whole numbers, and $2 - 3$ becomes well defined. In the same way to consider things which are like infinitely small (or infinitely big) sets we introduce *filters*.

An example of a filter is the infinitely small interval near 0 on the real line. To come to infinitely small, we consider all intervals $] - \epsilon; \epsilon[$ for all $\epsilon > 0$. This filter

consists of all intervals $] - \epsilon; \epsilon[$ for all $\epsilon > 0$ and also all subsets of \mathbb{R} containing such intervals as subsets. Informally speaking, this is the greatest filter contained in every interval $] - \epsilon; \epsilon[$ for all $\epsilon > 0$.

DEFINITION 405. A filter on a set \mathcal{U} is a $\mathcal{F} \in \mathcal{P}\mathcal{P}\mathcal{U}$ such that:

- 1°. $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$;
- 2°. $\forall A, B \in \mathcal{P}\mathcal{U} : (A \in \mathcal{F} \wedge B \supseteq A \Rightarrow B \in \mathcal{F})$.

EXERCISE 406. Verify that the above introduced infinitely small interval near 0 on the real line is a filter on \mathbb{R} .

EXERCISE 407. Describe “the neighborhood of positive infinity” filter on \mathbb{R} .

DEFINITION 408. A filter not containing empty set is called a *proper filter*.

OBVIOUS 409. The non-proper filter is $\mathcal{P}\mathcal{U}$.

REMARK 410. Some other authors require that all filters are proper. This is a stupid idea and we allow non-proper filters, in the same way as we allow to use the number 0.

5.2.2. Intro to filters on a meet-semilattice. A trivial generalization of the above:

DEFINITION 411. A filter on a meet-semilattice \mathfrak{J} is a $\mathcal{F} \in \mathcal{P}\mathfrak{J}$ such that:

- 1°. $\forall A, B \in \mathcal{F} : A \sqcap B \in \mathcal{F}$;
- 2°. $\forall A, B \in \mathfrak{J} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

5.2.3. Intro to filters on a poset.

DEFINITION 412. A filter on a poset \mathfrak{J} is a $\mathcal{F} \in \mathcal{P}\mathfrak{J}$ such that:

- 1°. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F} : C \sqsubseteq A, B$;
- 2°. $\forall A, B \in \mathfrak{J} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

It is easy to show (and there is a proof of it somewhere below) that this coincides with the above definition in the case if \mathfrak{J} is a meet-semilattice.

5.3. Filters on a poset

5.3.1. Filters on posets. Let \mathfrak{J} be a poset.

DEFINITION 413. *Filter base* is a nonempty subset F of \mathfrak{J} such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsubseteq X \wedge Z \sqsubseteq Y).$$

DEFINITION 414. *Ideal base* is a nonempty subset F of \mathfrak{J} such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsupseteq X \wedge Z \sqsupseteq Y).$$

OBVIOUS 415. Ideal base is the dual of filter base.

OBVIOUS 416.

- 1°. A poset with a lowest element is a filter base.
- 2°. A poset with a greatest element is an ideal base.

OBVIOUS 417.

- 1°. A meet-semilattice is a filter base.
- 2°. A join-semilattice is an ideal base.

OBVIOUS 418. A nonempty chain is a filter base and an ideal base.

DEFINITION 419. *Filter* is a subset of \mathfrak{J} which is both a filter base and an upper set.

I will denote the set of filters (for a given or implied poset \mathfrak{Z}) as \mathfrak{F} and call \mathfrak{F} the set of filters over the poset \mathfrak{Z} .

PROPOSITION 420. If \top is the maximal element of \mathfrak{Z} then $\top \in F$ for every filter F .

PROOF. If $\top \notin F$ then $\forall K \in \mathfrak{Z} : K \notin F$ and so F is empty what is impossible. \square

PROPOSITION 421. Let S be a filter base on a poset. If $A_0, \dots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : (C \sqsubseteq A_0 \wedge \dots \wedge C \sqsubseteq A_n).$$

PROOF. It can be easily proved by induction. \square

DEFINITION 422. A function f from a poset \mathfrak{A} to a poset \mathfrak{B} *preserves filtered meets* iff whenever $\prod S$ is defined for a filter base S on \mathfrak{A} we have $f \prod S = \prod \langle f \rangle^* S$.

5.3.2. Filters on meet-semilattices.

THEOREM 423. If \mathfrak{Z} is a meet-semilattice and F is a nonempty subset of \mathfrak{Z} then the following conditions are equivalent:

- 1°. F is a filter.
- 2°. $\forall X, Y \in F : X \sqcap Y \in F$ and F is an upper set.
- 3°. $\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F)$.

PROOF.

1° \Rightarrow 2°. Let F be a filter. Then F is an upper set. If $X, Y \in F$ then $Z \sqsubseteq X \wedge Z \sqsubseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \sqsubseteq X \sqcap Y$ then $X \sqcap Y \in F$.

2° \Rightarrow 1°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \sqcap Y$ (we have also taken into account that $F \neq \emptyset$).

2° \Rightarrow 3°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. Then

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Rightarrow X \sqcap Y \in F).$$

Let $X \sqcap Y \in F$; then $X, Y \in F$ because F is an upper set.

3° \Rightarrow 2°. Let

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F).$$

Then $\forall X, Y \in F : X \sqcap Y \in F$. Let $X \in F$ and $X \sqsubseteq Y \in \mathfrak{Z}$. Then $X \sqcap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set. \square

PROPOSITION 424. Let S be a filter base on a meet-semilattice. If $A_0, \dots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : C \sqsubseteq A_0 \sqcap \dots \sqcap A_n.$$

PROOF. It can be easily proved by induction. \square

PROPOSITION 425. If \mathfrak{Z} is a meet-semilattice and S is a filter base on it, $A \in \mathfrak{Z}$, then $\langle A \sqcap \rangle^* S$ is also a filter base.

PROOF. $\langle A \sqcap \rangle^* S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A \sqcap \rangle^* S$. Then $X = A \sqcap X'$ and $Y = A \sqcap Y'$ where $X', Y' \in S$. There exists $Z' \in S$ such that $Z' \sqsubseteq X' \sqcap Y'$. So $X \sqcap Y = A \sqcap X' \sqcap Y' \sqsupseteq A \sqcap Z' \in \langle A \sqcap \rangle^* S$. \square

5.3.3. Order of filters. Principal filters. I will make the set of filters \mathfrak{F} into a poset by the order defined by the formula: $a \sqsubseteq b \Leftrightarrow a \supseteq b$.

DEFINITION 426. The principal filter corresponding to an element $a \in \mathfrak{J}$ is

$$\uparrow a = \left\{ \frac{x \in \mathfrak{J}}{x \supseteq a} \right\}.$$

Elements of $\mathfrak{P} = \langle \uparrow \rangle^* \mathfrak{J}$ are called *principal filters*.

OBVIOUS 427. Principal filters are filters.

OBVIOUS 428. \uparrow is an order embedding from \mathfrak{J} to \mathfrak{F} .

COROLLARY 429. \uparrow is an order isomorphism between \mathfrak{J} and \mathfrak{P} .

We will equate principal filters with corresponding elements of the base poset (in the same way as we equate for example nonnegative whole numbers and natural numbers).

PROPOSITION 430. $\uparrow K \supseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$.

PROOF. $\uparrow K \supseteq \mathcal{A} \Leftrightarrow \uparrow K \subseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$. □

5.4. Filters on a Set

Consider filters on the poset $\mathfrak{J} = \mathcal{P}\mathfrak{U}$ (where \mathfrak{U} is some fixed set) with the order $A \sqsubseteq B \Leftrightarrow A \subseteq B$ (for $A, B \in \mathcal{P}\mathfrak{U}$).

In fact, it is a complete atomistic boolean lattice with $\prod S = \bigcap S$, $\sqcup S = \bigcup S$, $\overline{A} = \mathfrak{U} \setminus A$ for every $S \in \mathcal{P}\mathcal{P}\mathfrak{U}$ and $A \in \mathcal{P}\mathfrak{U}$, atoms being one-element sets.

DEFINITION 431. I will call a filter on the lattice of all subsets of a given set \mathfrak{U} as a *filter on set*.

DEFINITION 432. I will denote the set on which a filter \mathcal{F} is defined as $\text{Base}(\mathcal{F})$.

OBVIOUS 433. $\text{Base}(\mathcal{F}) = \bigcup \mathcal{F}$.

PROPOSITION 434. The following are equivalent for a non-empty set $F \in \mathcal{P}\mathcal{P}\mathfrak{U}$:

- 1°. F is a filter.
- 2°. $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.
- 3°. $\forall X, Y \in \mathcal{P}\mathfrak{U} : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

PROOF. By theorem 423. □

OBVIOUS 435. The minimal filter on $\mathcal{P}\mathfrak{U}$ is $\mathcal{P}\mathfrak{U}$.

OBVIOUS 436. The maximal filter on $\mathcal{P}\mathfrak{U}$ is $\{\mathfrak{U}\}$.

I will denote $\uparrow A = \uparrow^{\mathfrak{U}} A = \uparrow^{\mathcal{P}\mathfrak{U}} A$. (The distinction between conflicting notations $\uparrow^{\mathfrak{U}} A$ and $\uparrow^{\mathcal{P}\mathfrak{U}} A$ will be clear from the context.)

PROPOSITION 437. Every filter on a finite set is principal.

PROOF. Let \mathcal{F} be a filter on a finite set. Then obviously $\mathcal{F} = \prod^{\mathfrak{J}} \text{up } \mathcal{F}$ and thus \mathcal{F} is principal. □

5.5. Filtrators

$(\mathfrak{F}, \mathfrak{P})$ is a poset and its subset (with induced order on the subset). I call pairs of a poset and its subset like this *filtrators*.

DEFINITION 438. I will call a *filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a poset \mathfrak{A} and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$. I call \mathfrak{A} the *base* of the filtrator and \mathfrak{Z} the *core* of the filtrator. I will also say that $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator *over* poset \mathfrak{Z} .

I will denote $\text{base}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{A}$, $\text{core}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{Z}$ for a filtrator $(\mathfrak{A}, \mathfrak{Z})$.

While *filters* are customary and well known mathematical objects, the concept of *filtrators* is probably first researched by me.

When speaking about filters, we will imply that we consider the filtrator $(\mathfrak{F}, \mathfrak{P})$ or what is the same (as we equate principal filters with base elements) the filtrator $(\mathfrak{F}, \mathfrak{Z})$.

DEFINITION 439. I will call a *lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a lattice \mathfrak{A} and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

DEFINITION 440. I will call a *complete lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a complete lattice \mathfrak{A} and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

DEFINITION 441. I will call a *central filtrator* a filtrator $(\mathfrak{A}, Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice \mathfrak{A} .

DEFINITION 442. I will call *element* of a filtrator an element of its base.

DEFINITION 443. $\text{up}^{\mathfrak{Z}} a = \text{up } a = \left\{ \frac{c \sqsubseteq \mathfrak{Z}}{c \sqsubseteq a} \right\}$ for an element a of a filtrator.

DEFINITION 444. $\text{down}^{\mathfrak{Z}} a = \text{down } a = \left\{ \frac{c \sqsubseteq \mathfrak{Z}}{c \sqsubseteq a} \right\}$ for an element a of a filtrator.

OBVIOUS 445. “up” and “down” are dual.

Our main purpose here is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of $\text{up } a$ for every element a .

DEFINITION 446. I call a filtrator *with join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcup^{\mathfrak{Z}} S = \bigsqcup^{\mathfrak{A}} S$ whenever $\bigsqcup^{\mathfrak{Z}} S$ exists for $S \in \mathcal{P}\mathfrak{Z}$.

DEFINITION 447. I call a filtrator *with meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcap^{\mathfrak{Z}} S = \bigsqcap^{\mathfrak{A}} S$ whenever $\bigsqcap^{\mathfrak{Z}} S$ exists for $S \in \mathcal{P}\mathfrak{Z}$.

DEFINITION 448. I call a filtrator with *binarily join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcup^{\mathfrak{Z}} b = a \sqcup^{\mathfrak{A}} b$ whenever $a \sqcup^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

DEFINITION 449. I call a filtrator with *binarily meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcap^{\mathfrak{Z}} b = a \sqcap^{\mathfrak{A}} b$ whenever $a \sqcap^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

DEFINITION 450. *Prefiltered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that “up” is injective.

DEFINITION 451. *Filtered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that

$$\forall a, b \in \mathfrak{A} : (\text{up } a \supseteq \text{up } b \Rightarrow a \sqsubseteq b).$$

THEOREM 452. A filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered iff $\forall a \in \mathfrak{A} : a = \bigsqcap^{\mathfrak{A}} \text{up } a$.

PROOF.

$$\Leftarrow. \text{up } a \supseteq \text{up } b \Rightarrow \bigsqcap^{\mathfrak{A}} \text{up } a \sqsubseteq \bigsqcap^{\mathfrak{A}} \text{up } b \Rightarrow a \sqsubseteq b.$$

\Rightarrow . $a = \prod^{\mathfrak{A}}$ up a is equivalent to a is a greatest lower bound of up a . That is the implication that b is lower bound of up a implies $a \sqsupseteq b$.
 b is lower bound of up a implies up $b \supseteq$ up a . So as it is filtered $a \sqsupseteq b$.
 \square

OBVIOUS 453. Every filtered filtrator is prefiltered.

OBVIOUS 454. “up” is a straight map from \mathfrak{A} to the dual of the poset $\mathcal{P}\mathfrak{Z}$ if $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.

DEFINITION 455. An *isomorphism* between filtrators $(\mathfrak{A}_0, \mathfrak{Z}_0)$ and $(\mathfrak{A}_1, \mathfrak{Z}_1)$ is an isomorphism between posets \mathfrak{A}_0 and \mathfrak{A}_1 such that it maps \mathfrak{Z}_0 into \mathfrak{Z}_1 .

OBVIOUS 456. Isomorphism isomorphically maps the order on \mathfrak{Z}_0 into order on \mathfrak{Z}_1 .

DEFINITION 457. Two filtrators are *isomorphic* when there exists an isomorphism between them.

DEFINITION 458. I will call *primary filtrator* a filtrator isomorphic to the filtrator consisting of the set of filters on a poset and the set of principal filters on this poset.

OBVIOUS 459. The order on a primary filtrator is defined by the formula $a \sqsubseteq b \Leftrightarrow \text{up } a \supseteq \text{up } b$.

DEFINITION 460. I will call a primary filtrator over a poset isomorphic to a powerset as *powerset filtrator*.

OBVIOUS 461. up \mathcal{F} is a filter for every element \mathcal{F} of a primary filtrator. Reversely, there exists a filter \mathcal{F} if up \mathcal{F} is a filter.

THEOREM 462. For every poset \mathfrak{Z} there exists a poset $\mathfrak{A} \supseteq \mathfrak{Z}$ such that $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.

PROOF. See appendix A. \square

5.5.1. Filtrators with Separable Core.

DEFINITION 463. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with separable core* when

$$\forall x, y \in \mathfrak{A} : (x \succ^{\mathfrak{A}} y \Rightarrow \exists X \in \text{up } x : X \succ^{\mathfrak{A}} y).$$

PROPOSITION 464. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with separable core* iff

$$\forall x, y \in \mathfrak{A} : (x \succ^{\mathfrak{A}} y \Rightarrow \exists X \in \text{up } x, Y \in \text{up } y : X \succ^{\mathfrak{A}} Y).$$

PROOF.

\Rightarrow . Apply the definition twice.

\Leftarrow . Obvious. \square

DEFINITION 465. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with co-separable core* when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \text{down } x : X \equiv^{\mathfrak{A}} y).$$

OBVIOUS 466. Co-separability is the dual of separability.

DEFINITION 467. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with co-separable core* when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \text{down } x, Y \in \text{down } y : X \equiv^{\mathfrak{A}} Y).$$

PROOF. By duality. \square

5.6. Alternative primary filtrators

5.6.1. Lemmas.

LEMMA 468. A set F is a lower set iff \overline{F} is an upper set.

PROOF. $X \in \overline{F} \wedge Z \sqsupseteq X \Rightarrow Z \in \overline{F}$ is equivalent to $Z \in F \Rightarrow X \in F \vee Z \not\sqsupseteq X$ is equivalent $Z \in F \Rightarrow (Z \sqsupseteq X \Rightarrow X \in F)$ is equivalent $Z \in F \wedge X \sqsubseteq Z \Rightarrow X \in F$. \square

PROPOSITION 469. Let \mathfrak{Z} be a poset with least element \perp . Then for upper set F we have $F \neq \mathcal{P}\mathfrak{Z} \Leftrightarrow \perp \notin F$.

PROOF.

\Rightarrow . If $\perp \in F$ then $F = \mathcal{P}\mathfrak{Z}$ because F is an upper set.

\Leftarrow . Obvious. \square

5.6.2. Informal introduction. We have already defined filters on a poset. Now we will define three other sets which are order-isomorphic to the set of filters on a poset: ideals (\mathfrak{I}), free stars (\mathfrak{S}), and mixers (\mathfrak{M}).

These four kinds of objects are related through commutative diagrams. First we will paint an informal commutative diagram (it makes no formal sense because it is not pointed the poset for which the filters are defined):

$$\begin{array}{ccc} \mathfrak{F} & \xleftarrow{(\text{dual})^*} & \mathfrak{I} \\ \uparrow \neg & & \downarrow \neg \\ \mathfrak{M} & \xleftarrow{(\text{dual})^*} & \mathfrak{S} \end{array}$$

Then we can define ideals, free stars, and mixers as sets following certain formulas. You can check that the intuition behind these formulas follows the above commutative diagram. (That is transforming these formulas by the course of the above diagram, you get formulas of the other objects in this list.)

After this, we will paint some formal commutative diagrams similar to the above diagram but with particular posets at which filters, ideals, free stars, and mixers are defined.

5.6.3. Definitions of ideals, free stars, and mixers. *Filters* and *ideals* are well known concepts. The terms *free stars* and *mixers* are my new terminology.

Recall that *filters* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 470. *Ideals* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 471. *Free stars* are sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 472. *Mixers* are sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

By duality and and an above theorem about filters, we have:

PROPOSITION 473.

- Filters are nonempty upper sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Ideals are nonempty lower sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

- Free stars are upper sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

PROPOSITION 474. The following are equivalent:

- 1°. F is a free star.
- 2°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathcal{P}\mathfrak{Z}$.
- 3°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an upper set and $F \neq \mathcal{P}\mathfrak{Z}$.

PROOF.

1° \Leftrightarrow 2°. The following is a chain of equivalencies:

$$\begin{aligned} \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B) &\Leftrightarrow A \notin F \wedge B \notin F; \\ \forall Z \in \overline{F} : \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B) &\Leftrightarrow A \in F \vee B \in F; \\ \forall Z \in \mathfrak{Z} : (Z \notin F \Rightarrow \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B)) &\Leftrightarrow A \in F \vee B \in F; \\ \forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) &\Leftrightarrow A \in F \vee B \in F. \end{aligned}$$

2° \Rightarrow 3°. Let $A = B \in F$. Then $A \in F \vee B \in F$. So $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \Rightarrow Z \in F)$ that is F is an upper set.

3° \Rightarrow 2°. We need to prove that F is an upper set. let $A \in F$ and $A \sqsubseteq B \in \mathfrak{Z}$. Then $A \in F \vee B \in F$ and thus $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq B \Rightarrow Z \in F)$ and so $B \in F$.

□

COROLLARY 475. The following are equivalent:

- 1°. F is a mixer.
- 2°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathcal{P}\mathfrak{Z}$.
- 3°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is a lower set and $F \neq \mathcal{P}\mathfrak{Z}$.

OBVIOUS 476.

- 1°. A free star cannot contain the least element of the poset.
- 2°. A mixer cannot contain the greatest element of the poset.

5.6.4. Filters, ideals, free stars, and mixers on semilattices.

PROPOSITION 477.

- Free stars are sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Free stars are upper sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

PROOF. By duality. □

By duality and an above theorem about filters, we have:

PROPOSITION 478.

- Filters are nonempty sets F with $A \sqcap B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a meet-semilattice.
- Ideals are nonempty sets F with $A \sqcup B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a join-semilattice.
- Free stars are sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \sqcup B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a join-semilattice.
- Mixers are sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \sqcap B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a meet-semilattice.

By duality and and an above theorem about filters, we have:

PROPOSITION 479.

- Filters are nonempty upper sets F with $A \sqcap B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a meet-semilattice.
- Ideals are nonempty lower sets F with $A \sqcup B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a join-semilattice.
- Free stars are upper sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \sqcup B \in F \Rightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a join-semilattice.
- Mixers are lower sets F not equal to $\mathcal{P}\mathfrak{Z}$ with $A \sqcap B \in F \Rightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever \mathfrak{Z} is a meet-semilattice.

5.6.5. The general diagram. Let \mathfrak{A} and \mathfrak{B} be two posets connected by an order reversing isomorphism $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$. We have commutative diagram on the figure 3 in the category **Set**:

$$\begin{array}{ccc}
 \mathcal{P}\mathfrak{A} & \begin{array}{c} \xrightarrow{\langle \theta \rangle^*} \\ \xleftarrow{\langle \theta^{-1} \rangle^*} \end{array} & \mathcal{P}\mathfrak{B} \\
 \uparrow \neg & & \uparrow \neg \\
 \mathcal{P}\mathfrak{A} & \begin{array}{c} \xrightarrow{\langle \theta \rangle^*} \\ \xleftarrow{\langle \theta^{-1} \rangle^*} \end{array} & \mathcal{P}\mathfrak{B}
 \end{array}$$

FIGURE 3

THEOREM 480. This diagram is commutative, every arrow of this diagram is an isomorphism, every cycle in this diagrams is an identity (therefore “parallel” arrows are mutually inverse).

PROOF. That every arrow is an isomorphism is obvious.

Show that $\langle \theta \rangle^* \neg X = \neg \langle \theta \rangle^* X$ for every set $X \in \mathcal{P}\mathfrak{A}$.

Really,

$$\begin{aligned}
 p \in \langle \theta \rangle^* \neg X &\Leftrightarrow \exists q \in \neg X : p = \theta q \Leftrightarrow \exists q \in \neg X : \theta^{-1} p = q \Leftrightarrow \theta^{-1} p \in \neg X \Leftrightarrow \\
 &\nexists q \in X : q = \theta^{-1} p \Leftrightarrow \nexists q \in X : \theta q = p \Leftrightarrow p \notin \langle \theta \rangle^* X \Leftrightarrow p \in \neg \langle \theta \rangle^* X.
 \end{aligned}$$

Thus the theorem follows from lemma 197. \square

This diagram can be restricted to filters, ideals, free stars, and mixers, see figure 4:

THEOREM 481. It is a restriction of the above diagram. Every arrow of this diagram is an isomorphism, every cycle in these diagrams is an identity. (To prove that, is an easy application of duality and the above lemma.)

$$\begin{array}{ccc}
\mathfrak{F}(\mathfrak{A}) & \begin{array}{c} \xrightarrow{\langle \theta \rangle^*} \\ \xleftarrow{\langle \theta^{-1} \rangle^*} \end{array} & \mathfrak{J}(\mathfrak{B}) \\
\uparrow \lrcorner & & \uparrow \lrcorner \\
\mathfrak{M}(\mathfrak{A}) & \begin{array}{c} \xrightarrow{\langle \theta \rangle^*} \\ \xleftarrow{\langle \theta^{-1} \rangle^*} \end{array} & \mathfrak{S}(\mathfrak{B})
\end{array}$$

FIGURE 4

5.6.6. Special diagrams. Here are two important special cases of the above diagram:

$$\begin{array}{ccc}
\mathfrak{F}(\mathfrak{A}) & \xrightarrow{\langle \text{dual} \rangle^*} & \mathfrak{J}(\text{dual } \mathfrak{A}) & & \mathfrak{F}(\mathfrak{A}) & \xrightarrow{\langle \neg \rangle^*} & \mathfrak{J}(\mathfrak{A}) \\
\uparrow \lrcorner & & \uparrow \lrcorner & \text{and} & \uparrow \lrcorner & & \uparrow \lrcorner \\
\mathfrak{M}(\mathfrak{A}) & \xrightarrow{\langle \text{dual} \rangle^*} & \mathfrak{S}(\text{dual } \mathfrak{A}) & & \mathfrak{M}(\mathfrak{A}) & \xrightarrow{\langle \neg \rangle^*} & \mathfrak{S}(\mathfrak{A})
\end{array} \quad (1)$$

(the second diagram is defined for a boolean lattice \mathfrak{A}).

5.6.7. Order of ideals, free stars, mixers. Define order of ideals, free stars, mixers in such a way that the above diagrams isomorphically preserve order of filters:

- $A \sqsubseteq B \Leftrightarrow A \supseteq B$ for filters and ideals;
- $A \sqsubseteq B \Leftrightarrow A \subseteq B$ for free stars and mixers.

5.6.8. Principal ideals, free stars, mixers.

DEFINITION 482. *Principal* ideal generated by an element a of poset \mathfrak{A} is $\downarrow a = \left\{ \begin{array}{l} x \in \mathfrak{A} \\ x \sqsubseteq a \end{array} \right\}$.

DEFINITION 483. An ideal is *principal* iff it is generated by some poset element.

DEFINITION 484. The *filtrator of ideals* on a given poset is the pair consisting of the set of ideals and the set of principal ideals.

The above poset isomorphism maps principal filters into principal ideals and thus is an isomorphism between the filtrator of filters on a poset and the filtrator of ideals on the dual poset.

EXERCISE 485. Define principal free stars and mixers, filtrators of free stars and mixers and isomorphisms of these with the filtrator of filters (these isomorphisms exist because the posets of free stars and mixers are isomorphic to the poset of filters).

OBVIOUS 486. The following filtrators are primary:

- filtrators of filters;
- filtrators of ideals;
- filtrators of free stars;
- filtrators of mixers.

5.6.8.1. *Principal free stars.*

PROPOSITION 487. An upper set $F \in \mathscr{P}\mathfrak{J}$ is a principal filter iff $\exists Z \in F \forall P \in F : Z \sqsubseteq P$.

PROOF.

\Rightarrow . Obvious.

\Leftarrow . Let $Z \in F$ and $\forall P \in F : Z \sqsubseteq P$. F is nonempty because $Z \in F$. It remains to prove that $Z \sqsubseteq P \Leftrightarrow P \in F$. The reverse implication follows from $\forall P \in F : Z \sqsubseteq P$. The direct implication follows from that F is an upper set.

□

LEMMA 488. If $S \in \mathcal{P}\mathfrak{Z}$ is not the complement of empty set and for every $T \in \mathcal{P}\mathfrak{Z}$

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset,$$

then S is a free star.

PROOF. Take $T = \{A, B\}$. Then $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in S) \Leftrightarrow A \in S \vee B \in S$. So S is a free star. □

PROPOSITION 489. A set $S \in \mathcal{P}\mathfrak{Z}$ is a principal free star iff S is not the complement of empty set and for every $T \in \mathcal{P}\mathfrak{Z}$

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset.$$

PROOF. Let $S = \overline{\langle \text{dual} \rangle^* F}$. We need to prove that F is a principal filter iff the above formula holds. Really, we have the following chain of equivalencies:

$$\begin{aligned} \forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) &\Leftrightarrow T \cap S \neq \emptyset; \\ \forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \notin \langle \text{dual} \rangle^* F) &\Leftrightarrow T \cap \overline{\langle \text{dual} \rangle^* F} \neq \emptyset; \\ \forall Z \in \text{dual } \mathfrak{Z} : (\forall X \in T : Z \sqsubseteq X \Rightarrow Z \notin F) &\Leftrightarrow T \cap \overline{F} \neq \emptyset; \\ \forall Z \in \text{dual } \mathfrak{Z} : (\forall X \in T : Z \sqsubseteq X \Rightarrow Z \notin F) &\Leftrightarrow T \not\subseteq F; \\ T \subseteq F &\Leftrightarrow \neg \forall Z \in \text{dual } \mathfrak{Z} : (Z \in F \Rightarrow \neg \forall X \in T : Z \sqsubseteq X); \\ T \subseteq F &\Leftrightarrow \neg \forall Z \in \text{dual } \mathfrak{Z} : (Z \notin F \vee \neg \forall X \in T : Z \sqsubseteq X); \\ T \subseteq F &\Leftrightarrow \exists Z \in \text{dual } \mathfrak{Z} : (Z \in F \wedge \forall X \in T : Z \sqsubseteq X); \\ T \subseteq F &\Leftrightarrow \exists Z \in F \forall X \in T : Z \sqsubseteq X; \end{aligned}$$

$\exists Z \in F \forall X \in T : Z \sqsubseteq X$ that is F is a principal filter (S is an upper set because by the lemma it is a free star; thus F is also an upper set). □

PROPOSITION 490. $S \in \mathcal{P}\mathfrak{Z}$ where \mathfrak{Z} is a poset is a principal free star iff all the following:

- 1°. The least element (if it exists) is not in S .
- 2°. $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathcal{P}\mathfrak{Z}$.
- 3°. S is an upper set.

PROOF.

\Rightarrow . 1° and 2° are obvious. S is an upper set because S is a free star.

\Leftarrow . We need to prove that

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset.$$

Let $X' \in T \cap S$. Then $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq X' \Rightarrow Z \in S$ because S is an upper set.

□

PROPOSITION 491. Let \mathfrak{Z} be a complete lattice. $S \in \mathcal{P}\mathfrak{Z}$ is a principal free star iff all the following:

- 1°. The least element is not in S .
- 2°. $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathcal{P}\mathfrak{Z}$.
- 3°. S is an upper set.

PROOF.

\Rightarrow . We need to prove only $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. Let $\bigsqcup T \in S$. Because S is an upper set, we have $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq \bigsqcup T \Rightarrow Z \in S$ for every $Z \in \mathfrak{Z}$; from which we conclude $T \cap S \neq \emptyset$.

\Leftarrow . We need to prove only $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$. Really, if $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S)$ then $\bigsqcup T \in S$ and thus $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. □

PROPOSITION 492. Let \mathfrak{Z} be a complete lattice. $S \in \mathcal{P}\mathfrak{Z}$ is a principal free star iff the least element is not in S and for every $T \in \mathcal{P}\mathfrak{Z}$

$$\bigsqcup T \in S \Leftrightarrow T \cap S \neq \emptyset.$$

PROOF.

\Rightarrow . We need to prove only $\bigsqcup T \in S \Leftarrow T \cap S \neq \emptyset$ what follows from that S is an upper set.

\Leftarrow . We need to prove only that S is an upper set. To prove this we can use the fact that S is a free star. □

EXERCISE 493. Write down similar formulas for mixers.

5.6.9. Starrish posets.

DEFINITION 494. I will call a poset *starrish* when the full star $\star a$ is a free star for every element a of this poset.

PROPOSITION 495. Every distributive lattice is starrish.

PROOF. Let \mathfrak{A} be a distributive lattice, $a \in \mathfrak{A}$. Obviously $\perp \notin \star a$ (if \perp exists); obviously $\star a$ is an upper set. If $x \sqcup y \in \star a$, then $(x \sqcup y) \sqcap a$ is non-least that is $(x \sqcap a) \sqcup (y \sqcap a)$ is non-least what is equivalent to $x \sqcap a$ or $y \sqcap a$ being non-least that is $x \in \star a \vee y \in \star a$. □

THEOREM 496. If \mathfrak{A} is a starrish join-semilattice lattice then

$$\text{atoms}(a \sqcup b) = \text{atoms } a \cup \text{atoms } b$$

for every $a, b \in \mathfrak{A}$.

PROOF. For every atom c we have:

$$\begin{aligned} c \in \text{atoms}(a \sqcup b) &\Leftrightarrow \\ c \not\leq a \sqcup b &\Leftrightarrow \\ a \sqcup b \in \star c &\Leftrightarrow \\ a \in \star c \vee b \in \star c &\Leftrightarrow \\ c \not\leq a \vee c \not\leq b &\Leftrightarrow \\ c \in \text{atoms } a \vee c \in \text{atoms } b. & \end{aligned}$$

□

5.6.9.1. Completely starrish posets.

DEFINITION 497. I will call a poset *completely starrish* when the full star $\star a$ is a principal free star for every element a of this poset.

OBVIOUS 498. Every completely starrish poset is starrish.

PROPOSITION 499. Every complete join infinite distributive lattice is completely starrish.

PROOF. Let \mathfrak{A} be a join infinite distributive lattice, $a \in \mathfrak{A}$. Obviously $\perp \notin \star a$ (if \perp exists); obviously $\star a$ is an upper set. If $\bigsqcup T \in \star a$, then $(\bigsqcup T) \sqcap a$ is non-least that is $\bigsqcup \langle a \sqcap \rangle^* T$ is non-least what is equivalent to $a \sqcap x$ being non-least for some $x \in T$ that is $x \in \star a$. \square

THEOREM 500. If \mathfrak{A} is a completely starrish complete lattice then

$$\text{atoms} \bigsqcup T = \bigcup \langle \text{atoms} \rangle^* T.$$

for every $T \in \mathcal{P}\mathfrak{A}$.

PROOF. For every atom c we have:

$$\begin{aligned} c \in \text{atoms} \bigsqcup T &\Leftrightarrow c \not\prec \bigsqcup T \Leftrightarrow \bigsqcup T \in \star c \Leftrightarrow \exists X \in T : X \in \star c \Leftrightarrow \\ &\Leftrightarrow \exists X \in T : X \not\prec c \Leftrightarrow \exists X \in T : c \in \text{atoms} X \Leftrightarrow c \in \bigcup \langle \text{atoms} \rangle^* T. \end{aligned}$$

\square

5.7. Basic properties of filters

PROPOSITION 501. $\text{up } \mathcal{A} = \mathcal{A}$ for every filter \mathcal{A} (provided that we equate elements of the base poset \mathfrak{J} with corresponding principal filters).

PROOF. $A \in \text{up } \mathcal{A} \Leftrightarrow A \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow A \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow A \subseteq \mathcal{A} \Leftrightarrow A \in \mathcal{A}$. \square

5.7.1. Minimal and maximal filters.

OBVIOUS 502. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator.
- 3°. $\perp^{\mathfrak{A}}$ (equal to the principal filter for the least element of \mathfrak{J} if it exists) defined by the formula $\text{up } \perp^{\mathfrak{A}} = \mathfrak{J}$ is the least element of \mathfrak{A} .

PROPOSITION 503. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator with greatest element.
- 3°. $\top^{\mathfrak{A}}$ defined by the formula $\text{up } \top^{\mathfrak{A}} = \{\top^{\mathfrak{J}}\}$ is the greatest element of \mathfrak{A} .

PROOF. Take into account that filters are nonempty. \square

5.7.2. Alignment.

DEFINITION 504. I call *down-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{J})$ that \mathfrak{A} and \mathfrak{J} have common least element. (Let's denote it \perp .)

DEFINITION 505. I call *up-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{J})$ that \mathfrak{A} and \mathfrak{J} have common greatest element. (Let's denote it \top .)

OBVIOUS 506.

- 1°. If \mathfrak{J} has least element, the primary filtrator is down-aligned.
- 2°. If \mathfrak{J} has greatest element, the primary filtrator is up-aligned.

COROLLARY 507. Every powerset filtrator is both up and down-aligned.

We can also define (without requirement of having least and greatest elements, but coinciding with the above definitions if least/greatest elements are present):

DEFINITION 508. I call *weakly down-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{J})$ that whenever $\perp^{\mathfrak{J}}$ exists, $\perp^{\mathfrak{A}}$ also exists and $\perp^{\mathfrak{J}} = \perp^{\mathfrak{A}}$.

DEFINITION 509. I call *weakly up-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{J})$ that whenever $\top^{\mathfrak{J}}$ exists, $\top^{\mathfrak{A}}$ also exists and $\top^{\mathfrak{J}} = \top^{\mathfrak{A}}$.

OBVIOUS 510.

- 1°. Every up-aligned filtrator is weakly up-aligned.
- 2°. Every down-aligned filtrator is weakly down-aligned.

OBVIOUS 511.

- 1°. Every primary filtrator is weakly down-aligned.
- 2°. Every primary filtrator is weakly up-aligned.

5.8. More advanced properties of filters

5.8.1. Formulas for Meets and Joins of Filters.

LEMMA 512. If f is an order embedding from a poset \mathfrak{A} to a complete lattice \mathfrak{B} and $S \in \mathcal{P}\mathfrak{A}$ and there exists such $\mathcal{F} \in \mathfrak{A}$ that $f\mathcal{F} = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$.

PROOF. f is an order isomorphism from \mathfrak{A} to $\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$.

Consequently, $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S \in \mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$ and $\bigsqcup^{\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}} \langle f \rangle^* S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$.

$f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}} \langle f \rangle^* S$ because f is an order isomorphism.

Combining, $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$. \square

COROLLARY 513. If \mathfrak{B} is a complete lattice and \mathfrak{A} is its subset and $S \in \mathcal{P}\mathfrak{A}$ and $\bigsqcup^{\mathfrak{B}} S \in \mathfrak{A}$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} S$.

EXERCISE 514. The below theorem does not work for $S = \emptyset$. Formulate the general case.

THEOREM 515.

- 1°. If \mathfrak{Z} is a meet-semilattice, then $\bigsqcup^{\mathfrak{Z}(3)} S$ exists and $\bigsqcup^{\mathfrak{Z}(3)} S = \bigcap S$ for every bounded above set $S \in \mathcal{P}\mathfrak{Z}(3) \setminus \{\emptyset\}$.
- 2°. If \mathfrak{Z} is a join-semilattice, then $\prod^{\mathfrak{Z}(3)} S$ exists and $\prod^{\mathfrak{Z}(3)} S = \bigcap S$ for every bounded below set $S \in \mathcal{P}\mathfrak{Z}(3) \setminus \{\emptyset\}$.

PROOF.

1°. Taking into account the lemma, it is enough to prove that $\bigcap S$ is a filter. Let's prove that $\bigcap S$ is nonempty. There is an upper bound \mathcal{T} of S . Take arbitrary $T \in \mathcal{T}$. We have $T \in \mathcal{X}$ for every $\mathcal{X} \in S$. Thus S is nonempty.

For every $A, B \in \mathfrak{Z}$ we have:

$$A, B \in \bigcap S \Leftrightarrow \forall P \in S : A, B \in P \Leftrightarrow \forall P \in S : A \sqcap B \in P \Leftrightarrow A \sqcap B \in \bigcap S.$$

So $\bigcap S$ is a filter.

2°. By duality. \square

THEOREM 516.

- 1°. If \mathfrak{Z} is a meet-semilattice with greatest element, then $\bigsqcup^{\mathfrak{Z}(3)} S$ exists and $\bigsqcup^{\mathfrak{Z}(3)} S = \bigcap S$ for every $S \in \mathcal{P}\mathfrak{Z}(3) \setminus \{\emptyset\}$.
- 2°. If \mathfrak{Z} is a join-semilattice with least element, then $\prod^{\mathfrak{Z}(3)} S$ exists and $\prod^{\mathfrak{Z}(3)} S = \bigcap S$ for every $S \in \mathcal{P}\mathfrak{Z}(3) \setminus \{\emptyset\}$.
- 3°. If \mathfrak{Z} is a join-semilattice with least element, then $\bigsqcup^{\mathfrak{G}(3)} S$ exists and $\bigsqcup^{\mathfrak{G}(3)} S = \bigcup S$ for every $S \in \mathcal{P}\mathfrak{G}(3)$.
- 4°. If \mathfrak{Z} is a meet-semilattice with greatest element, then $\prod^{\mathfrak{M}(3)} S$ exists and $\prod^{\mathfrak{M}(3)} S = \bigcup S$ for every $S \in \mathcal{P}\mathfrak{M}(3)$.

PROOF.

- 1°. From the previous theorem.
 2°. By duality.
 3°. Taking into account the lemma, it is enough to prove that $\bigcup S$ is a free star. $\bigcup S$ is not the complement of empty set because $\perp \notin \bigcup S$. For every $A, B \in \mathfrak{F}$ we have:

$$A \in \bigcup S \vee B \in \bigcup S \Leftrightarrow \exists P \in S : (A \in P \vee B \in P) \Leftrightarrow \\ \exists P \in S : A \sqcup B \in P \Leftrightarrow A \sqcup B \in \bigcup S.$$

- 4°. By duality. □

COROLLARY 517. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a meet-semilattice with greatest element \top .
 3°. $\bigsqcup^{\mathfrak{A}} S$ exists and $\text{up} \bigsqcup^{\mathfrak{A}} S = \bigcap \langle \text{up} \rangle^* S$ for every $S \in \mathcal{P}\mathfrak{A} \setminus \{\emptyset\}$.

PROOF.

- 1° \Rightarrow 2°. Obvious.
 2° \Rightarrow 3°. By the theorem. □

COROLLARY 518. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a meet-semilattice with greatest element \top .
 3°. \mathfrak{A} is a complete lattice.

We will denote meets and joins on the lattice of filters just as \sqcap and \sqcup .

PROPOSITION 519. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over an ideal base.
 3°. \mathfrak{A} is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$

$$\text{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{B}) = \text{up} \mathcal{A} \sqcap \text{up} \mathcal{B}.$$

PROOF.

- 1° \Rightarrow 2°. Obvious.
 2° \Rightarrow 3°. Taking into account the lemma it is enough to prove that $R = \text{up} \mathcal{A} \sqcap \text{up} \mathcal{B}$ is a filter.

R is nonempty because we can take $X \in \text{up} \mathcal{A}$ and $Y \in \text{up} \mathcal{B}$ and $Z \sqsupseteq X \wedge Y \sqsupseteq Y$ and then $R \ni Z$.

Let $A, B \in R$. Then $A, B \in \text{up} \mathcal{A}$; so exists $C \in \text{up} \mathcal{A}$ such that $C \sqsubseteq A \wedge C \sqsubseteq B$. Analogously exists $D \in \text{up} \mathcal{B}$ such that $D \sqsubseteq A \wedge D \sqsubseteq B$. Take $E \sqsupseteq C \wedge D \sqsupseteq D$. Then $E \in \text{up} \mathcal{A}$ and $E \in \text{up} \mathcal{B}$; $E \in R$ and $E \sqsubseteq A \wedge E \sqsubseteq B$. So R is a filter base.

That R is an upper set is obvious. □

THEOREM 520. Let \mathfrak{F} be a distributive lattice. Then

- 1°. $\prod^{\mathfrak{F}(\mathfrak{F})} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{F}} \dots \sqcap^{\mathfrak{F}} K_n}{K_i \in \bigcup S \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathcal{P}\mathfrak{F}(\mathfrak{F}) \setminus \{\emptyset\}$;
 2°. $\bigsqcup^{\mathfrak{F}(\mathfrak{F})} S = \left\{ \frac{K_0 \sqcup^{\mathfrak{F}} \dots \sqcup^{\mathfrak{F}} K_n}{K_i \in \bigcup S \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathcal{P}\mathfrak{F}(\mathfrak{F}) \setminus \{\emptyset\}$.

PROOF. We will prove only the first, as the second is dual.

Let's denote the right part of the equality to be proven as R . First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^3 \dots \sqcap^3 X_k$, $B = Y_0 \sqcap^3 \dots \sqcap^3 Y_l$ where $X_i, Y_j \in \bigcup S$.

So

$$A \sqcap^3 B = X_0 \sqcap^3 \dots \sqcap^3 X_k \sqcap^3 Y_0 \sqcap^3 \dots \sqcap^3 Y_l \in R.$$

Let element $C \sqsupseteq A \in R$. Consequently (distributivity used)

$$C = C \sqcup^3 A = (C \sqcup^3 X_0) \sqcap^3 \dots \sqcap^3 (C \sqcup^3 X_k).$$

$X_i \in P_i$ for some $P_i \in S$; $C \sqcup^3 X_i \in P_i$; $C \sqcup^3 X_i \in \bigcup S$; consequently $C \in R$.

We have proved that that R is a filter base and an upper set. So R is a filter.

Let $\mathcal{A} \in S$. Then $\mathcal{A} \subseteq \bigcup S$;

$$R \supseteq \left\{ \frac{K_0 \sqcap^3 \dots \sqcap^3 K_n}{K_i \in \mathcal{A} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \mathcal{A}.$$

Consequently $\mathcal{A} \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{A}$ and $\forall \mathcal{A} \in S : \mathcal{A} \sqsupseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S : \mathcal{A} \subseteq \mathcal{B}$; $\mathcal{B} \supseteq \bigcup S$. Thus $\mathcal{B} \supseteq T$ for every finite set $T \subseteq \bigcup S$. Consequently $\text{up } \mathcal{B} \ni \prod^3 T$. Thus $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

Comparing we get $\prod^{\mathfrak{F}(3)} S = R$.

□

COROLLARY 521. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a distributive lattice.

3°. $\text{up } \prod^{\mathfrak{A}} S = \left\{ \frac{K_0 \sqcap^3 \dots \sqcap^3 K_n}{K_i \in \bigcup_{(\text{up})^* S} \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathcal{P}\mathfrak{A} \setminus \{\emptyset\}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. By the theorem.

□

THEOREM 522. Let \mathfrak{J} be a distributive lattice. Then:

1°. $\mathcal{F}_0 \sqcap^{\mathfrak{F}(3)} \dots \sqcap^{\mathfrak{F}(3)} \mathcal{F}_m = \left\{ \frac{K_0 \sqcap^3 \dots \sqcap^3 K_m}{K_i \in \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ for any $\mathcal{F}_0, \dots, \mathcal{F}_m \in \mathfrak{F}(\mathfrak{J})$;

2°. $\mathcal{F}_0 \sqcup^{\mathfrak{J}(3)} \dots \sqcup^{\mathfrak{J}(3)} \mathcal{F}_m = \left\{ \frac{K_0 \sqcup^3 \dots \sqcup^3 K_m}{K_i \in \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ for any $\mathcal{F}_0, \dots, \mathcal{F}_m \in \mathfrak{J}(\mathfrak{J})$.

PROOF. We will prove only the first as the second is dual.

Let's denote the right part of the equality to be proven as R . First we will prove that R is a filter. Obviously R is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^3 \dots \sqcap^3 X_m$, $B = Y_0 \sqcap^3 \dots \sqcap^3 Y_m$ where $X_i, Y_i \in \mathcal{F}_i$.

$$A \sqcap^3 B = (X_0 \sqcap^3 Y_0) \sqcap^3 \dots \sqcap^3 (X_m \sqcap^3 Y_m),$$

consequently $A \sqcap^3 B \in R$.

Let filter $C \sqsupseteq A \in R$

$$C = A \sqcup^3 C = (X_0 \sqcup^3 C) \sqcap^3 \dots \sqcap^3 (X_m \sqcup^3 C) \in R.$$

So R is a filter.

Let $P_i \in \mathcal{F}_i$. Then $P_i \in R$ because $P_i = (P_i \sqcup^3 P_0) \sqcap^3 \dots \sqcap^3 (P_i \sqcup^3 P_m)$. So $\mathcal{F}_i \subseteq R$; $\mathcal{F}_i \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{A}$ and $\forall i \in \{0, \dots, m\} : \mathcal{F}_i \sqsupseteq \mathcal{B}$. Then $\forall i \in \{0, \dots, m\} : \mathcal{F}_i \subseteq \mathcal{B}$.

Let $L_i \in \mathcal{B}$ for every $L_i \in \mathcal{F}_i$. $L_0 \sqcap^3 \dots \sqcap^3 L_m \in \mathcal{B}$. So $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

So $\mathcal{F}_0 \sqcap^{\mathfrak{F}(3)} \dots \sqcap^{\mathfrak{F}(3)} \mathcal{F}_m = R$.

□

COROLLARY 523. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
- 3°. $\text{up}(\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_m) = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \text{up } \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ for any $\mathcal{F}_0, \dots, \mathcal{F}_m \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. By the theorem. □

More general case of semilattices follows:

THEOREM 524.

- 1°. $\prod^{\mathfrak{Z}(\mathfrak{Z})} S = \bigcup \left\{ \frac{\uparrow(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n)}{K_i \in \bigcup S \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathcal{P}\mathfrak{Z}(\mathfrak{Z}) \setminus \{\emptyset\}$ if \mathfrak{Z} is a meet-semilattice;
- 2°. $\sqcup^{\mathfrak{J}(\mathfrak{Z})} S = \bigcup \left\{ \frac{\uparrow(K_0 \sqcup^{\mathfrak{Z}} \dots \sqcup^{\mathfrak{Z}} K_n)}{K_i \in \bigcup S \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathcal{P}\mathfrak{J}(\mathfrak{Z}) \setminus \{\emptyset\}$ if \mathfrak{Z} is a join-semilattice.

PROOF. We will prove only the first as the second is dual.

It follows from the fact that

$$\prod^{\mathfrak{Z}(\mathfrak{Z})} S = \prod^{\mathfrak{Z}(\mathfrak{Z})} \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$$

and that $\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ is a filter base. □

COROLLARY 525. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
- 3°. $\text{up} \prod S = \bigcup \left\{ \frac{\text{up}(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n)}{K_i \in \bigcup (\text{up})^* S \text{ where } i=0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for every $S \in \mathcal{P}\mathfrak{A} \setminus \{\emptyset\}$.

THEOREM 526.

- 1°. $\mathcal{F}_0 \sqcap^{\mathfrak{Z}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{Z}(\mathfrak{Z})} \mathcal{F}_m = \bigcup \left\{ \frac{\uparrow(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ for $S \in \mathcal{P}\mathfrak{Z}(\mathfrak{Z}) \setminus \{\emptyset\}$ if \mathfrak{Z} is a meet-semilattice;
- 2°. $\mathcal{F}_0 \sqcup^{\mathfrak{J}(\mathfrak{Z})} \dots \sqcup^{\mathfrak{J}(\mathfrak{Z})} \mathcal{F}_m = \bigcup \left\{ \frac{\uparrow(K_0 \sqcup^{\mathfrak{Z}} \dots \sqcup^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ for $S \in \mathcal{P}\mathfrak{J}(\mathfrak{Z}) \setminus \{\emptyset\}$ if \mathfrak{Z} is a join-semilattice.

PROOF. We will prove only the first as the second is dual.

It follows from the fact that

$$\mathcal{F}_0 \sqcap^{\mathfrak{Z}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{Z}(\mathfrak{Z})} \mathcal{F}_m = \prod^{\mathfrak{Z}(\mathfrak{Z})} \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \dots, m} \right\}$$

and that $\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ is a filter base. □

COROLLARY 527. $\text{up}(\mathcal{F}_0 \sqcap^{\mathfrak{Z}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{Z}(\mathfrak{Z})} \mathcal{F}_m) = \bigcup \left\{ \frac{\text{up}(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0, \dots, m} \right\}$ if \mathfrak{Z} is a meet-semilattice.

LEMMA 528. If $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator and \mathfrak{Z} is a meet-semilattice and an ideal base, then \mathfrak{A} is a lattice.

PROOF. It is a join-semilattice by proposition 519. It is a meet-semilattice by theorem 524. □

COROLLARY 529. If $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator and \mathfrak{J} is a lattice, then \mathfrak{A} is a lattice.

5.8.2. Distributivity of the Lattice of Filters.

THEOREM 530. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a distributive lattice.
- 3°. $\mathcal{A} \sqcup^{\mathfrak{A}} \prod^{\mathfrak{A}} S = \prod^{\mathfrak{A}} \langle \mathcal{A} \sqcup^{\mathfrak{A}} \rangle^* S$ for $S \in \mathcal{P}\mathfrak{A}$ and $\mathcal{A} \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Taking into account the previous section, we have:

$$\begin{aligned}
& \text{up} \left(\mathcal{A} \sqcup^{\mathfrak{A}} \prod^{\mathfrak{A}} S \right) = \\
& \text{up} \mathcal{A} \cap \text{up} \prod^{\mathfrak{A}} S = \\
& \text{up} \mathcal{A} \cap \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_i \in \bigcup \langle \text{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n \in \text{up} \mathcal{A}, K_i \in \bigcup \langle \text{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_i \in \text{up} \mathcal{A}, K_i \in \bigcup \langle \text{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_i \in \text{up} \mathcal{A} \cap \bigcup \langle \text{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_i \in \bigcup \langle \text{up} \mathcal{A} \cap \rangle^* \langle \text{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_i \in \bigcup \left\{ \frac{\text{up} \mathcal{A} \cap \text{up} \mathcal{X}}{\mathcal{X} \in S} \right\} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \left\{ \frac{K_0 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} K_n}{K_i \in \bigcup \left\{ \frac{\text{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{X})}{\mathcal{X} \in S} \right\} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\
& \text{up} \prod^{\mathfrak{A}} \left\{ \frac{\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{X}}{\mathcal{X} \in S} \right\} = \\
& \text{up} \prod^{\mathfrak{A}} \langle \mathcal{A} \sqcup^{\mathfrak{A}} \rangle^* S.
\end{aligned}$$

□

COROLLARY 531. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a distributive lattice which is an ideal base.
- 3°. \mathfrak{A} is a distributive and co-brouwerian lattice.

COROLLARY 532. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a distributive lattice with greatest element.

3°. \mathfrak{A} is a co-frame.

The below theorem uses the notation and results from section 3.9.

THEOREM 533. If \mathfrak{A} is a co-frame and L is a bounded distributive lattice which, then $\text{Join}(L, \mathfrak{A})$ is also a co-frame.

PROOF. Let $F = \uparrow \circ \sqcap : \text{Up}(\mathfrak{A}) \rightarrow \text{Up}(\mathfrak{A})$; F is a co-nucleus by above.

Since $\text{Up}(\mathfrak{A}) \cong \mathbf{Pos}(\mathfrak{A}, 2)$ by proposition 340, we may regard F as a co-nucleus on $\mathbf{Pos}(\mathfrak{A}, 2)$.

$\text{Join}(L, \mathfrak{A}) \cong \text{Join}(L, \text{Fix}(F))$ by corollary 343.

$\text{Join}(L, \text{Fix}(F)) \cong \text{Fix}(\text{Join}(L, F))$ by lemma 351.

By corollary 350 the function $\text{Join}(L, F)$ is a co-nucleus on $\text{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2))$.

$$\begin{aligned} \text{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2)) &\cong \quad (\text{by lemma 353}) \\ \mathbf{Pos}(\mathfrak{A}, \text{Join}(L, 2)) &\cong \\ \mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X)). & \end{aligned}$$

$\mathfrak{F}(X)$ is a co-frame by corollary 532. Thus $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$ is a co-frame by lemma 353.

Thus $\text{Join}(L, \mathfrak{A})$ is isomorphic to a poset of fixed points of a co-nucleus on the co-frame $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$. By lemma 335 $\text{Join}(L, \mathfrak{A})$ is also a co-frame. \square

5.9. Misc filtrator properties

THEOREM 534. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is a filtered filtrator.
- 4°. $(\mathfrak{A}, \mathfrak{F})$ is a filtrator with join-closed core.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. The formula $\forall a, b \in \mathfrak{A} : (\text{up } a \supseteq \text{up } b \Rightarrow a \sqsubseteq b)$ is obvious for primary filtrators.

3° \Rightarrow 4°. Let $(\mathfrak{A}, \mathfrak{F})$ be a filtered filtrator. Let $S \in \mathcal{P}\mathfrak{F}$ and $\bigsqcup^3 S$ be defined. We need to prove $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^3 S$. That $\bigsqcup^3 S$ is an upper bound for S is obvious. Let $a \in \mathfrak{A}$ be an upper bound for S . It's enough to prove that $\bigsqcup^3 S \sqsubseteq a$. Really,

$$c \in \text{up } a \Rightarrow c \sqsubseteq a \Rightarrow \forall x \in S : c \sqsupseteq x \Rightarrow c \sqsupseteq \bigsqcup^3 S \Rightarrow c \in \text{up } \bigsqcup^3 S;$$

so $\text{up } a \subseteq \text{up } \bigsqcup^3 S$ and thus $a \sqsupseteq \bigsqcup^3 S$ because it is filtered. \square

5.10. Characterization of Binarily Meet-Closed Filtrators

THEOREM 535. The following are equivalent for a filtrator $(\mathfrak{A}, \mathfrak{F})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A} : \text{up } a \neq \emptyset$:

- 1°. The filtrator is with binarily meet-closed core.
- 2°. $\text{up } a$ is a filter for every $a \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Let $X, Y \in \text{up } a$. Then $X \sqcap^3 Y = X \sqcap^{\mathfrak{A}} Y \sqsupseteq a$. That $\text{up } a$ is an upper set is obvious. So taking into account that $\text{up } a \neq \emptyset$, $\text{up } a$ is a filter.

$2^\circ \Rightarrow 1^\circ$. It is enough to prove that $a \sqsubseteq A, B \Rightarrow a \sqsubseteq A \sqcap^3 B$ for every $A, B \in \mathfrak{A}$.
Really:

$$a \sqsubseteq A, B \Rightarrow A, B \in \text{up } a \Rightarrow A \sqcap^3 B \in \text{up } a \Rightarrow a \sqsubseteq A \sqcap^3 B.$$

□

COROLLARY 536. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a meet semilattice.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is with binarily meet-closed core.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. From the theorem.

□

5.10.1. Separability of Core for Primary Filtrators.

THEOREM 537. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a meet semilattice with least element.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is with separable core.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. Let $\mathcal{A} \simeq^{\mathfrak{A}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$.

$$\text{up}(\mathcal{A} \sqcap^{\mathfrak{A}} \mathcal{B}) = \bigcup \left\{ \frac{\text{up}(A \sqcap^3 B)}{A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\}.$$

So

$$\begin{aligned} \perp \in \text{up}(\mathcal{A} \sqcap^{\mathfrak{A}} \mathcal{B}) &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : \perp \in \text{up}(A \sqcap^3 B) &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : A \sqcap^3 B = \perp &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : A \sqcap^{\mathfrak{A}} B = \perp^{\mathfrak{A}} & \end{aligned}$$

(used proposition 536).

□

5.11. Core Part

Let $(\mathfrak{A}, \mathfrak{F})$ be a filtrator.

DEFINITION 538. The *core part* of an element $a \in \mathfrak{A}$ is $\text{Cor } a = \prod^3 \text{up } a$.

DEFINITION 539. The *dual core part* of an element $a \in \mathfrak{A}$ is $\text{Cor}' a = \prod^3 \text{down } a$.

OBVIOUS 540. Cor' is dual of Cor .

OBVIOUS 541. $\text{Cor } a = \text{Cor}' a = a$ for every element a of the core of a filtrator.

THEOREM 542. The following is an implications tuple:

- 1°. a is a filter on a set.
- 2°. a is a filter on a complete lattice.
- 3°. a is an element of a filtered filtrator and $\text{Cor } a$ exists.
- 4°. $\text{Cor } a \sqsubseteq a$ and $\text{Cor } a \in \text{down } a$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. Theorem 534.

$3^\circ \Rightarrow 4^\circ$. $\text{Cor } a = \prod^3 \text{ up } a \subseteq \prod^{\mathfrak{A}} \text{ up } a = a$. Then obviously $\text{Cor } a \in \text{down } a$.

□

THEOREM 543. The following is an implications tuple:

1° . a is a filter on a set.

2° . a is a filter on a complete lattice.

3° . a is an element a of a filtrator with join-closed core and $\text{Cor}' a$ exists.

4° . $\text{Cor}' a \subseteq a$ and $\text{Cor}' a \in \text{down } a$ and $\text{Cor}' a = \max \text{down } a$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. It is join closed by 534. $\text{Cor}' a$ exists because our filtrator is join-closed.

$3^\circ \Rightarrow 4^\circ$. $\text{Cor}' a = \prod^3 \text{ down } a = \prod^{\mathfrak{A}} \text{ down } a \subseteq a$. Now $\text{Cor}' a \in \text{down } a$ is obvious.
Thus $\text{Cor}' a = \max \text{down } a$.

□

PROPOSITION 544. $\text{Cor}' a \subseteq \text{Cor } a$ whenever both $\text{Cor } a$ and $\text{Cor}' a$ exist for any element a of a filtrator with join-closed core.

PROOF. $\text{Cor } a = \prod^3 \text{ up } a \supseteq \text{Cor}' a$ because $\forall A \in \text{up } a : \text{Cor}' a \subseteq A$.

□

THEOREM 545. The following is an implications tuple:

1° . a is a filter on a set.

2° . a is a filter on a complete lattice.

3° . a is an element of a filtered filtrator and both $\text{Cor } a$ and $\text{Cor}' a$ exist.

4° . $\text{Cor}' a = \text{Cor } a$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. By theorem 534.

$3^\circ \Rightarrow 4^\circ$. It is with join-closed core because it is filtered. So $\text{Cor}' a \subseteq \text{Cor } a$. $\text{Cor } a \in \text{down } a$. So $\text{Cor } a \subseteq \prod^3 \text{ down } a = \text{Cor}' a$.

□

COROLLARY 546. $\text{Cor}' a = \text{Cor } a = \bigcap a$ for every filter a on a set.

5.12. Intersection and Joining with an Element of the Core

DEFINITION 547. A filtrator $(\mathfrak{A}; \mathfrak{B})$ is with *correct intersection* iff $\forall a, b \in \mathfrak{B} : (a \not\leq^{\mathfrak{B}} b \Leftrightarrow a \not\leq^{\mathfrak{A}} b)$.

DEFINITION 548. A filtrator $(\mathfrak{A}; \mathfrak{B})$ is with *correct joining* iff $\forall a, b \in \mathfrak{B} : (a \equiv^{\mathfrak{B}} b \Leftrightarrow a \equiv^{\mathfrak{A}} b)$.

PROPOSITION 549. The following is an implications tuple:

1° . $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.

2° . $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a meet-semilattice.

3° . $(\mathfrak{A}, \mathfrak{B})$ is with binarily meet-closed core, weakly down-aligned filtrator, and \mathfrak{B} is a meet-semilattice.

4° . $(\mathfrak{A}, \mathfrak{B})$ is with correct intersection.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. Corollary 536.

$3^\circ \Rightarrow 4^\circ$. $a \not\prec^{\mathfrak{Z}} b \Rightarrow a \not\prec^{\mathfrak{A}} b$ is obvious. Let $a \prec^{\mathfrak{Z}} b$. Then $a \sqcap^{\mathfrak{Z}} b$ exists; so $\perp^{\mathfrak{Z}}$ exists and $a \sqcap^{\mathfrak{Z}} b = \perp^{\mathfrak{Z}}$ (as otherwise $a \sqcap^{\mathfrak{Z}} b$ is non-least). So $\perp^{\mathfrak{Z}} = \perp^{\mathfrak{A}}$. We have $a \sqcap^{\mathfrak{A}} b = \perp^{\mathfrak{A}}$. Thus $a \prec^{\mathfrak{A}} b$.

□

PROPOSITION 550. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a join-semilattice.
- 3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily join-closed core, weakly up-aligned filtrator, and \mathfrak{Z} is a join-semilattice.
- 4°. $(\mathfrak{A}, \mathfrak{Z})$ is with correct joining.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. Corollary 534.

$3^\circ \Rightarrow 4^\circ$. Dual of the previous proposition.

□

LEMMA 551. For a filtrator $(\mathfrak{A}, \mathfrak{Z})$ where \mathfrak{Z} is a boolean lattice, for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$:

- 1°. $B \prec^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsupseteq \mathcal{A}$ if it is with separable core and with correct intersection;
- 2°. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ if it is with co-separable core and with correct joining.

PROOF. We will prove only the first as the second is dual.

$$\begin{aligned}
 B \prec^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow \\
 \exists A \in \text{up } \mathcal{A} : B \prec^{\mathfrak{A}} A &\Leftrightarrow \\
 \exists A \in \text{up } \mathcal{A} : B \prec^{\mathfrak{Z}} A &\Leftrightarrow \\
 \exists A \in \text{up } \mathcal{A} : \overline{B} \sqsupseteq A &\Leftrightarrow \\
 \overline{B} \in \text{up } \mathcal{A} &\Leftrightarrow \\
 \overline{B} \sqsupseteq \mathcal{A}. &
 \end{aligned}$$

□

5.13. Stars of Elements of Filtrators

DEFINITION 552. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. *Core star* of an element a of the filtrator is

$$\partial a = \left\{ \frac{x \in \mathfrak{Z}}{x \not\prec^{\mathfrak{A}} a} \right\}.$$

PROPOSITION 553. $\text{up } a \subseteq \partial a$ for any non-least element a of a filtrator.

PROOF. For any element $X \in \mathfrak{Z}$

$$X \in \text{up } a \Rightarrow a \sqsubseteq X \wedge a \sqsubseteq a \Rightarrow X \not\prec^{\mathfrak{A}} a \Rightarrow X \in \partial a.$$

□

THEOREM 554. Let $(\mathfrak{A}, \mathfrak{Z})$ be a distributive lattice filtrator with least element and binarily join-closed core which is a join-semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF. For every $A, B \in \mathfrak{J}$

$$\begin{aligned}
A \sqcup^{\mathfrak{J}} B \in \partial a &\Leftrightarrow \\
A \sqcup^{\mathfrak{A}} B \in \partial a &\Leftrightarrow \\
(A \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} a \neq \perp^{\mathfrak{A}} &\Leftrightarrow \\
(A \sqcap^{\mathfrak{A}} a) \sqcup^{\mathfrak{A}} (B \sqcap^{\mathfrak{A}} a) \neq \perp^{\mathfrak{A}} &\Leftrightarrow \\
A \sqcap^{\mathfrak{A}} a \neq \perp^{\mathfrak{A}} \vee B \sqcap^{\mathfrak{A}} a \neq \perp^{\mathfrak{A}} &\Leftrightarrow \\
A \in \partial a \vee B \in \partial a. &
\end{aligned}$$

That ∂a doesn't contain $\perp^{\mathfrak{A}}$ is obvious. \square

DEFINITION 555. I call a filtrator *star-separable* when its core is a separation subset of its base.

5.14. Atomic Elements of a Filtrator

See [4, 9] for more detailed treatment of ultrafilters and prime filters.

PROPOSITION 556. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice with greatest element.
- 3°. \mathfrak{A} is a complete lattice.
- 4°. $\text{atoms} \prod S = \prod (\text{atoms})^* S$ for every $S \in \mathcal{P}\mathfrak{A}$.
- 5°. $\text{atoms}(a \sqcap b) = \text{atoms } a \cap \text{atoms } b$ for $a, b \in \mathfrak{A}$.

PROOF.

- 1° \Rightarrow 2°. Obvious.
- 2° \Rightarrow 3°. Corollary 518.
- 3° \Rightarrow 4°. Theorem 108.
- 4° \Rightarrow 5°. Obvious. \square

PROPOSITION 557. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a distributive lattice which is and ideal base.
- 3°. \mathfrak{A} is a starrish join-semilattice.
- 4°. $\text{atoms}(a \sqcup b) = \text{atoms } a \cup \text{atoms } b$ for $a, b \in \mathfrak{A}$.

PROOF.

- 1° \Rightarrow 2°. Obvious.
- 2° \Rightarrow 3°. Corollary 531.
- 3° \Rightarrow 4°. Corollary 496. \square

THEOREM 558. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is a filtered weakly down-aligned filtrator with binarily meet-closed core \mathfrak{J} which is a meet-semilattice.
- 4°. a is an atom of \mathfrak{J} iff $a \in \mathfrak{J}$ and a is an atom of \mathfrak{A} .

PROOF.

- 1° \Rightarrow 2°. Obvious.
- 2° \Rightarrow 3°. It is filtered by the theorem 534, binarily meet-closed by corollary 536.

$3^\circ \Rightarrow 4^\circ$.

- \Leftarrow . Let a be an atom of \mathfrak{A} and $a \in \mathfrak{Z}$. Then either a is an atom of \mathfrak{Z} or a is the least element of \mathfrak{Z} . But if a is the least element of \mathfrak{Z} then a is also least element of \mathfrak{A} and thus is not an atom of \mathfrak{A} . So the only possible outcome is that a is an atom of \mathfrak{Z} .
- \Rightarrow . We need to prove that if a is an atom of \mathfrak{Z} then a is an atom of \mathfrak{A} . Suppose the contrary that a is not an atom of \mathfrak{A} . Then there exists $x \in \mathfrak{A}$ such that $x \sqsubset a$ and x is not least element of \mathfrak{A} . Because “up” is a straight monotone map to the dual of the poset $\mathcal{P}\mathfrak{Z}$ (obvious 454), $\text{up } a \subset \text{up } x$. So there exists $K \in \text{up } x$ such that $K \notin \text{up } a$. Also $a \in \text{up } x$. We have $K \sqcap^{\mathfrak{Z}} a = K \sqcap^{\mathfrak{A}} a \in \text{up } x$; $K \sqcap^{\mathfrak{Z}} a$ is not least of \mathfrak{Z} (Suppose for the contrary that $K \sqcap^{\mathfrak{Z}} a = \perp^{\mathfrak{Z}}$, then $K \sqcap^{\mathfrak{Z}} a = \perp^{\mathfrak{A}} \notin \text{up } x$.) and $K \sqcap^{\mathfrak{Z}} a \sqsubset a$. So a is not an atom of \mathfrak{Z} . \square

THEOREM 559. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
- 3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
- 4°. $a \in \mathfrak{A}$ is an atom of \mathfrak{A} iff $\text{up } a = \partial a$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. By the theorem 534.

$3^\circ \Rightarrow 4^\circ$.

\Rightarrow . For any $K \in \mathfrak{A}$

$$K \in \text{up } a \Leftrightarrow K \sqsupseteq a \Leftrightarrow K \not\prec^{\mathfrak{A}} a \Leftrightarrow K \in \partial a.$$

\Leftarrow . Let $\text{up } a = \partial a$. Then a is not least element of \mathfrak{A} . Consequently for every $x \in \mathfrak{A}$ if x is not the least element of \mathfrak{A} we have

$$\begin{aligned} x \sqsubset a &\Rightarrow \\ x \not\prec^{\mathfrak{A}} a &\Rightarrow \\ \forall K \in \text{up } x : K \in \partial a &\Rightarrow \\ \forall K \in \text{up } x : K \in \text{up } a &\Rightarrow \\ \text{up } x \subseteq \text{up } a &\Rightarrow \\ x \sqsupseteq a. & \end{aligned}$$

So a is an atom of \mathfrak{A} . \square

PROPOSITION 560. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
- 3°. Coatoms of \mathfrak{A} are exactly coatoms of \mathfrak{Z} .

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. Suppose a is a coatom of \mathfrak{Z} . Then a is the only non-greatest element in $\text{up } a$. Suppose $b \sqsubset a$ for some $b \in \mathfrak{A}$. Then a cannot be in $\text{up } b$ and thus the only possible element of $\text{up } b$ is the greatest element of \mathfrak{Z} (if it exists) from what follows $b = \top^{\mathfrak{A}}$. So a is a coatom of \mathfrak{A} .

Suppose now that a is a coatom of \mathfrak{A} . To finish the proof it is enough to show that a is principal. (Then a is non-greatest and thus is a coatom of \mathfrak{J} .)

Suppose a is non-principal. Then obviously exist two distinct elements x and y of the core such that $x, y \in \text{up } a$. Thus a is not an atom of \mathfrak{A} . \square

COROLLARY 561. Coatoms of the set of filters on a set U are exactly sets $U \setminus \{x\}$ where $x \in U$.

PROPOSITION 562. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a coatomic poset.
- 3°. \mathfrak{A} is coatomic.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Suppose $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{A} \neq \top^{\mathfrak{A}}$. Then there exists $A \in \text{up } \mathcal{A}$ such that A is not greatest element of \mathfrak{J} . Consequently there exists a coatom $a \in \mathfrak{J}$ such that $a \sqsupseteq A$. Thus $a \in \text{up } \mathcal{A}$ and a is not greatest. \square

5.15. Prime Filtrator Elements

DEFINITION 563. Let $(\mathfrak{A}, \mathfrak{J})$ be a filtrator. *Prime* filtrator elements are such $a \in \mathfrak{A}$ that $\text{up } a$ is a free star (in lattice \mathfrak{J}).

PROPOSITION 564. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a distributive lattice which is an ideal base.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is a filtrator with binarily join-closed core, where \mathfrak{A} is a starrish join-semilattice and \mathfrak{J} is a join-semilattice.
- 4°. Atomic elements of this filtrator are prime.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. $(\mathfrak{A}, \mathfrak{J})$ is with binarily join-closed core by the theorem 534, \mathfrak{A} is a distributive lattice by theorem 531.

3° \Rightarrow 4°. Let a be an atom of the lattice \mathfrak{A} . We have for every $X, Y \in \mathfrak{J}$

$$\begin{aligned}
 X \sqcup^{\mathfrak{J}} Y \in \text{up } a &\Leftrightarrow \\
 X \sqcup^{\mathfrak{A}} Y \in \text{up } a &\Leftrightarrow \\
 X \sqcup^{\mathfrak{A}} Y \sqsupseteq a &\Leftrightarrow \\
 X \sqcup^{\mathfrak{A}} Y \not\neq^{\mathfrak{A}} a &\Leftrightarrow \\
 X \not\neq^{\mathfrak{A}} a \vee Y \not\neq^{\mathfrak{A}} a &\Leftrightarrow \\
 X \sqsupseteq a \vee Y \sqsupseteq a &\Leftrightarrow \\
 X \in \text{up } a \vee Y \in \text{up } a. &
 \end{aligned}$$

\square

The following theorem is essentially borrowed from [19]:

THEOREM 565. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.

- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
 3°. Let $a \in \mathfrak{A}$. Then the following are equivalent:
 (a) a is prime.
 (b) For every $A \in \mathfrak{F}$ exactly one of $\{A, \bar{A}\}$ is in $\text{up } a$.
 (c) a is an atom of \mathfrak{A} .

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°.

3°a \Rightarrow 3°b. Let a be prime. Then $A \sqcup^3 \bar{A} = \top^{\mathfrak{A}} \in \text{up } a$. Therefore $A \in \text{up } a \vee \bar{A} \in \text{up } a$. But since $A \cap^3 \bar{A} = \perp^3$ it is impossible $A \in \text{up } a \wedge \bar{A} \in \text{up } a$.

3°b \Rightarrow 3°c. Obviously $a \neq \perp^{\mathfrak{A}}$.

Let a filter $b \sqsubset a$. Take $X \in \text{up } b$ such that $X \notin \text{up } a$. Then $\bar{X} \in \text{up } a$ because a is prime and thus $\bar{X} \in \text{up } b$. So $\perp^3 = X \cap^3 \bar{X} \in \text{up } b$ and thus $b = \perp^{\mathfrak{A}}$. So a is atomic.

3°c \Rightarrow 3°a. By the previous proposition.

□

5.16. Stars for filters

THEOREM 566. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a distributive lattice which is an ideal base and has least element.
 3°. ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. \mathfrak{A} is a distributive lattice by the corollary 531. The filtrator $(\mathfrak{A}, \mathfrak{F})$ is binarily join-closed by corollary 534. So we can apply the theorem 554.

□

5.16.1. Stars of Filters on Boolean Lattices. In this section we will consider the set of filters \mathfrak{A} on a boolean lattice \mathfrak{F} .

THEOREM 567. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
 3°. $\partial \mathcal{A} = \neg \langle \neg \rangle^* \text{up } \mathcal{A} = \langle \neg \rangle^* \neg \text{up } \mathcal{A}$ and $\text{up } \mathcal{A} = \neg \langle \neg \rangle^* \partial \mathcal{A} = \langle \neg \rangle^* \neg \partial \mathcal{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Because of properties of diagram (1), it is enough to prove just $\partial \mathcal{A} = \neg \langle \neg \rangle^* \text{up } \mathcal{A}$. Really,

$$X \in \text{up } \mathcal{A} \Leftrightarrow X \sqsupseteq \mathcal{A} \Leftrightarrow \bar{X} \prec^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \bar{X} \notin \partial \mathcal{A}$$

for any $X \in \mathfrak{F}$ (taking into account theorems 535, 537, and lemma 551).

□

COROLLARY 568. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
 3°. ∂ is an order isomorphism from \mathfrak{A} to $\mathfrak{S}(\mathfrak{F})$.

PROOF. By properties of the diagram (1).

□

COROLLARY 569. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a boolean lattice.
- 3°. $\partial \sqcup^{\mathfrak{A}} S = \bigcup \langle \partial \rangle^* S$ for every $S \in \mathcal{P}\mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. $\partial \sqcup^{\mathfrak{A}} S = \sqcup^{\mathfrak{S}(\mathfrak{J})} \langle \partial \rangle^* S = \bigcup \langle \partial \rangle^* S$.

□

5.17. Generalized Filter Base

DEFINITION 570. *Generalized filter base* is a filter base on the set \mathfrak{A} where $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator.

DEFINITION 571. If S is a generalized filter base and $\mathcal{A} = \prod^{\mathfrak{A}} S$ for some $\mathcal{A} \in \mathfrak{A}$, then we call S a generalized filter base of \mathcal{A} .

THEOREM 572. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice.
- 3°. For a generalized filter base S of $\mathcal{F} \in \mathfrak{A}$ and $K \in \mathfrak{J}$ we have

$$K \in \text{up } \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : K \in \text{up } \mathcal{L}.$$

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°.

\Leftarrow . Because $\mathcal{F} = \prod^{\mathfrak{A}} S$.

\Rightarrow . Let $K \in \text{up } \mathcal{F}$. Then (taken into account corollary 525 and that S is nonempty) there exist $X_1, \dots, X_n \in \bigcup \langle \text{up} \rangle^* S$ such that $K \in \text{up}(X_1 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} X_n)$ that is $K \in \text{up}(\uparrow X_1 \sqcap^{\mathfrak{J}} \dots \sqcap^{\mathfrak{J}} \uparrow X_n)$. Consequently (by theorem 535) $K \in \text{up}(\uparrow X_1 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \uparrow X_n)$. Replacing every $\uparrow X_i$ with such $\mathcal{X}_i \in S$ that $X_i \in \text{up } \mathcal{X}_i$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $K \in \text{up} \prod^{\mathfrak{A}} T_0$. From this there exists $\mathcal{C} \in S$ such that $\mathcal{C} \sqsubseteq \prod^{\mathfrak{A}} T_0$ and so $K \in \text{up } \mathcal{C}$.

□

COROLLARY 573. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice with least element.
- 3°. For a generalized filter base S of a $\mathcal{F} \in \mathfrak{A}$ we have

$$\perp^{\mathfrak{A}} \in S \Leftrightarrow \mathcal{F} = \perp^{\mathfrak{A}}.$$

PROOF. Substitute $\perp^{\mathfrak{A}}$ as K .

□

THEOREM 574. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice with least element.
- 3°. Let $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_n \neq \perp^{\mathfrak{A}}$ for every $\mathcal{F}_0, \dots, \mathcal{F}_n \in S$, where S is a nonempty set of elements of \mathfrak{A} . Then $\prod^{\mathfrak{A}} S \neq \perp^{\mathfrak{A}}$.

PROOF. Consider the set

$$S' = \left\{ \frac{\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_n}{\mathcal{F}_0, \dots, \mathcal{F}_n \in S} \right\}.$$

Obviously S' is nonempty and binarily meet-closed. So S' is a generalized filter base. Obviously $\perp^{\mathfrak{A}} \notin S$. So by properties of generalized filter bases $\prod^{\mathfrak{A}} S' \neq \perp^{\mathfrak{A}}$. But obviously $\prod^{\mathfrak{A}} S = \prod^{\mathfrak{A}} S'$. So $\prod^{\mathfrak{A}} S \neq \perp^{\mathfrak{A}}$. \square

COROLLARY 575. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a meet-semilattice with least element.
- 3°. Let $S \in \mathcal{P}\mathfrak{B}$ such that $S \neq \emptyset$ and $A_0 \sqcap^{\mathfrak{B}} \dots \sqcap^{\mathfrak{B}} A_n \neq \perp^{\mathfrak{B}}$ for every $A_0, \dots, A_n \in S$. Then $\prod^{\mathfrak{A}} S \neq \perp^{\mathfrak{A}}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Because $(\mathfrak{A}, \mathfrak{B})$ is binarily meet-closed (by the theorem 535). \square

THEOREM 576. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a bounded meet-semilattice.
- 3°. \mathfrak{A} is an atomic lattice.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Let $\mathcal{F} \in \mathfrak{A}$. Let choose (by Kuratowski's lemma) a maximal chain S from $\perp^{\mathfrak{A}}$ to \mathcal{F} . Let $S' = S \setminus \{\perp^{\mathfrak{A}}\}$. $a = \prod^{\mathfrak{A}} S' \neq \perp^{\mathfrak{A}}$ by properties of generalized filter bases (the corollary 573 which uses the fact that \mathfrak{B} is a meet-semilattice with least element). If $a \notin S$ then the chain S can be extended adding there element a because $\perp^{\mathfrak{A}} \sqsubset a \sqsubseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S' . Consequently (taking into account maximality of the chain) there is no $\mathcal{Y} \in \mathfrak{A}$ such that $\perp^{\mathfrak{A}} \sqsubset \mathcal{Y} \sqsubset a$. So a is an atomic filter. Obviously $a \sqsubseteq \mathcal{F}$. \square

DEFINITION 577. A complete lattice is *co-compact* iff $\prod S = \perp$ for a set S of elements of this lattice implies that there is its finite subset $T \subseteq S$ such that $\prod T = \perp$.

THEOREM 578. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a bounded meet-semilattice.
- 3°. \mathfrak{A} is co-compact.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Poset \mathfrak{A} is complete by corollary 518.

If $\perp \in \text{up} \prod^{\mathfrak{A}} S$ then there are $K_i \in \text{up} \bigcup S$ such that $\perp \in \text{up}(K_0 \sqcap^{\mathfrak{B}} \dots \sqcap^{\mathfrak{B}} K_n)$ that is $K_0 \sqcap^{\mathfrak{B}} \dots \sqcap^{\mathfrak{B}} K_n = \perp$ from which easily follows $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_n = \perp$ for some $\mathcal{F}_i \in S$. \square

5.18. Separability of filters

PROPOSITION 579. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a boolean lattice.

3°. \mathfrak{A} is strongly separable.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. By properties of stars of filters.

□

REMARK 580. [14] seems to show that the above theorem cannot be generalized for a wider class of lattices.

THEOREM 581. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
- 3°. \mathfrak{A} is an atomistic poset.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Because (used theorem 232) \mathfrak{A} is atomic (theorem 576) and separable.

□

COROLLARY 582. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
- 3°. \mathfrak{A} is atomically separable.

PROOF. By theorem 230.

□

5.19. Some Criteria

THEOREM 583. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.
- 4°. The following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:
 - (a) $\mathcal{F} \in \mathfrak{F}$;
 - (b) $\forall S \in \mathcal{P}\mathfrak{A} : (\mathcal{F} \cap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \perp \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \cap^{\mathfrak{A}} \mathcal{K} \neq \perp)$;
 - (c) $\forall S \in \mathcal{P}\mathfrak{F} : (\mathcal{F} \cap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \perp \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \cap^{\mathfrak{A}} \mathcal{K} \neq \perp)$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. The filtrator $(\mathfrak{A}, \mathfrak{F})$ is with with join-closed core by theorem 534, binarily meet-closed core by corollary 536, with separable core by theorem 537.

3° \Rightarrow 4°.

4°a \Rightarrow 4°b. Let $\mathcal{F} \in \mathfrak{F}$. Then (taking into account the lemma 551)

$$\mathcal{F} \cap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \perp \Leftrightarrow \overline{\mathcal{F}} \not\sqsupseteq \bigsqcup^{\mathfrak{A}} S \Rightarrow \exists \mathcal{K} \in S : \overline{\mathcal{F}} \not\sqsupseteq \mathcal{K} \Leftrightarrow \exists \mathcal{K} \in S : \mathcal{F} \cap^{\mathfrak{A}} \mathcal{K} \neq \perp.$$

4°b \Rightarrow 4°c. Obvious.

$4^\circ \mathbf{c} \Rightarrow 4^\circ \mathbf{a}$.

$$\begin{aligned}
\forall S \in \mathcal{P}\mathfrak{Z} : \left(\mathcal{F} \cap^{\mathfrak{A}} \bigsqcup S \neq \perp \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq \perp \right) &\Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{Z} : \left(\mathcal{F} \not\leq^{\mathfrak{A}} \bigsqcup^3 S \Rightarrow \exists K \in S : \mathcal{F} \not\leq^{\mathfrak{A}} K \right) &\Leftrightarrow \text{(lemma 551)} \\
\forall S \in \mathcal{P}\mathfrak{Z} : \left(\overline{\bigsqcup^3 S} \not\supseteq \mathcal{F} \Rightarrow \exists K \in S : \overline{K} \not\supseteq \mathcal{F} \right) &\Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \supseteq \mathcal{F} \Rightarrow \overline{\bigsqcup^3 S} \supseteq \mathcal{F} \right) &\Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \supseteq \mathcal{F} \Rightarrow \overline{\bigsqcap^3 \langle \neg \rangle^* S} \supseteq \mathcal{F} \right) &\Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{Z} : \left(\forall K \in S : K \supseteq \mathcal{F} \Rightarrow \overline{\bigsqcap^3 S} \supseteq \mathcal{F} \right) &\Rightarrow \\
\overline{\bigsqcap^3 \text{up } \mathcal{F}} \supseteq \mathcal{F} &\Leftrightarrow \\
\overline{\bigsqcap^3 \text{up } \mathcal{F}} \in \text{up } \mathcal{F} &\Rightarrow \\
\mathcal{F} \in \mathfrak{Z}. &
\end{aligned}$$

□

REMARK 584. The above theorem strengthens theorem 53 in [30]. Both the formulation of the theorem and the proof are considerably simplified.

DEFINITION 585. Let S be a subset of a meet-semilattice. The *filter base generated by S* is the set

$$[S]_{\cap} = \left\{ \frac{a_0 \cap \cdots \cap a_n}{a_i \in S, n = 0, 1, \dots} \right\}.$$

LEMMA 586. The set of all finite subsets of an infinite set A has the same cardinality as A .

PROOF. Let denote the number of n -element subsets of A as s_n . Obviously $s_n \leq \text{card } A^n = \text{card } A$. Then the number S of all finite subsets of A is equal to

$$s_0 + s_1 + \cdots \leq \text{card } A + \text{card } A + \cdots = \text{card } A.$$

That $S \geq \text{card } A$ is obvious. So $S = \text{card } A$. □

LEMMA 587. A filter base generated by an infinite set has the same cardinality as that set.

PROOF. From the previous lemma. □

DEFINITION 588. Let \mathfrak{A} be a complete lattice. A set $S \in \mathcal{P}\mathfrak{A}$ is *filter-closed* when for every filter base $T \in \mathcal{P}S$ we have $\bigsqcap T \in S$.

THEOREM 589. A subset S of a complete lattice is filter-closed iff for every nonempty chain $T \in \mathcal{P}S$ we have $\bigsqcap T \in S$.

PROOF. (proof sketch by JOEL DAVID HAMKINS)

\Rightarrow . Because every nonempty chain is a filter base.

\Leftarrow . We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha < \beta \Leftrightarrow \alpha \in \beta$ for ordinals α, β .

We will take it as given that for every nonempty chain $T \in \mathcal{P}S$ we have $\prod T \in S$.

We will prove the following statement: If $\text{card } S = n$ then S is filter closed, for any cardinal n .

Instead we will prove it not only for cardinals but for wider class of ordinals: If $\text{card } S = n$ then S is filter-closed, for any ordinal n .

We will prove it using transfinite induction by n .

For finite n we have $\prod T \in S$ because $T \subseteq S$ has minimal element.

Let $\text{card } T = n$ be an infinite ordinal.

Let the assumption hold for every $m \in \text{card } T$.

We can assign $T = \left\{ \frac{a_\alpha}{\alpha \in \text{card } T} \right\}$ for some a_α because $\text{card } \text{card } T = \text{card } T$.

Consider $\beta \in \text{card } T$.

Let $P_\beta = \left\{ \frac{a_\alpha}{\alpha \in \beta} \right\}$. Let $b_\beta = \prod P_\beta$. Obviously $b_\beta = \prod [P_\beta]_\square$. We have

$$\text{card}[P_\beta]_\square = \text{card } P_\beta = \text{card } \beta < \text{card } T$$

(used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_\beta \in S$.

$\forall \beta \in \text{card } T : P_\beta \subseteq T$ and thus $b_\beta \sqsupseteq \prod T$.

It is easy to see that the set $\left\{ \frac{P_\beta}{\beta \in \text{card } T} \right\}$ is a chain. Consequently $\left\{ \frac{b_\beta}{\beta \in \text{card } T} \right\}$ is a chain.

By the theorem conditions $b = \prod_{\beta \in \text{card } T} b_\beta \in S$ (taken into account that $b_\beta \in S$ by the assumption of induction).

Obviously $b \sqsupseteq \prod T$.

$b \sqsubseteq b_\beta$ and so $\forall \beta \in \text{card } T, \alpha \in \beta : b \sqsubseteq a_\alpha$. Let $\alpha \in \text{card } T$. Then (because $\text{card } T$ is a limit ordinal, see [44]) there exists $\beta \in \text{card } T$ such that $\alpha \in \beta \in \text{card } T$. So $b \sqsubseteq a_\alpha$ for every $\alpha \in \text{card } T$. Thus $b \sqsubseteq \prod T$.

Finally $\prod T = b \in S$.

□

5.20. Co-Separability of Core

THEOREM 590. The following is an implications tuple.

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet infinite distributive complete lattice.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is an up-aligned filtered filtrator whose core is a meet infinite distributive complete lattice.
- 4°. This filtrator is with co-separable core.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. It is obviously up-aligned, and filtered by theorem 534.

3° \Rightarrow 4°. Our filtrator is with join-closed core (theorem 534).

Let $a, b \in \mathfrak{A}$. $\text{Cor } a$ and $\text{Cor } b$ exist since \mathfrak{J} is a complete lattice.

$\text{Cor } a \in \text{down } a$ and $\text{Cor } b \in \text{down } b$ by the theorem 542 since our filtrator is filtered. So we have

$$\begin{aligned}
\exists x \in \text{down } a, y \in \text{down } b : x \sqcup^{\mathfrak{A}} y = \top &\Leftarrow \\
\text{Cor } a \sqcup^{\mathfrak{A}} \text{Cor } b = \top &\Leftrightarrow (\text{by finite join-closedness of the core}) \\
\text{Cor } a \sqcup^{\mathfrak{B}} \text{Cor } b = \top &\Leftrightarrow \\
\prod^{\mathfrak{B}} \text{up } a \sqcup^{\mathfrak{B}} \prod^{\mathfrak{B}} \text{up } b = \top &\Leftrightarrow (\text{by infinite distributivity}) \\
\prod^{\mathfrak{B}} \left\{ \frac{x \sqcup^{\mathfrak{B}} y}{x \in \text{up } a, y \in \text{up } b} \right\} = \top &\Leftarrow \\
\forall x \in \text{up } a, y \in \text{up } b : x \sqcup^{\mathfrak{B}} y = \top &\Leftrightarrow (\text{by binary join-closedness of the core}) \\
\forall x \in \text{up } a, y \in \text{up } b : x \sqcup^{\mathfrak{A}} y = \top &\Leftarrow \\
a \sqcup^{\mathfrak{A}} b = \top. &
\end{aligned}$$

□

5.21. Complements and Core Parts

LEMMA 591. If $(\mathfrak{A}, \mathfrak{B})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a, c \in \mathfrak{A}$

$$c \equiv^{\mathfrak{A}} a \Leftrightarrow c \equiv^{\mathfrak{A}} \text{Cor } a.$$

PROOF.

\Rightarrow . If $c \equiv^{\mathfrak{A}} a$ then by co-separability of the core exists $K \in \text{down } a$ such that $c \equiv^{\mathfrak{A}} K$. To finish the proof we will show that $K \sqsubseteq \text{Cor } a$. To show this is enough to show that $\forall X \in \text{up } a : K \sqsubseteq X$ what is obvious.
 \Leftarrow . $\text{Cor } a \sqsubseteq a$ (by theorem 542 using that our filtrator is filtered).

□

THEOREM 592. If $(\mathfrak{A}, \mathfrak{B})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ = \overline{\text{Cor } a}$ for every $a \in \mathfrak{A}$.

PROOF. Our filtrator is with join-closed core (theorem 534).

$$\begin{aligned}
a^+ &= \\
\prod^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}} \right\} &= \\
\prod^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} \text{Cor } a = \top^{\mathfrak{A}}} \right\} &= \\
\prod^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqsupseteq \overline{\text{Cor } a}} \right\} &= \\
\overline{\text{Cor } a} &
\end{aligned}$$

(used the lemma above and lemma 551).

□

COROLLARY 593. If $(\mathfrak{A}, \mathfrak{B})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ \in \mathfrak{B}$ for every $a \in \mathfrak{A}$.

THEOREM 594.

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete boolean lattice.

- 3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered complete lattice filtrator with down-aligned, binarily meet-closed, separable core which is a complete boolean lattice.
 4°. $a^* = \overline{\text{Cor } a} = \overline{\text{Cor}' a}$ for every $a \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2° Obvious.

2° \Rightarrow 3° It is filtered by theorem 534. It is complete lattice filtrator by 518. It is with binarily meet-closed core (proposition 536), with separable core (theorem 537).

3° \Rightarrow 4° Our filtrator is with join-closed core (theorem 534).

$$a^* = \bigsqcup^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\}.$$

But

$$c \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}} \Rightarrow \exists C \in \text{up } c : C \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}.$$

So

$$\begin{aligned} a^* &= \\ & \bigsqcup^{\mathfrak{A}} \left\{ \frac{C \in \mathfrak{Z}}{C \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\} = \\ & \bigsqcup^{\mathfrak{A}} \left\{ \frac{C \in \mathfrak{Z}}{a \sqsubseteq \overline{C}} \right\} = \\ & \bigsqcup^{\mathfrak{A}} \left\{ \frac{\overline{C}}{C \in \mathfrak{Z}, a \sqsubseteq C} \right\} = \\ & \bigsqcup^{\mathfrak{A}} \left\{ \frac{\overline{C}}{C \in \text{up } a} \right\} = \\ & \bigsqcup^{\mathfrak{Z}} \left\{ \frac{\overline{C}}{C \in \text{up } a} \right\} = \\ & \overline{\bigsqcup^{\mathfrak{Z}} \left\{ \frac{C}{C \in \text{up } a} \right\}} = \\ & \overline{\bigsqcup^{\mathfrak{Z}} \text{up } a} = \\ & \overline{\text{Cor } a} \end{aligned}$$

(used lemma 551).

$\text{Cor } a = \text{Cor}' a$ by theorem 545. □

THEOREM 595. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
 3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete boolean lattice.
 4°. $a^* = a^+ = \overline{\text{Cor } a} = \overline{\text{Cor}' a} \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. \mathfrak{A} is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is with co-separable core by theorem 590. $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed by proposition 536, with separable core by theorem 537.

3°⇒4°. Comparing two last theorems. □

THEOREM 596. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a complete lattice.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a complete lattice filtrator with join-closed separable core which is a complete lattice.
- 3°. $a^* \in \mathfrak{J}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. \mathfrak{A} is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{J})$ is a filtrator with join-closed core by theorem 534. $(\mathfrak{A}, \mathfrak{J})$ is a filtrator with separable core by theorem 537.

2°⇒3°. $\left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\} \supseteq \left\{ \frac{A \in \mathfrak{J}}{A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\}$; consequently $a^* \sqsupseteq \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{J}}{A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\}$.
 But if $c \in \left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\}$ then there exists $A \in \mathfrak{J}$ such that $A \sqsupseteq c$ and $A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}$ that is $A \in \left\{ \frac{A \in \mathfrak{J}}{A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\}$. Consequently $a^* \sqsubseteq \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{J}}{A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\}$.

We have $a^* = \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{J}}{A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\} = \bigsqcup^{\mathfrak{J}} \left\{ \frac{A \in \mathfrak{J}}{A \sqcap^{\mathfrak{A}} a = \perp^{\mathfrak{A}}} \right\} \in \mathfrak{J}$. □

THEOREM 597. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is an up-aligned filtered complete lattice filtrator with co-separable core which is a complete boolean lattice.
- 4°. a^+ is dual pseudocomplement of a , that is

$$a^+ = \min \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}} \right\}$$

for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{J})$ is filtered by the theorem 534. It is with co-separable core by theorem 590. \mathfrak{A} is a complete lattice by corollary 518.

3°⇒4°. Our filtrator is with join-closed core (theorem 534). It's enough to prove that $a^+ \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}$. But $a^+ \sqcup^{\mathfrak{A}} a = \overline{\text{Cor } a} \sqcup^{\mathfrak{A}} a \sqsupseteq \overline{\text{Cor } a} \sqcup^{\mathfrak{A}} \text{Cor } a = \overline{\text{Cor } a} \sqcup^{\mathfrak{J}} \text{Cor } a = \top^{\mathfrak{A}}$ (used the theorem 542 and the fact that our filtrator is filtered). □

DEFINITION 598. The *edge part* of an element $a \in \mathfrak{A}$ is $\text{Edg } a = a \setminus \text{Cor } a$, the *dual edge part* is $\text{Edg}' a = a \setminus \text{Cor}' a$.

Knowing core part and edge part or dual core part and dual edge part of an element of a filtrator, the filter can be restored by the formulas:

$$a = \text{Cor } a \sqcup^{\mathfrak{A}} \text{Edg } a \quad \text{and} \quad a = \text{Cor}' a \sqcup^{\mathfrak{A}} \text{Edg}' a.$$

5.22. Core Part and Atomic Elements

PROPOSITION 599. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over an atomistic lattice.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is a filtrator with join-closed core and \mathfrak{J} be an atomistic lattice.

4°. $\text{Cor}' a = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\}$ for every $a \in \mathfrak{A}$ such that $\text{Cor}' a$ exists.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. $(\mathfrak{A}, \mathfrak{Z})$ is with join-closed core by corollary 534.

3° \Rightarrow 4°.

$$\begin{aligned} \text{Cor}' a &= \\ &= \bigsqcup^{\mathfrak{Z}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqsubseteq a} \right\} = \\ &= \bigsqcup^{\mathfrak{Z}} \left\{ \frac{\bigsqcup^{\mathfrak{Z}} \text{atoms}^{\mathfrak{Z}} A}{A \in \mathfrak{Z}, A \sqsubseteq a} \right\} = \\ &= \bigsqcup^{\mathfrak{Z}} \bigcup \left\{ \frac{\text{atoms}^{\mathfrak{Z}} A}{A \in \mathfrak{Z}, A \sqsubseteq a} \right\} = \\ &= \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\}. \end{aligned}$$

□

COROLLARY 600. $\text{Cor} a = \uparrow \left\{ \frac{p \in \mathfrak{U}}{\uparrow \{p\} \sqsubseteq a} \right\}$ and $\cap a = \left\{ \frac{p \in \mathfrak{U}}{\uparrow \{p\} \sqsubseteq a} \right\}$ for every filter a on a set \mathfrak{U} .

PROOF. By proposition 546. □

5.23. Distributivity of Core Part over Lattice Operations

THEOREM 601. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
- 3°. $(\mathfrak{A}, \mathfrak{Z})$ is a join-closed filtrator and \mathfrak{A} is a meet-semilattice and \mathfrak{Z} is a meet-semilattice.
- 4°. $\text{Cor}'(a \cap^{\mathfrak{A}} b) = \text{Cor}' a \cap^{\mathfrak{Z}} \text{Cor}' b$ for every $a, b \in \mathfrak{A}$. whenever $\text{Cor}'(a \cap^{\mathfrak{A}} b)$, $\text{Cor}' a$, and $\text{Cor}' b$ exist

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. $(\mathfrak{A}, \mathfrak{Z})$ is with join-closed core by corollary 534. \mathfrak{A} is a meet-semilattice by corollary 518.

3° \Rightarrow 4°. We have $\text{Cor}' p \sqsubseteq p$ for every $p \in \mathfrak{A}$ whenever $\text{Cor}' p$ exists, because our filtrator is with join-closed core (theorem 543).

Obviously $\text{Cor}'(a \cap^{\mathfrak{A}} b) \sqsubseteq \text{Cor}' a$ and $\text{Cor}'(a \cap^{\mathfrak{A}} b) \sqsubseteq \text{Cor}' b$.

If $x \sqsubseteq \text{Cor}' a$ and $x \sqsubseteq \text{Cor}' b$ for some $x \in \mathfrak{Z}$ then $x \sqsubseteq a$ and $x \sqsubseteq b$, thus $x \sqsubseteq a \cap^{\mathfrak{A}} b$ and $x \sqsubseteq \text{Cor}'(a \cap^{\mathfrak{A}} b)$.

□

THEOREM 602. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
- 3°. $(\mathfrak{A}, \mathfrak{Z})$ is a join-closed filtrator.
- 4°. $\text{Cor}' \prod^{\mathfrak{A}} S = \prod^{\mathfrak{Z}} \langle \text{Cor}' \rangle^* S$ for every $S \in \mathcal{P}\mathfrak{A}$ whenever both sides of the equality are defined. Also $\text{Cor}' \prod^{\mathfrak{A}} T = \prod^{\mathfrak{Z}} T$ for every $T \in \mathcal{P}\mathfrak{Z}$ whenever both sides of the equality are defined.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. It is with join-closed core by theorem 534. \mathfrak{A} is a complete lattice by corollary 518.

3° \Rightarrow 4°. We have $\text{Cor}' p \sqsubseteq p$ for every $p \in \mathfrak{A}$ because our filtrator is with join-closed core (theorem 543).

Obviously $\text{Cor}' \prod^{\mathfrak{A}} S \sqsubseteq \text{Cor}' a$ for every $a \in S$.

If $x \sqsubseteq \text{Cor}' a$ for every $a \in S$ for some $x \in \mathfrak{J}$ then $x \sqsubseteq a$, thus $x \sqsubseteq \prod^{\mathfrak{A}} S$ and $x \sqsubseteq \text{Cor}' \prod^{\mathfrak{A}} S$.

So $\text{Cor}' \prod^{\mathfrak{A}} S = \prod^{\mathfrak{J}} \langle \text{Cor}' \rangle^* S$. $\text{Cor}' \prod^{\mathfrak{A}} T = \prod^{\mathfrak{J}} T$ trivially follows from this. □

THEOREM 603. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a complete atomistic distributive lattice.

3°. $(\mathfrak{A}, \mathfrak{J})$ is a filtered down-aligned filtrator with binarily meet-closed core \mathfrak{J} which is a complete atomistic lattice and \mathfrak{A} is a complete starrish lattice.

4°. $\text{Cor}'(a \sqcup^{\mathfrak{A}} b) = \text{Cor}' a \sqcup^{\mathfrak{J}} \text{Cor}' b$ for every $a, b \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. $(\mathfrak{A}, \mathfrak{J})$ is filtered by theorem 534. It is with binarily meet-close core by corollary 536. \mathfrak{A} is starrish by corollary 531. \mathfrak{A} is complete by corollary 518.

3° \Rightarrow 4°. From theorem conditions it follows that $\text{Cor}'(a \sqcup^{\mathfrak{A}} b)$ exists.

$$\text{Cor}'(a \sqcup^{\mathfrak{A}} b) = \prod^{\mathfrak{J}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{J}, x \sqsubseteq a \sqcup^{\mathfrak{A}} b} \right\} \text{ (used proposition 599).}$$

By theorem 558 we have

$$\begin{aligned} \text{Cor}'(a \sqcup^{\mathfrak{A}} b) &= \\ &= \prod^{\mathfrak{J}} ((\text{atoms}^{\mathfrak{A}}(a \sqcup^{\mathfrak{A}} b) \cap \mathfrak{J}) \cap \mathfrak{J}) = \\ &= \prod^{\mathfrak{J}} ((\text{atoms}^{\mathfrak{A}} a \cup \text{atoms}^{\mathfrak{A}} b) \cap \mathfrak{J}) = \\ &= \prod^{\mathfrak{J}} ((\text{atoms}^{\mathfrak{A}} a \cap \mathfrak{J}) \cup (\text{atoms}^{\mathfrak{A}} b \cap \mathfrak{J})) = \\ &= \prod^{\mathfrak{J}} (\text{atoms}^{\mathfrak{A}} a \cap \mathfrak{J}) \sqcup^{\mathfrak{J}} \prod^{\mathfrak{J}} (\text{atoms}^{\mathfrak{A}} b \cap \mathfrak{J}) \end{aligned}$$

(used the theorem 496). Again using theorem 558, we get

$$\begin{aligned} \text{Cor}'(a \sqcup^{\mathfrak{A}} b) &= \\ &= \prod^{\mathfrak{J}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{J}, x \sqsubseteq a} \right\} \sqcup^{\mathfrak{J}} \prod^{\mathfrak{J}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{J}, x \sqsubseteq b} \right\} = \\ &= \text{Cor}' a \sqcup^{\mathfrak{J}} \text{Cor}' b \end{aligned}$$

(again used proposition 599). □

See also theorem 167 above.

5.24. Separability criteria

THEOREM 604. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtrator with correct intersection, with binarily meet-closed and separable core.
- 4°. $B \succ^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsupseteq \mathcal{A}$ for every $B \in \mathfrak{B}$, $\mathcal{A} \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Using proposition 549, corollary 536, theorem 537.

3° \Rightarrow 4°. By the lemma 551. □

THEOREM 605. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtrator over a boolean lattice with correct joining and co-separable core.
- 4°. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ for every $B \in \mathfrak{B}$, $\mathcal{A} \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Using obvious 550, theorem 590.

3° \Rightarrow 4°. By the lemma 551. □

5.25. Filtrators over Boolean Lattices

PROPOSITION 606. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a down-aligned and up-aligned binarily meet-closed and binarily join-closed distributive lattice filtrator and \mathfrak{B} is a boolean lattice.
- 4°. $a \setminus^{\mathfrak{A}} B = a \sqcap^{\mathfrak{A}} \overline{B}$ for every $a \in \mathfrak{A}$, $B \in \mathfrak{B}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. \mathfrak{A} is a distributive lattice by corollary 531. Our filtrator is binarily meet-closed by the corollary 536 and with join-closed core by the theorem 534. It is also up and down aligned.

3° \Rightarrow 4°.

$$\begin{aligned} (a \sqcap^{\mathfrak{A}} \overline{B}) \sqcup^{\mathfrak{A}} B &= (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} (\overline{B} \sqcup^{\mathfrak{A}} B) = \\ (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} (\overline{B} \sqcup^{\mathfrak{B}} B) &= (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} \top = a \sqcup^{\mathfrak{A}} B. \\ (a \sqcap^{\mathfrak{A}} \overline{B}) \sqcap^{\mathfrak{A}} B &= a \sqcap^{\mathfrak{A}} (\overline{B} \sqcap^{\mathfrak{A}} B) = a \sqcap^{\mathfrak{A}} (\overline{B} \sqcap^{\mathfrak{B}} B) = a \sqcap^{\mathfrak{A}} \perp = \perp. \end{aligned}$$

So $a \sqcap^{\mathfrak{A}} \overline{B}$ is the difference of a and B . □

PROPOSITION 607. For a primary filtrator over a complete boolean lattice both edge part and dual edge part are always defined.

PROOF. Core part and dual core part are defined because the core is a complete lattice. Using the theorem 606. □

THEOREM 608. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a boolean lattice.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a complete co-brouwerian atomistic down-aligned lattice filtrator with binarily meet-closed and separable boolean core.
- 3°. The three expressions of pseudodifference of a and b in theorem 247 are also equal to $\bigsqcup \left\{ \frac{a \sqcap \bar{B}}{B \in \text{up } b} \right\}$.

PROOF.

1° \Rightarrow 2°. The filtrator of filters on a boolean lattice is:

- complete by corollary 518;
- atomistic by theorem 581;
- co-brouwerian by corollary 531;
- with separable core by theorem 537;
- with binarily meet-closed core by corollary 536.

2° \Rightarrow 3°. $\bigsqcup \left\{ \frac{z \in \mathcal{F}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\} \sqsubseteq \bigsqcup \left\{ \frac{a \sqcap \bar{B}}{B \in \text{up } b} \right\}$ because

$$\begin{aligned} z \in \left\{ \frac{z \in \mathcal{F}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\} &\Leftrightarrow z \sqsubseteq a \wedge z \sqcap b = \perp \Leftrightarrow (\text{separability}) \\ z \sqsubseteq a \wedge \exists B \in \text{up } b : z \sqcap B = \perp &\Leftrightarrow (\text{theorem 604}) \Leftrightarrow z \sqsubseteq a \wedge \exists B \in \text{up } b : z \sqsubseteq \bar{B} \Leftrightarrow \\ &\exists B \in \text{up } b : (z \sqsubseteq a \wedge z \sqsubseteq \bar{B}) \Leftrightarrow \exists B \in \text{up } b : z \sqsubseteq a \sqcap \bar{B} \Rightarrow \\ &z \sqsubseteq \bigsqcup \left\{ \frac{a \sqcap \bar{B}}{B \in \text{up } b} \right\}. \end{aligned}$$

But $a \sqcap \bar{B} \in \left\{ \frac{z \in \mathcal{F}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$ because

$$(a \sqcap \bar{B}) \sqcap b = a \sqcap (\bar{B} \sqcap b) \sqsubseteq a \sqcap (\bar{B} \sqcap^{\mathfrak{A}} B) = a \sqcap (\bar{B} \sqcap^{\mathfrak{B}} B) = a \sqcap \perp = \perp$$

and thus

$$a \sqcap \bar{B} \sqsubseteq \bigsqcup \left\{ \frac{z \in \mathcal{F}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$$

so $\bigsqcup \left\{ \frac{z \in \mathcal{F}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\} \supseteq \bigsqcup \left\{ \frac{a \sqcap \bar{B}}{B \in \text{up } b} \right\}$.

□

5.26. Distributivity for an Element of Boolean Core

LEMMA 609. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
- 4°. $A \sqcap^{\mathfrak{A}}$ is a lower adjoint of $\bar{A} \sqcup^{\mathfrak{A}}$ for every $A \in \mathfrak{B}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. It is binarily join closed by theorem 534. It is binarily meet-closed by corollary 536. It is distributive by corollary 531.

3° \Rightarrow 4°. We will use the theorem 126.

That $A \sqcap^{\mathfrak{A}}$ and $\bar{A} \sqcup^{\mathfrak{A}}$ are monotone is obvious.

We need to prove (for every $x, y \in \mathfrak{A}$) that

$$x \sqsubseteq \bar{A} \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} x) \quad \text{and} \quad A \sqcap^{\mathfrak{A}} (\bar{A} \sqcup^{\mathfrak{A}} y) \sqsubseteq y.$$

Really,

$$\begin{aligned}\bar{A} \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} x) &= (\bar{A} \sqcup^{\mathfrak{A}} A) \sqcap^{\mathfrak{A}} (\bar{A} \sqcup^{\mathfrak{A}} x) = \\ &= (\bar{A} \sqcup^{\mathfrak{A}} A) \sqcap^{\mathfrak{A}} (\bar{A} \sqcup^{\mathfrak{A}} x) = \top \sqcap^{\mathfrak{A}} (\bar{A} \sqcup^{\mathfrak{A}} x) = \bar{A} \sqcup^{\mathfrak{A}} x \sqsupseteq x\end{aligned}$$

and

$$\begin{aligned}A \sqcap^{\mathfrak{A}} (\bar{A} \sqcup^{\mathfrak{A}} y) &= (A \sqcap^{\mathfrak{A}} \bar{A}) \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = (A \sqcap^{\mathfrak{A}} \bar{A}) \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = \\ &= \perp \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = A \sqcap^{\mathfrak{A}} y \sqsubseteq y.\end{aligned}$$

□

THEOREM 610. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
- 4°. $A \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle A \sqcap^{\mathfrak{A}} \rangle^* S$ for every $A \in \mathfrak{F}$ and every set $S \in \mathcal{P}\mathfrak{A}$.

PROOF.

1° ⇒ 2°. Obvious.

2° ⇒ 3°. It is binarily join-closed by theorem 534. It is binarily meet-closed by corollary 536. It is distributive by corollary 531.

3° ⇒ 4°. Direct consequence of the lemma.

□

5.27. More about the Lattice of Filters

DEFINITION 611. Atoms of \mathfrak{F} are called *ultrafilters*.

DEFINITION 612. Principal ultrafilters are also called *trivial ultrafilters*.

THEOREM 613. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
- 3°. The filtrator $(\mathfrak{A}, \mathfrak{F})$ is central.

PROOF.

1° ⇒ 2°. Obvious.

2° ⇒ 3°. We can conclude that \mathfrak{A} is atomically separable (the corollary 582), with separable core (the theorem 537), and with join-closed core (theorem 534), binarily meet-closed by corollary 536.

We need to prove $Z(\mathfrak{A}) = \mathfrak{F}$.

Let $\mathcal{X} \in Z(\mathfrak{A})$. Then there exists $\mathcal{Y} \in Z(\mathfrak{A})$ such that $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = \perp^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Consequently there is $X \in \text{up}\mathcal{X}$ such that $X \sqcap^{\mathfrak{A}} \mathcal{Y} = \perp^{\mathfrak{A}}$; we also have $X \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Suppose $X \sqsupset \mathcal{X}$. Then there exists $a \in \text{atoms}^{\mathfrak{A}} X$ such that $a \notin \text{atoms}^{\mathfrak{A}} \mathcal{X}$. We can conclude also $a \notin \text{atoms}^{\mathfrak{A}} \mathcal{Y}$ (otherwise $X \sqcap^{\mathfrak{A}} \mathcal{Y} \neq \perp^{\mathfrak{A}}$). Thus $a \notin \text{atoms}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y})$ and consequently $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \neq \top^{\mathfrak{A}}$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{F}$.

Let now $X \in \mathfrak{F}$. Let $Y = \bar{X}$. We have $X \sqcap^{\mathfrak{A}} Y = \perp^{\mathfrak{A}}$ and $X \sqcup^{\mathfrak{A}} Y = \top^{\mathfrak{A}}$. Thus $X \sqcap^{\mathfrak{A}} Y = \bigsqcap^{\mathfrak{A}} \{X \sqcap^{\mathfrak{A}} Y\} = \perp^{\mathfrak{A}}$; $X \sqcap^{\mathfrak{A}} Y = X \sqcap^{\mathfrak{A}} Y = \top^{\mathfrak{A}}$. We have shown that $X \in Z(\mathfrak{A})$.

□

5.28. More Criteria

THEOREM 614. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a boolean lattice.
- 3°. For every $S \in \mathcal{P}\mathfrak{A}$ the condition $\exists \mathcal{F} \in \mathfrak{A} : S = \star\mathcal{F}$ is equivalent to conjunction of the following items:
 - (a) S is a free star on \mathfrak{A} ;
 - (b) S is filter-closed.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°.

\Rightarrow .

3°a. That $\perp^{\mathfrak{A}} \notin \star\mathcal{F}$ is obvious. For every $a, b \in \mathfrak{A}$

$$\begin{aligned}
 a \sqcup^{\mathfrak{A}} b \in \star\mathcal{F} &\Leftrightarrow \\
 (a \sqcup^{\mathfrak{A}} b) \sqcap^{\mathfrak{A}} \mathcal{F} \neq \perp^{\mathfrak{A}} &\Leftrightarrow \\
 (a \sqcap^{\mathfrak{A}} \mathcal{F}) \sqcup^{\mathfrak{A}} (b \sqcap^{\mathfrak{A}} \mathcal{F}) \neq \perp^{\mathfrak{A}} &\Leftrightarrow \\
 a \sqcap^{\mathfrak{A}} \mathcal{F} \neq \perp^{\mathfrak{A}} \vee b \sqcap^{\mathfrak{A}} \mathcal{F} \neq \perp^{\mathfrak{A}} &\Leftrightarrow \\
 a \in \star\mathcal{F} \vee b \in \star\mathcal{F} &
 \end{aligned}$$

(taken into account corollary 531). So $\star\mathcal{F}$ is a free star on \mathfrak{A} .

3°b. We have a filter base $T \subseteq S$ and need to prove that $\prod^{\mathfrak{A}} T \sqcap \mathcal{F} \neq \perp^{\mathfrak{A}}$. Because $\langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T$ is a generalized filter base, $\perp^{\mathfrak{A}} \in \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T \Leftrightarrow \prod^{\mathfrak{A}} \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T = \perp^{\mathfrak{A}} \Leftrightarrow \prod^{\mathfrak{A}} T \sqcap^{\mathfrak{A}} \mathcal{F} = \perp^{\mathfrak{A}}$. So it is left to prove $\perp^{\mathfrak{A}} \notin \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T$ what follows from $T \subseteq S$.

\Leftarrow . Let S be a free star on \mathfrak{A} . Then for every $A, B \in \mathfrak{F}$

$$\begin{aligned}
 A, B \in S \cap \mathfrak{F} &\Leftrightarrow \\
 A, B \in S &\Leftrightarrow \\
 A \sqcup^{\mathfrak{A}} B \in S &\Leftrightarrow \\
 A \sqcup^{\mathfrak{F}} B \in S &\Leftrightarrow \\
 A \sqcup^{\mathfrak{F}} B \in S \cap \mathfrak{F} &
 \end{aligned}$$

(taken into account the theorem 534). So $S \cap \mathfrak{F}$ is a free star on \mathfrak{F} .

Thus there exists $\mathcal{F} \in \mathfrak{A}$ such that $\partial\mathcal{F} = S \cap \mathfrak{F}$. We have up $\mathcal{X} \subseteq S \Leftrightarrow \mathcal{X} \in S$ (because S is filter-closed) for every $\mathcal{X} \in \mathfrak{A}$; then (taking

into account properties of generalized filter bases)

$$\begin{aligned}
& \mathcal{X} \in S \Leftrightarrow \\
& \text{up } \mathcal{X} \subseteq S \Leftrightarrow \\
& \text{up } \mathcal{X} \subseteq \partial \mathcal{F} \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{X} : X \cap^{\mathfrak{A}} \mathcal{F} \neq \perp^{\mathfrak{A}} \Leftrightarrow \\
& \perp^{\mathfrak{A}} \notin \langle \mathcal{F} \cap^{\mathfrak{A}} \rangle^* \text{up } \mathcal{X} \Leftrightarrow \\
& \prod^{\mathfrak{A}} \langle \mathcal{F} \cap^{\mathfrak{A}} \rangle^* \text{up } \mathcal{X} \neq \perp^{\mathfrak{A}} \Leftrightarrow \\
& \mathcal{F} \cap^{\mathfrak{A}} \prod^{\mathfrak{A}} \text{up } \mathcal{X} \neq \perp^{\mathfrak{A}} \Leftrightarrow \\
& \mathcal{F} \cap^{\mathfrak{A}} \mathcal{X} \neq \perp^{\mathfrak{A}} \Leftrightarrow \\
& \mathcal{X} \in \star \mathcal{F}.
\end{aligned}$$

□

5.29. Filters and a Special Sublattice

Remind that $Z(X)$ is the center of lattice X and Da is the lattice $\left\{ \begin{array}{l} x \in \mathfrak{A} \\ x \sqsubseteq a \end{array} \right\}$.

THEOREM 615. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a boolean lattice.
- 3°. Let $\mathcal{A} \in \mathfrak{A}$. Then for each $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \in Z(D\mathcal{A}) \Leftrightarrow \exists X \in \mathfrak{J} : \mathcal{X} = X \cap^{\mathfrak{A}} \mathcal{A}.$$

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°.

\Leftarrow . Let $\mathcal{X} = X \cap^{\mathfrak{A}} \mathcal{A}$ where $X \in \mathfrak{J}$. Let also $\mathcal{Y} = \overline{X} \cap^{\mathfrak{A}} \mathcal{A}$. Then

$$\mathcal{X} \cap^{\mathfrak{A}} \mathcal{Y} = X \cap^{\mathfrak{A}} \overline{X} \cap^{\mathfrak{A}} \mathcal{A} = (X \cap^{\mathfrak{J}} \overline{X}) \cap^{\mathfrak{A}} \mathcal{A} = \perp^{\mathfrak{A}} \cap^{\mathfrak{A}} \mathcal{A} = \perp^{\mathfrak{A}}$$

(used corollary 536) and

$$\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = (X \sqcup^{\mathfrak{A}} \overline{X}) \cap^{\mathfrak{A}} \mathcal{A} = (X \sqcup^{\mathfrak{J}} \overline{X}) \cap^{\mathfrak{A}} \mathcal{A} = \top^{\mathfrak{A}} \cap^{\mathfrak{A}} \mathcal{A} = \mathcal{A}$$

(used theorem 534 and corollary 531). So $\mathcal{X} \in Z(D\mathcal{A})$.

\Rightarrow . Let $\mathcal{X} \in Z(D\mathcal{A})$. Then there exists $\mathcal{Y} \in Z(D\mathcal{A})$ such that $\mathcal{X} \cap^{\mathfrak{A}} \mathcal{Y} = \perp^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \mathcal{A}$. Then (used theorem 537) there exists $X \in \text{up } \mathcal{X}$ such that $X \cap^{\mathfrak{A}} \mathcal{Y} = \perp^{\mathfrak{A}}$. We have

$$\mathcal{X} = \mathcal{X} \sqcup (X \cap^{\mathfrak{A}} \mathcal{Y}) = X \cap^{\mathfrak{A}} (\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) = X \cap^{\mathfrak{A}} \mathcal{A}.$$

□

THEOREM 616. The following is an implication tuple:

- 1°. $(\mathfrak{A}; \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}; \mathfrak{J})$ is a primary filtrator over a boolean lattice.
- 3°. $\mathfrak{F}(Z(D\mathcal{A}))$ is order-isomorphic to $D\mathcal{A}$ by the formulas
 - $\mathcal{Y} = \prod \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(Z(D\mathcal{A}))$;
 - $\mathcal{X} = \left\{ \begin{array}{l} \mathcal{F} \in Z(D\mathcal{A}) \\ \mathcal{F} \sqsupseteq \mathcal{Y} \end{array} \right\}$ for every $\mathcal{Y} \in D\mathcal{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

$2^\circ \Rightarrow 3^\circ$. We need to prove that the above formulas define a bijection, then it becomes evident that it's an order isomorphism (take into account that the order of filters is *reverse* to set inclusion).

First prove that these formulas describe correspondences between $\mathfrak{F}(Z(D\mathcal{A}))$ and $D\mathcal{A}$.

Let $\mathcal{X} \in \mathfrak{F}(Z(D\mathcal{A}))$. Consider $\mathcal{Y} = \prod \mathcal{X}$. Every element of \mathcal{X} is below \mathcal{A} , consequently $\mathcal{Y} \in D\mathcal{A}$.

Let now $\mathcal{Y} \in D\mathcal{A}$. Then $\left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \supseteq \mathcal{Y}} \right\}$ is a filter.

It remains to prove that these correspondences are mutually inverse.

Let $\mathcal{X} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \supseteq \mathcal{Y}_0} \right\}$ and $\mathcal{Y}_1 = \prod \mathcal{X}$ for some $\mathcal{Y}_0 \in D\mathcal{A}$.

$\mathcal{Y}_1 \supseteq \mathcal{Y}_0$ is obvious. By theorem 615 and the condition 2° we have $\mathcal{Y}_1 = \prod \mathcal{X} \sqsubseteq \prod^{\mathfrak{A}} \left\{ \frac{\mathcal{F} \cap \mathcal{A}}{\mathcal{F} \in \text{up } \mathcal{Y}_0} \right\} = \prod^{\mathfrak{A}} \left\{ \frac{\mathcal{F}}{\mathcal{F} \in \text{up } \mathcal{Y}_0} \right\} \cap \mathcal{A} = \mathcal{Y}_0 \cap \mathcal{A} = \mathcal{Y}_0$. So $\mathcal{Y}_1 = \mathcal{Y}_0$.

Let now $\mathcal{Y} = \prod \mathcal{X}_0$ and $\mathcal{X}_1 = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \supseteq \mathcal{Y}} \right\}$ for some $\mathcal{X}_0 \in \mathfrak{F}(Z(D\mathcal{A}))$.

$\mathcal{X}_1 = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \supseteq \prod \mathcal{X}_0} \right\} =$ (by generalized filter bases) $= \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\exists X \in \mathcal{X}_0 : \mathcal{F} \supseteq X} \right\} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \in \mathcal{X}_0} \right\} = \mathcal{X}_0$ because $\mathcal{F} \in \mathcal{X}_0 \Leftrightarrow \exists X \in \mathcal{X}_0 : \mathcal{F} \supseteq X$ if $\mathcal{F} \in Z(D\mathcal{A})$.

□

5.30. Distributivity of quasicomplements

THEOREM 617. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete boolean lattice.
- 4°. $(a \cap^{\mathfrak{A}} b)^* = (a \cap^{\mathfrak{A}} b)^+ = a^* \sqcup^{\mathfrak{A}} b^* = a^+ \sqcup^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. The filtrator $(\mathfrak{A}, \mathfrak{B})$ is filtered by the theorem 534. \mathfrak{A} is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{B})$ is with co-separable core by theorem 590. $(\mathfrak{A}, \mathfrak{B})$ is binarily meet-closed by proposition 536, with separable core by theorem 537.

$3^\circ \Rightarrow 4^\circ$. Theorem 595 apply. Also theorem 601 apply because every filtered filtrator is join-closed. So

$$(a \cap^{\mathfrak{A}} b)^* = (a \cap^{\mathfrak{A}} b)^+ = \overline{\text{Cor}(a \cap^{\mathfrak{A}} b)} = \overline{\text{Cor } a \cap^{\mathfrak{A}} \text{Cor } b} = \overline{\text{Cor } a} \sqcup^{\mathfrak{A}} \overline{\text{Cor } b} = a^+ \sqcup^{\mathfrak{A}} b^+ = a^* \sqcup^{\mathfrak{A}} b^*.$$

□

THEOREM 618. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered starrish down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete atomistic boolean lattice.
- 3°. $(a \sqcup^{\mathfrak{A}} b)^* = (a \sqcup^{\mathfrak{A}} b)^+ = a^* \cap^{\mathfrak{A}} b^* = a^+ \cap^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. $(\mathfrak{A}, \mathfrak{B})$ is a filtered (theorem 534), distributive (corollary 531) complete lattice filtrator (corollary 518), with binarily meet-closed core (corollary 536), with separable core (theorem 537), with co-separable core (theorem 590).

$$2^\circ \Rightarrow 3^\circ. (a \sqcup^{\mathfrak{A}} b)^+ = (a \sqcup^{\mathfrak{A}} b)^* = \overline{\text{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\text{Cor}' a \sqcup^{\mathfrak{A}} \text{Cor}' b} = \overline{\text{Cor}' a \sqcap^{\mathfrak{A}} \text{Cor}' b} = a^* \sqcap^{\mathfrak{A}} b^* = a^* \sqcap^{\mathfrak{A}} b^* = a^+ \sqcap^{\mathfrak{A}} b^+ \text{ (used theorems 594, 603, 595).}$$

□

THEOREM 619. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered complete lattice filtrator with down-aligned, binarily meet-closed, separable core which is a complete boolean lattice.
- 4°. $(a \sqcap^{\mathfrak{A}} b)^* = a^* \sqcup^{\mathfrak{A}} b^*$ for every $a, b \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. It is filtered by theorem 534. It is complete lattice filtrator by 518. It is with binarily meet-closed core (corollary 536), with separable core (theorem 537).

3° \Rightarrow 4°. It is join closed because it is filtered.

$$(a \sqcap^{\mathfrak{A}} b)^* = \overline{\text{Cor}'(a \sqcap^{\mathfrak{A}} b)} = \overline{\text{Cor}' a \sqcap^{\mathfrak{A}} \text{Cor}' b} = \overline{\text{Cor}' a \sqcup^{\mathfrak{A}} \text{Cor}' b} = a^* \sqcup^{\mathfrak{A}} b^* = a^* \sqcup^{\mathfrak{A}} b^*$$

(theorems 601, 594).

□

THEOREM 620. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered starrish down-aligned complete lattice filtrator with binarily meet-closed, separable core which is a complete atomistic boolean lattice.
- 3°. $(a \sqcup^{\mathfrak{A}} b)^* = a^* \sqcap^{\mathfrak{A}} b^*$ for every $a, b \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered (theorem 534), distributive (corollary 531) complete lattice filtrator (corollary 518), with binarily meet-closed core (corollary 536), with separable core (theorem 537).

2° \Rightarrow 3°.

$$(a \sqcup^{\mathfrak{A}} b)^* = \overline{\text{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\text{Cor}' a \sqcup^{\mathfrak{A}} \text{Cor}' b} = \overline{\text{Cor}' a \sqcap^{\mathfrak{A}} \text{Cor}' b} = a^* \sqcap^{\mathfrak{A}} b^* = a^* \sqcap^{\mathfrak{A}} b^*$$

(used theorems 594, 603).

□

THEOREM 621. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice.
- 4°. $(a \sqcap^{\mathfrak{A}} b)^+ = a^+ \sqcup^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. It is filtered by theorem 534, is a complete lattice by corollary 518, is with co-separable core by theorem 590.

$3^\circ \Rightarrow 4^\circ$.

$$(a \sqcap^{\mathfrak{A}} b)^+ = \overline{\text{Cor}(a \sqcap^{\mathfrak{A}} b)} = \overline{\text{Cor}'(a \sqcap^{\mathfrak{A}} b)} = \overline{\text{Cor}' a \sqcap^{\mathfrak{J}} \text{Cor}' b} = \\ \overline{\text{Cor}' a} \sqcup^{\mathfrak{J}} \overline{\text{Cor}' b} = \overline{\text{Cor}' a} \sqcup^{\mathfrak{A}} \overline{\text{Cor}' b} = a^+ \sqcup^{\mathfrak{A}} b^+$$

using theorems 592, 545, 601 and the fact that filtered filtrator is join-closed.

□

THEOREM 622. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a filtered down-aligned and up-aligned filtrator with binarily meet-closed core, with co-separable core \mathfrak{J} which is a complete atomistic boolean lattice and \mathfrak{A} is a complete starrish lattice.
- 3°. $(a \sqcup^{\mathfrak{A}} b)^+ = a^+ \sqcap^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$.

$$(a \sqcup^{\mathfrak{A}} b)^+ = \overline{\text{Cor}(a \sqcup^{\mathfrak{A}} b)} = \overline{\text{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\text{Cor}' a \sqcup^{\mathfrak{J}} \text{Cor}' b} = \\ \overline{\text{Cor}' a} \sqcap^{\mathfrak{J}} \overline{\text{Cor}' b} = \overline{\text{Cor}' a} \sqcap^{\mathfrak{A}} \overline{\text{Cor}' b} = a^+ \sqcap^{\mathfrak{A}} b^+$$

using theorems 592, 545, 603.

□

5.31. Complementive Filters and Factoring by a Filter

DEFINITION 623. Let \mathfrak{A} be a meet-semilattice and $\mathcal{A} \in \mathfrak{A}$. The relation \sim on \mathfrak{A} is defined by the formula

$$\forall X, Y \in \mathfrak{A} : (X \sim Y \Leftrightarrow X \sqcap^{\mathfrak{A}} \mathcal{A} = Y \sqcap^{\mathfrak{A}} \mathcal{A}).$$

PROPOSITION 624. The relation \sim is an equivalence relation.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Obvious.

□

DEFINITION 625. When $X, Y \in \mathfrak{J}$ and $\mathcal{A} \in \mathfrak{A}$ we define $X \sim Y \Leftrightarrow \uparrow X \sim \uparrow Y$.

THEOREM 626. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a distributive lattice.
- 3°. For every $\mathcal{A} \in \mathfrak{A}$ and $X, Y \in \mathfrak{J}$ we have

$$X \sim Y \Leftrightarrow \exists A \in \text{up } \mathcal{A} : X \sqcap^{\mathfrak{J}} A = Y \sqcap^{\mathfrak{J}} A.$$

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$.

$$\begin{aligned}
& \exists A \in \text{up } \mathcal{A} : X \sqcap^3 A = Y \sqcap^3 A \Leftrightarrow \text{(corollary 536)} \\
& \exists A \in \text{up } \mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \uparrow A = \uparrow Y \sqcap^{\mathfrak{A}} \uparrow A \Rightarrow \\
& \exists A \in \text{up } \mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \uparrow A \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \uparrow A \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \\
& \exists A \in \text{up } \mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \\
& \quad \uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \\
& \quad \quad \uparrow X \sim \uparrow Y \Leftrightarrow \\
& \quad \quad X \sim Y.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \\
& \quad \left\{ \frac{X \sqcap^3 A_0}{A_0 \in \mathcal{A}} \right\} = \left\{ \frac{Y \sqcap^3 A_1}{A_1 \in \mathcal{A}} \right\} \Rightarrow \\
& \quad \exists A_0, A_1 \in \text{up } \mathcal{A} : X \sqcap^3 A_0 = Y \sqcap^3 A_1 \Rightarrow \\
& \quad \exists A_0, A_1 \in \text{up } \mathcal{A} : X \sqcap^3 A_0 \sqcap^3 A_1 = Y \sqcap^3 A_0 \sqcap^3 A_1 \Rightarrow \\
& \quad \exists A \in \text{up } \mathcal{A} : Y \sqcap^3 A = X \sqcap^3 A.
\end{aligned}$$

□

PROPOSITION 627. The relation \sim is a congruence¹ for each of the following:

- 1°. a meet-semilattice \mathfrak{A} ;
- 2°. a distributive lattice \mathfrak{A} .

PROOF. Let $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ and $a_0 \sim a_1$ and $b_0 \sim b_1$.

1°. $a_0 \sqcap b_0 \sim a_1 \sqcap b_1$ because $(a_0 \sqcap b_0) \sqcap \mathcal{A} = a_0 \sqcap (b_0 \sqcap \mathcal{A}) = a_0 \sqcap (b_1 \sqcap \mathcal{A}) = b_1 \sqcap (a_0 \sqcap \mathcal{A}) = b_1 \sqcap (a_1 \sqcap \mathcal{A}) = (a_1 \sqcap b_1) \sqcap \mathcal{A}$.

2°. Taking the above into account, we need to prove only $a_0 \sqcup b_0 \sim a_1 \sqcup b_1$. We have

$$(a_0 \sqcup b_0) \sqcap \mathcal{A} = (a_0 \sqcap \mathcal{A}) \sqcup (b_0 \sqcap \mathcal{A}) = (a_1 \sqcap \mathcal{A}) \sqcup (b_1 \sqcap \mathcal{A}) = (a_1 \sqcup b_1) \sqcap \mathcal{A}.$$

□

DEFINITION 628. We will denote $A/(\sim) = A/((\sim) \cap A \times A)$ for a set A and an equivalence relation \sim on a set $B \supseteq A$. I will call \sim a congruence on A when $(\sim) \cap (A \times A)$ is a congruence on A .

THEOREM 629. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
- 3°. Let $\mathcal{A} \in \mathfrak{A}$. Consider the function $\gamma : Z(D\mathcal{A}) \rightarrow \mathfrak{Z}/\sim$ defined by the formula (for every $p \in Z(D\mathcal{A})$)

$$\gamma p = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = p} \right\}.$$

Then:

- (a) γ is a lattice isomorphism.
- (b) $\forall Q \in q : \gamma^{-1} q = Q \sqcap^{\mathfrak{A}} \mathcal{A}$ for every $q \in \mathfrak{Z}/\sim$.

PROOF.

1° \Rightarrow 2°. Obvious.

¹See Wikipedia for a definition of congruence.

$2^\circ \Rightarrow 3^\circ$. $\forall p \in Z(D\mathcal{A}) : \gamma p \neq \emptyset$ because of theorem 615. Thus it is easy to see that $\gamma p \in \mathfrak{Z}/\sim$ and that γ is an injection.

Let's prove that γ is a lattice homomorphism:

$$\begin{aligned} \gamma(p_0 \sqcap^{\mathfrak{A}} p_1) &= \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \sqcap^{\mathfrak{A}} p_1} \right\}; \\ \gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1 &= \\ &= \left\{ \frac{X_0 \in \mathfrak{Z}}{X_0 \sqcap^{\mathfrak{A}} \mathcal{A} = p_0} \right\} \sqcap^{\mathfrak{Z}/\sim} \left\{ \frac{X_1 \in \mathfrak{Z}}{X_1 \sqcap^{\mathfrak{A}} \mathcal{A} = p_1} \right\} = \\ &= \left\{ \frac{X_0 \sqcap^{\mathfrak{A}} X_1}{X_0, X_1 \in \mathfrak{Z}, X_0 \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \wedge X_1 \sqcap^{\mathfrak{A}} \mathcal{A} = p_1} \right\} \subseteq \\ &= \left\{ \frac{X' \in \mathfrak{Z}}{X' \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \sqcap^{\mathfrak{A}} p_1} \right\} = \\ &= \gamma(p_0 \sqcap^{\mathfrak{A}} p_1). \end{aligned}$$

Because $\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1$ and $\gamma(p_0 \sqcap^{\mathfrak{A}} p_1)$ are equivalence classes, thus follows $\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1 = \gamma(p_0 \sqcap^{\mathfrak{A}} p_1)$.

To finish the proof it is enough to show that $\forall Q \in q : q = \gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A})$ for every $q \in \mathfrak{Z}/\sim$. (From this it follows that γ is surjective because q is not empty and thus $\exists Q \in q : q = \gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A})$.) Really,

$$\gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A}) = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = Q \sqcap^{\mathfrak{A}} \mathcal{A}} \right\} = [Q] = q.$$

□

This isomorphism is useful in both directions to reveal properties of both lattices $Z(D\mathcal{A})$ and $q \in \mathfrak{Z}/\sim$.

COROLLARY 630. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
- 3°. \mathfrak{Z}/\sim is a boolean lattice

PROOF. Because $Z(D\mathcal{A})$ is a boolean lattice (theorem 98). □

5.32. Pseudodifference of filters

PROPOSITION 631. The following is an implications tuple:

- 1°. \mathfrak{A} is a lattice of filters on a set.
- 2°. \mathfrak{A} is a lattice of filters over a boolean lattice.
- 3°. \mathfrak{A} is an atomistic co-brouwerian lattice.
- 4°. For every $a, b \in \mathfrak{A}$ the following expressions are always equal:

- (a) $a \setminus^* b = \sqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);
- (b) $a \# b = \sqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \perp} \right\}$ (second quasidifference of a and b);
- (c) $\sqcup(\text{atoms } a \setminus \text{atoms } b)$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. By corollary 531 and theorem 581.

$3^\circ \Rightarrow 4^\circ$. Theorem 247.

CONJECTURE 632. $a \setminus^* b = a \# b$ for arbitrary filters a, b on powersets is not provable in ZF (without axiom of choice). □

5.33. Function spaces of posets

DEFINITION 633. Let \mathfrak{A}_i be a family of posets indexed by some set $\text{dom } \mathfrak{A}$. We will define order of indexed families of elements of posets by the formula

$$a \sqsubseteq b \Leftrightarrow \forall i \in \text{dom } \mathfrak{A} : a_i \sqsubseteq b_i.$$

I will call this new poset $\prod \mathfrak{A}$ *the function space* of posets and the above order *product order*.

PROPOSITION 634. The function space for posets is also a poset.

PROOF.

Reflexivity. Obvious.

Antisymmetry. Obvious.

Transitivity. Obvious. □

OBVIOUS 635. \mathfrak{A} has least element iff each \mathfrak{A}_i has a least element. In this case

$$\perp \prod \mathfrak{A} = \prod_{i \in \text{dom } \mathfrak{A}} \perp^{\mathfrak{A}_i}.$$

PROPOSITION 636. $a \not\leq b \Leftrightarrow \exists i \in \text{dom } \mathfrak{A} : a_i \not\leq b_i$ for every $a, b \in \prod \mathfrak{A}$ if every \mathfrak{A}_i has least element.

PROOF. If $\text{dom } \mathfrak{A} = \emptyset$, then $a = b = \perp$, $a \asymp b$ and thus the theorem statement holds. Assume $\text{dom } \mathfrak{A} \neq \emptyset$.

$$\begin{aligned} a \not\leq b &\Leftrightarrow \\ \exists c \in \prod \mathfrak{A} \setminus \{\perp \prod \mathfrak{A}\} : (c \sqsubseteq a \wedge c \not\sqsubseteq b) &\Leftrightarrow \\ \exists c \in \prod \mathfrak{A} \setminus \{\perp \prod \mathfrak{A}\} \forall i \in \text{dom } \mathfrak{A} : (c_i \sqsubseteq a_i \wedge c_i \not\sqsubseteq b_i) &\Leftrightarrow \\ \text{(for the reverse implication take } c_j = \perp^{\mathfrak{A}_j} \text{ for } i \neq j) & \\ \exists i \in \text{dom } \mathfrak{A}, c \in \mathfrak{A}_i \setminus \{\perp^{\mathfrak{A}_i}\} : (c \sqsubseteq a_i \wedge c \not\sqsubseteq b_i) &\Leftrightarrow \\ \exists i \in \text{dom } \mathfrak{A} : a_i \not\leq b_i. & \end{aligned}$$

□

PROPOSITION 637.

1°. If \mathfrak{A}_i are join-semilattices then \mathfrak{A} is a join-semilattice and

$$A \sqcup B = \lambda i \in \text{dom } \mathfrak{A} : A_i \sqcup B_i. \quad (2)$$

2°. If \mathfrak{A}_i are meet-semilattices then \mathfrak{A} is a meet-semilattice and

$$A \sqcap B = \lambda i \in \text{dom } \mathfrak{A} : A_i \sqcap B_i.$$

PROOF. It is enough to prove the formula (2).

It's obvious that $\lambda i \in \text{dom } \mathfrak{A} : A_i \sqcup B_i \sqsupseteq A, B$.

Let $C \sqsupseteq A, B$. Then (for every $i \in \text{dom } \mathfrak{A}$) $C_i \sqsupseteq A_i$ and $C_i \sqsupseteq B_i$. Thus $C_i \sqsupseteq A_i \sqcup B_i$ that is $C \sqsupseteq \lambda i \in \text{dom } \mathfrak{A} : A_i \sqcup B_i$. □

COROLLARY 638. If \mathfrak{A}_i are lattices then $\prod \mathfrak{A}$ is a lattice.

OBVIOUS 639. If \mathfrak{A}_i are distributive lattices then $\prod \mathfrak{A}$ is a distributive lattice.

PROPOSITION 640. If \mathfrak{A}_i are boolean lattices then $\prod \mathfrak{A}$ is a boolean lattice.

PROOF. We need to prove only that every element $a \in \prod \mathfrak{A}$ has a complement. But this complement is evidently $\lambda i \in \text{dom } \mathfrak{A} : \bar{a}_i$. □

PROPOSITION 641. If every \mathfrak{A}_i is a poset then for every $S \in \mathcal{P} \prod \mathfrak{A}$

- 1°. $\bigsqcup S = \lambda i \in \text{dom } \mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever every $\bigsqcup_{x \in S} x_i$ exists;
 2°. $\prod S = \lambda i \in \text{dom } \mathfrak{A} : \prod_{x \in S} x_i$ whenever every $\prod_{x \in S} x_i$ exists.

PROOF. It's enough to prove the first formula.

$(\lambda i \in \text{dom } \mathfrak{A} : \bigsqcup_{x \in S} x_i)_i = \bigsqcup_{x \in S} x_i \supseteq x_i$ for every $x \in S$ and $i \in \text{dom } \mathfrak{A}$.

Let $y \supseteq x$ for every $x \in S$. Then $y_i \supseteq x_i$ for every $i \in \text{dom } \mathfrak{A}$ and thus $y_i \supseteq \bigsqcup_{x \in S} x_i = (\lambda i \in \text{dom } \mathfrak{A} : \bigsqcup_{x \in S} x_i)_i$ that is $y \supseteq \lambda i \in \text{dom } \mathfrak{A} : \bigsqcup_{x \in S} x_i$.

Thus $\bigsqcup S = \lambda i \in \text{dom } \mathfrak{A} : \bigsqcup_{x \in S} x_i$ by the definition of join. \square

COROLLARY 642. If \mathfrak{A}_i are posets then for every $S \in \mathcal{P} \prod \mathfrak{A}$

- 1°. $\bigsqcup S = \lambda i \in \text{dom } \mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever $\bigsqcup S$ exists;
 2°. $\prod S = \lambda i \in \text{dom } \mathfrak{A} : \prod_{x \in S} x_i$ whenever $\prod S$ exists.

PROOF. It is enough to prove that (for every i) $\bigsqcup_{x \in S} x_i$ exists whenever $\bigsqcup S$ exists.

Fix $i \in \text{dom } \mathfrak{A}$.

Take $y_i = (\bigsqcup S)_i$ and let prove that y_i is the least upper bound of $\{\frac{x_i}{x \in S}\}$.

y_i is it's upper bound because $\bigsqcup S \supseteq x$ and thus $(\bigsqcup S)_i \supseteq x_i$ for every $x \in S$.

Let $x \in S$ and for some $t \in \mathfrak{A}_i$

$$T(t) = \lambda j \in \text{dom } \mathfrak{A} : \begin{cases} t & \text{if } i = j \\ x_j & \text{if } i \neq j. \end{cases}$$

Let $t \supseteq x_i$. Then $T(t) \supseteq x$ for every $x \in S$. So $T(t) \supseteq \bigsqcup S$ and consequently $t = T(t)_i \supseteq y_i$.

So y_i is the least upper bound of $\{\frac{x_i}{x \in S}\}$. \square

COROLLARY 643. If \mathfrak{A}_i are complete lattices then \mathfrak{A} is a complete lattice.

OBVIOUS 644. If \mathfrak{A}_i are complete (co-)brouwerian lattices then \mathfrak{A} is a (co-)brouwerian lattice.

PROPOSITION 645. If each \mathfrak{A}_i is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

PROOF. Let $a \neq b$. Then $\exists i \in \text{dom } \mathfrak{A} : a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in \text{dom } \mathfrak{A} : \begin{cases} x & \text{if } j = i; \\ \perp^{\mathfrak{A}_j} & \text{if } j \neq i. \end{cases}$ Then $y \not\asymp a$ and $y \asymp b$. \square

OBVIOUS 646. If every \mathfrak{A}_i is a poset with least element, then the set of atoms of $\prod \mathfrak{A}$ is

$$\left\{ \frac{\lambda i \in \text{dom } \mathfrak{A} : \left(\begin{cases} a & \text{if } i = k; \\ \perp^{\mathfrak{A}_i} & \text{if } i \neq k \end{cases} \right)}{k \in \text{dom } \mathfrak{A}, a \in \text{atoms}^{\mathfrak{A}_k}} \right\}.$$

PROPOSITION 647. If every \mathfrak{A}_i is an atomistic poset with least element, then $\prod \mathfrak{A}$ is an atomistic poset.

PROOF. $x_i = \sqcup \text{atoms } x_i$ for every $x_i \in \mathfrak{A}_i$. Thus

$$\begin{aligned} x = \lambda i \in \text{dom } x : x_i = \lambda i \in \text{dom } x : \sqcup \text{atoms } x_i &= \\ \sqcup_{i \in \text{dom } x} \lambda j \in \text{dom } x : \begin{cases} x_i & \text{if } j = i \\ \perp^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} &= \\ \sqcup_{i \in \text{dom } x} \lambda j \in \text{dom } x : \begin{cases} \sqcup \text{atoms } x_i & \text{if } j = i \\ \perp^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} &= \\ \sqcup_{i \in \text{dom } x} \sqcup_{q \in \text{atoms } x_i} \lambda j \in \text{dom } x : \begin{cases} q & \text{if } j = i \\ \perp^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} \end{aligned}$$

Thus x is a join of atoms of $\prod \mathfrak{A}$. \square

COROLLARY 648. If \mathfrak{A}_i are atomistic posets with least elements, then $\prod \mathfrak{A}$ is atomically separable.

PROOF. Proposition 230. \square

PROPOSITION 649. Let $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a family of filtrators. Then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtrator.

PROOF. We need to prove that $\prod \mathfrak{Z}$ is a sub-poset of $\prod \mathfrak{A}$. First $\prod \mathfrak{Z} \subseteq \prod \mathfrak{A}$ because $\mathfrak{Z}_i \subseteq \mathfrak{A}_i$ for each $i \in n$.

Let $A, B \in \prod \mathfrak{Z}$ and $A \sqsubseteq \prod \mathfrak{Z} B$. Then $\forall i \in n : A_i \sqsubseteq^{\mathfrak{Z}_i} B_i$; consequently $\forall i \in n : A_i \sqsubseteq^{\mathfrak{A}_i} B_i$ that is $A \sqsubseteq \prod \mathfrak{A} B$. \square

PROPOSITION 650. Let $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a family of filtrators.

- 1°. The filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is (binarily) join-closed if every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is (binarily) join-closed.
- 2°. The filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is (binarily) meet-closed if every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is (binarily) meet-closed.

PROOF. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be binarily join-closed. Let $A, B \in \prod \mathfrak{Z}$ and $A \sqcup \prod \mathfrak{Z} B$ exist. Then (by corollary 642)

$$A \sqcup \prod \mathfrak{Z} B = \lambda i \in n : A_i \sqcup^{\mathfrak{Z}_i} B_i = \lambda i \in n : A_i \sqcup^{\mathfrak{A}_i} B_i = A \sqcup \prod \mathfrak{A} B.$$

Let now every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be join-closed. Let $S \in \mathscr{P} \prod \mathfrak{Z}$ and $\sqcup \prod \mathfrak{Z} S$ exist. Then (by corollary 642)

$$\sqcup \prod \mathfrak{Z} S = \lambda i \in \text{dom } \mathfrak{A} : \sqcup^{\mathfrak{Z}_i} \left\{ \frac{x_i}{x \in S} \right\} = \lambda i \in \text{dom } \mathfrak{A} : \sqcup^{\mathfrak{A}_i} \left\{ \frac{x_i}{x \in S} \right\} = \sqcup \prod \mathfrak{A} S.$$

The rest follows from symmetry. \square

PROPOSITION 651. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ where $i \in n$ (for some index set n) is a down-aligned filtrator with separable core then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is with separable core.

PROOF. Let $a \neq b$. Then $\exists i \in n : a_i \neq b_i$. So $\exists x \in \mathfrak{Z}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in n : \begin{cases} x & \text{if } j = i \\ \perp^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases}$. Then we have $y \not\asymp a$ and $y \asymp b$ and $y \in \mathfrak{Z}$. \square

PROPOSITION 652. Let every \mathfrak{A}_i be a bounded lattice. Every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a central filtrator iff $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a central filtrator.

PROOF.

$$\begin{aligned}
x \in Z\left(\prod \mathfrak{A}\right) &\Leftrightarrow \\
\exists y \in \prod \mathfrak{A} : (x \sqcap y = \perp \prod \mathfrak{A} \wedge x \sqcup y = \top \prod \mathfrak{A}) &\Leftrightarrow \\
\exists y \in \prod \mathfrak{A} \forall i \in \text{dom } \mathfrak{A} : (x_i \sqcap y_i = \perp \mathfrak{A}_i \wedge x_i \sqcup y_i = \top \mathfrak{A}_i) &\Leftrightarrow \\
\forall i \in \text{dom } \mathfrak{A} \exists y \in \mathfrak{A}_i : (x_i \sqcap y = \perp \mathfrak{A}_i \wedge x_i \sqcup y = \top \mathfrak{A}_i) &\Leftrightarrow \\
\forall i \in \text{dom } \mathfrak{A} : x_i \in Z(\mathfrak{A}_i). &
\end{aligned}$$

So

$$\begin{aligned}
Z\left(\prod \mathfrak{A}\right) = \prod \mathfrak{Z} &\Leftrightarrow \prod_{i \in \text{dom } \mathfrak{A}} Z(\mathfrak{A}_i) = \prod \mathfrak{Z} \Leftrightarrow \\
&\text{(because every } \mathfrak{Z}_i \text{ is nonempty)} \Leftrightarrow \forall i \in \text{dom } \mathfrak{A} : Z(\mathfrak{A}_i) = \mathfrak{Z}_i.
\end{aligned}$$

□

PROPOSITION 653. For every element a of a product filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$:

- 1°. $\text{up } a = \prod_{i \in \text{dom } a} \text{up } a_i$;
- 2°. $\text{down } a = \prod_{i \in \text{dom } a} \text{down } a_i$.

PROOF. We will prove only the first as the second is dual.

$$\begin{aligned}
\text{up } a = \left\{ \frac{c \in \prod \mathfrak{Z}}{c \supseteq a} \right\} &= \left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \text{dom } a : c_i \supseteq a_i} \right\} = \\
&= \left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \text{dom } a : c_i \in \text{up } a_i} \right\} = \prod_{i \in \text{dom } a} \text{up } a_i.
\end{aligned}$$

□

PROPOSITION 654. If every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ is a prefiltered filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a prefiltered filtrator.

PROOF. Let $a, b \in \prod \mathfrak{A}$ and $a \neq b$. Then there exists $i \in n$ such that $a_i \neq b_i$ and so $\text{up } a_i \neq \text{up } b_i$. Consequently $\prod_{i \in \text{dom } a} \text{up } a_i \neq \prod_{i \in \text{dom } a} \text{up } b_i$ that is $\text{up } a \neq \text{up } b$. □

PROPOSITION 655. Let every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a filtered filtrator with $\text{up } x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). Then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtered filtrator.

PROOF. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be a filtered filtrator. Let $\text{up } a \supseteq \text{up } b$ for some $a, b \in \prod \mathfrak{A}$. Then $\prod_{i \in \text{dom } a} \text{up } a_i \supseteq \prod_{i \in \text{dom } a} \text{up } b_i$ and consequently (taking into account that $\text{up } x \neq \emptyset$ for every $x \in \mathfrak{A}_i$) $\text{up } a_i \supseteq \text{up } b_i$ for every $i \in n$. Then $\forall i \in n : a_i \sqsubseteq b_i$ that is $a \sqsubseteq b$. □

PROPOSITION 656. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be filtrators and each \mathfrak{Z}_i be a complete lattice with $\text{up } x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). For $a \in \prod \mathfrak{A}$:

- 1°. $\text{Cor } a = \lambda i \in \text{dom } a : \text{Cor } a_i$;
- 2°. $\text{Cor}' a = \lambda i \in \text{dom } a : \text{Cor}' a_i$.

PROOF. We will prove only the first, because the second is dual.

$$\begin{aligned}
\text{Cor } a &= \\
&\prod \mathfrak{z} \\
&\prod \uparrow \text{up } a = \\
\lambda i \in \text{dom } a : \prod \left\{ \frac{x_i}{x \in \text{up } a} \right\} &= (\text{up } x \neq \emptyset \text{ taken into account}) \\
\lambda i \in \text{dom } a : \prod \left\{ \frac{x}{x \in \text{up } a_i} \right\} &= \\
\lambda i \in \text{dom } a : \prod \uparrow \text{up } a_i &= \\
\lambda i \in \text{dom } a : \text{Cor } a_i. &
\end{aligned}$$

□

PROPOSITION 657. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a filtrator with (co)separable core and each \mathfrak{A}_i has a least (greatest) element, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtrator with (co)separable core.

PROOF. We will prove only for separable core, as co-separable core is dual.

$$\begin{aligned}
x \asymp \prod \mathfrak{A} y &\Leftrightarrow \\
(\text{used the fact that } \mathfrak{A}_i \text{ has a least element}) & \\
\forall i \in \text{dom } \mathfrak{A} : x_i \asymp^{\mathfrak{A}_i} y_i &\Rightarrow \\
\forall i \in \text{dom } \mathfrak{A} \exists X \in \text{up } x_i : X \asymp^{\mathfrak{A}_i} y_i &\Leftrightarrow \\
\exists X \in \text{up } x \forall i \in \text{dom } \mathfrak{A} : X_i \asymp^{\mathfrak{A}_i} y_i &\Leftrightarrow \\
\exists X \in \text{up } x : X \asymp \prod \mathfrak{A} y &
\end{aligned}$$

for every $x, y \in \prod \mathfrak{A}$.

□

OBVIOUS 658.

- 1°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a down-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a down-aligned filtrator.
- 2°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is an up-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is an up-aligned filtrator.

OBVIOUS 659.

- 1°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a weakly down-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a weakly down-aligned filtrator.
- 2°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a weakly up-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a weakly up-aligned filtrator.

PROPOSITION 660. If every b_i is subtractive from a_i where a and b are n -indexed families of elements of distributive lattices with least elements (where n is an index set), then $a \setminus b = \lambda i \in n : a_i \setminus b_i$.

PROOF. We need to prove $(\lambda i \in n : a_i \setminus b_i) \sqcap b = \perp$ and $a \sqcup b = b \sqcup (\lambda i \in n : a_i \setminus b_i)$. Really

$$\begin{aligned}
(\lambda i \in n : a_i \setminus b_i) \sqcap b &= \lambda i \in n : (a_i \setminus b_i) \sqcap b_i = \perp; \\
b \sqcup (\lambda i \in n : a_i \setminus b_i) &= \lambda i \in n : b_i \sqcup (a_i \setminus b_i) = \lambda i \in n : b_i \sqcup a_i = a \sqcup b.
\end{aligned}$$

□

PROPOSITION 661. If every \mathfrak{A}_i is a distributive lattice, then $a \setminus^* b = \lambda i \in \text{dom } \mathfrak{A} : a_i \setminus^* b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \setminus^* b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \text{dom } \mathfrak{A} : a_i \setminus^* b_i = \prod \left\{ \frac{z \in \prod \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$.

To prove it is enough to show $a_i \setminus^* b_i = \prod \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\}$ that is $a_i \setminus^* b_i = \prod \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ because $z' \in \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\} \Leftrightarrow z' \in \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ (for the reverse implication take $z_j = a_i$ for $j \neq i$), but $a_i \setminus^* b_i = \prod \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ is true by definition. \square

PROPOSITION 662. If every \mathfrak{A}_i is a distributive lattice with least element, then $a \# b = \lambda i \in \text{dom } \mathfrak{A} : a_i \# b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \# b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \text{dom } \mathfrak{A} : a_i \# b_i = \sqcup \left\{ \frac{z \in \prod \mathfrak{A}}{z \sqsubseteq a \wedge z \succ b} \right\}$.

To prove it is enough to show $a_i \# b_i = \sqcup \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, z \sqsubseteq a \wedge z \succ b} \right\}$ that is $a_i \# b_i = \sqcup \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, z_i \sqsubseteq a_i \wedge \forall j \in \text{dom } \mathfrak{A} : z_j \succ b_j} \right\}$ (take $z_j = \perp^{\mathfrak{A}_j}$ for $j \neq i$) what is true by definition. \square

PROPOSITION 663. Let every \mathfrak{A}_i be a poset with least element and a_i^* is defined. Then $a^* = \lambda i \in \text{dom } \mathfrak{A} : a_i^*$.

PROOF. We need to prove that $\lambda i \in \text{dom } \mathfrak{A} : a_i^* = \sqcup \left\{ \frac{c \in \prod \mathfrak{A}}{c \succ a} \right\}$. To prove this it is enough to show that $a_i^* = \sqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, c \succ a} \right\}$ that is $a_i^* = \sqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, \forall j \in \text{dom } \mathfrak{A} : c_j \succ a_j} \right\}$ that is $a_i^* = \sqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, c_i \succ a_i} \right\}$ (take $c_j = \perp^{\mathfrak{A}_j}$ for $j \neq i$) that is $a_i^* = \sqcup \left\{ \frac{c \in \mathfrak{A}_i}{c \succ a_i} \right\}$ what is true by definition. \square

COROLLARY 664. Let every \mathfrak{A}_i be a poset with greatest element and a_i^+ is defined. Then $a^+ = \lambda i \in \text{dom } \mathfrak{A} : a_i^+$.

PROOF. By duality. \square

5.34. Filters on a Set

In this section we will fix a powerset filtrator $(\mathfrak{A}, \mathfrak{F}) = (\mathfrak{A}, \mathcal{P}\mathfrak{A})$ for some set \mathfrak{A} .

The consideration below is about filters on a set \mathfrak{A} , but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorphic to algebras of sets on some set \mathfrak{A} .

5.34.1. Fréchet Filter.

DEFINITION 665. $\Omega = \left\{ \frac{\mathfrak{A} \setminus X}{X \text{ is a finite subset of } \mathfrak{A}} \right\}$ is called either *Fréchet filter* or *cofinite filter*.

It is trivial that Fréchet filter is a filter.

PROPOSITION 666. $\text{Cor } \Omega = \perp^3; \bigcap \Omega = \emptyset$.

PROOF. This can be deduced from the formula $\forall \alpha \in \mathfrak{A} \exists X \in \Omega : \alpha \notin X$. \square

THEOREM 667. $\max \left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\text{Cor } \mathcal{X} = \perp^3} \right\} = \max \left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\bigcap \mathcal{X} = \emptyset} \right\} = \Omega$.

PROOF. Due the last proposition, it is enough to show that $\text{Cor } \mathcal{X} = \perp^3 \Rightarrow \mathcal{X} \sqsubseteq \Omega$ for every filter \mathcal{X} .

Let $\text{Cor } \mathcal{X} = \perp^3$ for some filter \mathcal{X} . Let $X \in \Omega$. We need to prove that $X \in \mathcal{X}$.

$X = \mathfrak{U} \setminus \{\alpha_0, \dots, \alpha_n\}$. $\mathfrak{U} \setminus \{\alpha_i\} \in \mathcal{X}$ because otherwise $\alpha_i \in \uparrow^{-1} \text{Cor } \mathcal{X}$. So $X \in \mathcal{X}$. \square

THEOREM 668. $\Omega = \bigsqcup^{\mathfrak{A}} \left\{ \frac{x}{x \text{ is a non-trivial ultrafilter}} \right\}$.

PROOF. It follows from the facts that $\text{Cor } x = \perp^3$ for every non-trivial ultrafilter x , that \mathfrak{A} is an atomistic lattice, and the previous theorem. \square

THEOREM 669. Cor is the lower adjoint of $\Omega \sqcup^{\mathfrak{A}} -$.

PROOF. Because both Cor and $\Omega \sqcup^{\mathfrak{A}} -$ are monotone, it is enough (theorem 126) to prove (for every filters \mathcal{X} and \mathcal{Y})

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{A}} \text{Cor } \mathcal{X} \quad \text{and} \quad \text{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$$\begin{aligned} \text{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) &= \text{Cor } \Omega \sqcup^{\mathfrak{A}} \text{Cor } \mathcal{Y} = \perp^3 \sqcup^{\mathfrak{A}} \text{Cor } \mathcal{Y} = \text{Cor } \mathcal{Y} \sqsubseteq \mathcal{Y}. \\ \Omega \sqcup^{\mathfrak{A}} \text{Cor } \mathcal{X} &\supseteq \text{Edg } \mathcal{X} \sqcup^{\mathfrak{A}} \text{Cor } \mathcal{X} = \mathcal{X}. \end{aligned}$$

\square

COROLLARY 670. $\text{Cor } \mathcal{X} = \mathcal{X} \setminus^* \Omega$ for every filter on a set.

PROOF. By theorem 154. \square

COROLLARY 671. $\text{Cor } \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} (\text{Cor})^* S$ for any set S of filters on a powerset.

This corollary can be rewritten in elementary terms and proved elementarily:

PROPOSITION 672. $\bigcap \bigcap S = \bigcup_{F \in S} \bigcap F$ for a set S of filters on some set.

PROOF. (by ANDREAS BLASS) The \supseteq direction is rather formal. Consider any one of the sets being intersected on the left side, i.e., any set X that is in all the filters in S , and consider any of the sets being unioned (that's not a word, but you know what I mean) on the right, i.e., $\bigcap F$ for some $F \in S$. Then, since $X \in F$, we have $\bigcap F \subseteq X$. Taking the union over all $F \in S$ (while keeping X fixed), we get that the right side of your equation is $\subseteq X$. Since that's true for all $X \in \bigcap S$, we infer that the right side is a subset of the left side. (This argument seems to work in much greater generality; you just need that the relevant infima (in place of intersections) exist in your poset.)

For the \subseteq direction, consider any element $x \in \bigcap \bigcap S$, and suppose, toward a contradiction, that it is not an element of the union on the right side of your equation. So, for each $F \in S$, we have $x \notin \bigcap F$, and therefore we can find a set $A_F \in F$ with $x \notin A_F$. Let $B = \bigcup_{F \in S} A_F$ and notice that $B \in F$ for every $F \in S$ (because $B \supseteq A_F$). So $B \in \bigcap S$. But, by choice of the A_F 's, we have $x \notin B$, contrary to the assumption that $x \in \bigcap \bigcap S$. \square

PROPOSITION 673. $\partial\Omega(U)$ is the set of infinite subsets of U .

PROOF. $\partial\Omega(U) = \neg \langle \neg \rangle^* \Omega(U)$.

$\langle \neg \rangle^* \Omega$ is the set of finite subsets of U . Thus $\neg \langle \neg \rangle^* \Omega(U)$ is the set of infinite subsets of U . \square

5.34.2. Number of Filters on a Set.

DEFINITION 674. A collection Y of sets has finite intersection property iff intersection of any finite subcollection of Y is non-empty.

The following was borrowed from [7]. Thanks to ANDREAS BLASS for email support about his proof.

LEMMA 675. (by HAUSDORFF) For an infinite set X there is a family \mathcal{F} of $2^{\text{card } X}$ many subsets of X such that given any disjoint finite subfamilies \mathcal{A}, \mathcal{B} , the intersection of sets in \mathcal{A} and complements of sets in \mathcal{B} is nonempty.

PROOF. Let

$$X' = \left\{ \frac{(P, Q)}{P \in \mathcal{P}X \text{ is finite, } Q \in \mathcal{P}\mathcal{P}P} \right\}.$$

It's easy to show that $\text{card } X' = \text{card } X$. So it is enough to show this for X' instead of X . Let

$$\mathcal{F} = \left\{ \frac{\left\{ \frac{(P, Q) \in X'}{Y \cap P \in Q} \right\}}{Y \in \mathcal{P}X} \right\}.$$

To finish the proof we show that for every disjoint finite $Y_+ \in \mathcal{P}\mathcal{P}X$ and finite $Y_- \in \mathcal{P}\mathcal{P}X$ there exist $(P, Q) \in X'$ such that

$$\forall Y \in Y_+ : (P, Q) \in \left\{ \frac{(P, Q) \in X'}{Y \cap P \in Q} \right\} \quad \text{and} \quad \forall Y \in Y_- : (P, Q) \notin \left\{ \frac{(P, Q) \in X'}{Y \cap P \in Q} \right\}$$

what is equivalent to existence $(P, Q) \in X'$ such that

$$\forall Y \in Y_+ : Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_- : Y \cap P \notin Q.$$

For existence of this (P, Q) , it is enough existence of P such that intersections $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$.

Really, for each pair of distinct $Y_0, Y_1 \in Y_+ \cup Y_-$ choose a point which lies in one of the sets Y_0, Y_1 and not in an other, and call the set of such points P . Then $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$. \square

COROLLARY 676. For an infinite set X there is a family \mathcal{F} of $2^{\text{card } X}$ many subsets of X such that for arbitrary disjoint subfamilies \mathcal{A} and \mathcal{B} the set $\mathcal{A} \cup \left\{ \frac{X \setminus A}{A \in \mathcal{B}} \right\}$ has finite intersection property.

THEOREM 677. Let X be a set. The number of ultrafilters on X is $2^{2^{\text{card } X}}$ if X is infinite and $\text{card } X$ if X is finite.

PROOF. The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let X be infinite. From the lemma, there exists a family \mathcal{F} of $2^{\text{card } X}$ many subsets of X such that for every $\mathcal{G} \in \mathcal{P}\mathcal{F}$ we have $\Phi(\mathcal{F}, \mathcal{G}) = \prod^{\mathfrak{A}} \mathcal{G} \cap \prod^{\mathfrak{A}} \left\{ \frac{X \setminus A}{A \in \mathcal{F} \setminus \mathcal{G}} \right\} \neq \perp^{\mathfrak{A}(X)}$.

This filter contains all sets from \mathcal{G} and does not contain any sets from $\mathcal{F} \setminus \mathcal{G}$. So for every suitable pairs $(\mathcal{F}_0, \mathcal{G}_0)$ and $(\mathcal{F}_1, \mathcal{G}_1)$ there is $A \in \Phi(\mathcal{F}_0, \mathcal{G}_0)$ such that $\bar{A} \in \Phi(\mathcal{F}_1, \mathcal{G}_1)$. Consequently all filters $\Phi(\mathcal{F}, \mathcal{G})$ are disjoint. So for every pair $(\mathcal{F}, \mathcal{G})$ where $\mathcal{G} \in \mathcal{P}\mathcal{F}$ there exist a distinct ultrafilter under $\Phi(\mathcal{F}, \mathcal{G})$, but the number of such pairs $(\mathcal{F}, \mathcal{G})$ is $2^{2^{\text{card } X}}$. Obviously the number of all filters is not above $2^{2^{\text{card } X}}$. \square

COROLLARY 678. The number of filters on \mathfrak{U} is $2^{2^{\text{card } X}}$ if \mathfrak{U} is infinite and $2^{\text{card } \mathfrak{U}}$ if \mathfrak{U} is finite.

PROOF. The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than $2^{2^{\text{card } \mathfrak{U}}}$ collections of sets on \mathfrak{U} . \square

5.35. Bases on filtrators

DEFINITION 679. A set S of binary relations is a *base* on a filtrator $(\mathfrak{A}, \mathfrak{B})$ of $f \in \mathfrak{A}$ when all elements of S are above f and $\forall X \in \text{up } f \exists T \in S : T \sqsubseteq X$.

OBVIOUS 680. Every base on an up-aligned filtrator is nonempty.

PROPOSITION 681. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered filtrator.
- 4°. A set $S \in \mathcal{P}\mathfrak{B}$ is a base of a filtrator element iff $\prod^{\mathfrak{A}} S$ exists and S is a base of $\prod^{\mathfrak{A}} S$.

PROOF.

1° \Rightarrow 2°, 2° \Rightarrow 3°. Obvious.

3° \Rightarrow 4°.

\Leftarrow . Obvious.

\Rightarrow . Let S be a base of an $f \in \mathfrak{A}$. f is obviously a lower bound of S . Let g be a lower bound of S . Then for every $X \in \text{up } f$ we have $g \sqsubseteq X$ that is $X \in \text{up } g$. Thus $\text{up } f \subseteq \text{up } g$ and thus $f \sqsupseteq g$ that is f is the greatest upper bound of S .

□

PROPOSITION 682. There exists an $f \in \mathfrak{A}$ such that $\text{up } f = S$ iff S is a base and is an upper set (for every set $S \in \mathcal{P}\mathfrak{B}$).

PROOF.

\Rightarrow . If $\text{up } f = S$ then S is an upper set and S is a base of f because $\forall X \in \text{up } f \exists T \in S : T = X$.

\Leftarrow . Let S be a base of some filtrator element f and is an upper set. Then for every $X \in \text{up } f$ there is $T \in S$ such that $T \sqsubseteq X$. Thus $X \in S$. We have $\text{up } f \subseteq S$. But $S \subseteq \text{up } f$ is obvious. We have $\text{up } f = S$.

□

PROPOSITION 683. $\text{up } f$ is a base of f for every $f \in \mathfrak{A}$.

PROOF. Denote $S = \text{up } f$. That f is a lower bound of S is obvious. If $X \in \text{up } f$ then $\exists T \in S : T = X$. Thus S is a base of f .

□

PROPOSITION 684. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator.
- 3°. $(\mathfrak{A}, \mathfrak{B})$ is a filtered filtrator.
- 4°. $f = \prod^{\mathfrak{A}} S$ for every base S of an $f \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°, 2° \Rightarrow 3°. Obvious.

3° \Rightarrow 4°. f is a lower bound of S by definition.

Let g be a lower bound of S . Then for every $X \in \text{up } f$ there we have $g \sqsubseteq X$ that is $X \in \text{up } g$. Thus $\text{up } f \subseteq \text{up } g$ and thus $f \sqsupseteq g$ that is f is the greatest lower bound of S .

□

PROPOSITION 685. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.

- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is a filtered filtrator.
- 4°. If S is a base on a filtrator, then $\prod^{\mathfrak{A}} S$ exists and $\text{up } \prod^{\mathfrak{A}} S = \bigcup_{K \in S} \text{up } K$.

PROOF.

1° \Rightarrow 2°, 2° \Rightarrow 3°. Obvious.

3° \Rightarrow 4°. $\prod^{\mathfrak{A}} S$ exists because our filtrator is filtered. Above we proved that S is a base of $\prod^{\mathfrak{A}} S$. That $\bigcup_{K \in S} \text{up } K \subseteq \text{up } \prod^{\mathfrak{A}} S$ is obvious. If $X \in \text{up } \prod^{\mathfrak{A}} S$ then by properties of bases we have $K \in S$ such that $K \sqsubseteq X$. Thus $X \in \text{up } K$ and so $X \in \bigcup_{K \in S} \text{up } K$. So $\text{up } \prod^{\mathfrak{A}} S \subseteq \bigcup_{K \in S} \text{up } K$. □

PROPOSITION 686. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{F})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{F})$ is a primary filtrator over a meet-semilattice.
- 3°. $(\mathfrak{A}, \mathfrak{F})$ is a filtrator with binarily meet-closed core such that $\forall a \in \mathfrak{A} : \text{up } a \neq \emptyset$.
- 4°. A base on the filtrator $(\mathfrak{A}; \mathfrak{F})$ is the same as base of a filter (on \mathfrak{F}).

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Corollary 536.

3° \Rightarrow 4°.

- \Rightarrow . Let S be a base of f on the filtrator $(\mathfrak{A}; \mathfrak{F})$. Then for every $a, b \in S$ we have $a, b \in \text{up } f$ and thus $a \sqcap^{\mathfrak{F}} b = a \sqcap^{\mathfrak{A}} b \in \text{up } f$. Thus $\exists x \in S : x \sqsubseteq a \sqcap^{\mathfrak{F}} b$ that is $x \sqsubseteq a \wedge x \sqsubseteq b$. It remains to show that S is nonempty, but this follows from $\text{up } a$ being nonempty.
- \Leftarrow . Let S be a base of filter f (on \mathfrak{F}). Let $X \in \text{up } f$. Then there is $T \in S$ such that $T \sqsubseteq X$. □

5.36. Some Counter-Examples

EXAMPLE 687. There exist a bounded distributive lattice which is not lattice with separable center.

PROOF. The lattice with the Hasse diagram² on figure 5 is bounded and distributive because it does not contain “diamond lattice” nor “pentagon lattice” as a sublattice [43].

It’s center is $\{0, 1\}$. $x \sqcap y = 0$ despite $\text{up } x = \{x, a, 1\}$ but $y \sqcap 1 \neq 0$ consequently the lattice is not with separable center. □

In this section \mathfrak{A} denotes the set of filters on a set.

EXAMPLE 688. There is a separable poset (that is a set with \star being an injection) which is not strongly separable (that is \star isn’t order reflective).

PROOF. (with help of sci.math partakers) Consider a poset with the Hasse diagram 6.

Then $\star p = \{p, a, b\}$, $\star q = \{q, a, b\}$, $\star r = \{r, b\}$, $\star a = \{p, q, a, b\}$, $\star b = \{p, q, a, b, r\}$.

Thus $\star x = \star y \Rightarrow x = y$ for any x, y in our poset.

$\star a \subseteq \star b$ but not $a \subseteq b$. □

EXAMPLE 689. There is a prefiltered filtrator which is not filtered.

²See Wikipedia for a definition of Hasse diagrams.

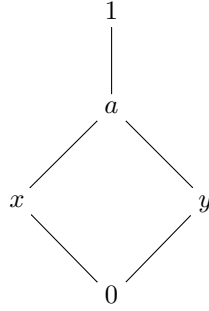


FIGURE 5

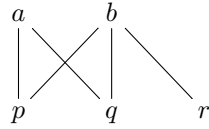


FIGURE 6

PROOF. (MATTHIAS KLUPSCH) Take $\mathfrak{A} = \{a, b\}$ with the order being equality and $\mathfrak{B} = \{b\}$. Then $\text{up } a = \emptyset \sqsubseteq \{b\} = \text{up } b$, so up is injective, hence the filtrator is prefiltered, but because of $a \not\sqsubseteq b$ the filtrator is not filtered. \square

For further examples we will use the filter Δ defined by the formula

$$\Delta = \prod^{\mathfrak{A}} \left\{ \frac{] - \epsilon; \epsilon[}{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}$$

and more general

$$\Delta + a = \prod^{\mathfrak{A}} \left\{ \frac{]a - \epsilon; a + \epsilon[}{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}.$$

EXAMPLE 690. There exists $A \in \mathcal{P}U$ such that $\prod^{\mathfrak{A}} A \neq \prod A$.

PROOF. $\prod^{\mathfrak{B}} \left\{ \frac{] - \epsilon; \epsilon[}{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\} = \uparrow \{0\} \neq \Delta$. \square

EXAMPLE 691. There exists a set U and a filter a and a set S of filters on the set U such that $a \prod^{\mathfrak{A}} \prod^{\mathfrak{A}} S \neq \prod^{\mathfrak{A}} \langle a \prod^{\mathfrak{A}} \rangle^* S$.

PROOF. Let $a = \Delta$ and $S = \left\{ \frac{]^{\mathfrak{B}} \epsilon; +\infty[}{ \epsilon > 0 } \right\}$. Then $a \prod^{\mathfrak{A}} \prod^{\mathfrak{A}} S = \Delta \prod^{\mathfrak{A}}]0; +\infty[\neq \perp^{\mathfrak{A}}$ while $\prod^{\mathfrak{A}} \langle a \prod^{\mathfrak{A}} \rangle^* S = \prod^{\mathfrak{A}} \{ \perp^{\mathfrak{A}} \} = \perp^{\mathfrak{A}}$. \square

EXAMPLE 692. There are tornings which are not weak partitions.

PROOF. $\left\{ \frac{\Delta + a}{a \in \mathbb{R}} \right\}$ is a torning but not weak partition of the real line. \square

LEMMA 693. Let \mathfrak{A} be the set of filters on a set U . Then $X \prod^{\mathfrak{A}} \Omega \sqsubseteq Y \prod^{\mathfrak{A}} \Omega$ iff $X \setminus Y$ is a finite set, having fixed sets $X, Y \in \mathcal{P}U$.

PROOF. Let M be the set of finite subsets of U .

$$\begin{aligned} X \sqcap^{\mathfrak{A}} \Omega \subseteq Y \sqcap^{\mathfrak{A}} \Omega &\Leftrightarrow \\ \left\{ \frac{X \cap K_X}{K_X \in \Omega} \right\} \supseteq \left\{ \frac{Y \cap K_Y}{K_Y \in \Omega} \right\} &\Leftrightarrow \\ \forall K_Y \in \Omega \exists K_X \in \Omega : Y \cap K_Y = X \cap K_X &\Leftrightarrow \\ \forall L_Y \in M \exists L_X \in M : Y \setminus L_Y = X \setminus L_X &\Leftrightarrow \\ \forall L_Y \in M : X \setminus (Y \setminus L_Y) \in M &\Leftrightarrow \\ X \setminus Y \in M. & \end{aligned}$$

□

EXAMPLE 694. There exists a filter \mathcal{A} on a set U such that $(\mathcal{P}U)/\sim$ and $Z(D\mathcal{A})$ are not complete lattices.

PROOF. Due to the isomorphism it is enough to prove for $(\mathcal{P}U)/\sim$.

Let take $U = \mathbb{N}$ and $\mathcal{A} = \Omega$ be the Fréchet filter on \mathbb{N} .

Partition \mathbb{N} into infinitely many infinite sets A_0, A_1, \dots . To withhold our example we will prove that the set $\{[A_0], [A_1], \dots\}$ has no supremum in $(\mathcal{P}U)/\sim$.

Let $[X]$ be an upper bound of $[A_0], [A_1], \dots$ that is $\forall i \in \mathbb{N} : X \sqcap^{\mathfrak{A}} \Omega \supseteq A_i \sqcap^{\mathfrak{A}} \Omega$ that is $A_i \setminus X$ is finite. Consequently X is infinite. So $X \cap A_i \neq \emptyset$.

Choose for every $i \in \mathbb{N}$ some $z_i \in X \cap A_i$. The $\{z_0, z_1, \dots\}$ is an infinite subset of X (take into account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \dots\}$. Then $Y \sqcap^{\mathfrak{A}} \Omega \supseteq A_i \sqcap^{\mathfrak{A}} \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \dots\}$.

Suppose $Y \sqcap^{\mathfrak{A}} \Omega = X \sqcap^{\mathfrak{A}} \Omega$. Then $Y \setminus X$ is finite what is not true. So $Y \sqcap^{\mathfrak{A}} \Omega \subset X \sqcap^{\mathfrak{A}} \Omega$ that is $[Y]$ is below $[X]$. □

5.36.1. Weak and Strong Partition.

DEFINITION 695. A family S of subsets of a countable set is *independent* iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.

LEMMA 696. The “infinite” at the end of the definition could be equivalently replaced with “nonempty” if we assume that S is infinite.

PROOF. Suppose that some sets from the above definition has a finite intersection J of cardinality n . Then (thanks S is infinite) get one more set $X \in S$ and we have $J \cap X \neq \emptyset$ and $J \cap (\mathbb{N} \setminus X) \neq \emptyset$. So $\text{card}(J \cap X) < n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. □

LEMMA 697. There exists an independent family on \mathbb{N} of cardinality \mathfrak{c} .

PROOF. Let C be the set of finite subsets of \mathbb{Q} . Since $\text{card } C = \text{card } \mathbb{N}$, it suffices to find \mathfrak{c} independent subsets of C . For each $r \in \mathbb{R}$ let

$$E_r = \left\{ \frac{F \in C}{\text{card}(F \cap \cdot) - \infty; r \mid \cdot \text{ is even}} \right\}.$$

All E_{r_1} and E_{r_2} are distinct for distinct $r_1, r_2 \in \mathbb{R}$ since we may consider $F = \{r'\} \in C$ where a rational number r' is between r_1 and r_2 and thus F is a member of exactly one of the sets E_{r_1} and E_{r_2} . Thus $\text{card} \left\{ \frac{E_r}{r \in \mathbb{R}} \right\} = \mathfrak{c}$.

We will show that $\left\{ \frac{E_r}{r \in \mathbb{R}} \right\}$ is independent. Let $r_1, \dots, r_k, s_1, \dots, s_k$ be distinct reals. It is enough to show that these have a nonempty intersection, that is existence of some F such that F belongs to all the E_{r_i} and none of E_{s_i} .

But this can be easily accomplished taking F having zero or one element in each of intervals to which $r_1, \dots, r_k, s_1, \dots, s_k$ split the real line. \square

EXAMPLE 698. There exists a weak partition of a filter on a set which is not a strong partition.

PROOF. (suggested by ANDREAS BLASS) Let $\{\frac{X_r}{r \in \mathbb{R}}\}$ be an independent family of subsets of \mathbb{N} . We can assume $a \neq b \Rightarrow X_a \neq X_b$ due the above lemma.

Let \mathcal{F}_a be a filter generated by X_a and the complements $\mathbb{N} \setminus X_b$ for all $b \in \mathbb{R}$, $b \neq a$. Independence implies that $\mathcal{F}_a \neq \perp^{\aleph_1}$ (by properties of filter bases).

Let $S = \{\frac{\mathcal{F}_r}{r \in \mathbb{R}}\}$. We will prove that S is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\} : \mathbb{N} \setminus X_a \in \mathcal{F}_b$ and therefore $\mathbb{N} \setminus X_a \in \bigsqcup^{\aleph_1} \{\frac{\mathcal{F}_b}{\mathbb{R} \ni b \neq a}\}$. Therefore $\mathcal{F}_a \cap^{\aleph_1} \bigsqcup^{\aleph_1} \{\frac{\mathcal{F}_b}{\mathbb{R} \ni b \neq a}\} = \perp^{\aleph_1}$. Thus S is a weak partition.

Suppose S is a strong partition. Then for each set $Z \in \mathcal{P}\mathbb{R}$

$$\bigsqcup^{\aleph_1} \left\{ \frac{\mathcal{F}_b}{b \in Z} \right\} \cap^{\aleph_1} \bigsqcup^{\aleph_1} \left\{ \frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z} \right\} = \perp^{\aleph_1}$$

what is equivalent to existence of $M(Z) \in \mathcal{P}\mathbb{N}$ such that

$$M(Z) \in \bigsqcup^{\aleph_1} \left\{ \frac{\mathcal{F}_b}{b \in Z} \right\} \quad \text{and} \quad \mathbb{N} \setminus M(Z) \in \bigsqcup^{\aleph_1} \left\{ \frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z} \right\}$$

that is

$$\forall b \in Z : M(Z) \in \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathbb{R} \setminus Z : \mathbb{N} \setminus M(Z) \in \mathcal{F}_b.$$

Suppose $Z \neq Z' \in \mathcal{P}\mathbb{N}$. Without loss of generality we may assume that some $b \in Z$ but $b \notin Z'$. Then $M(Z) \in \mathcal{F}_b$ and $\mathbb{N} \setminus M(Z') \in \mathcal{F}_b$. If $M(Z) = M(Z')$ then $\mathcal{F}_b = \perp^{\aleph_1}$ what contradicts to the above.

So M is an injective function from $\mathcal{P}\mathbb{R}$ to $\mathcal{P}\mathbb{N}$ what is impossible due cardinality issues. \square

LEMMA 699. (by NIELS DIEPEVEEN, with help of KARL KRONENFELD) Let K be a collection of nontrivial ultrafilters. We have $\bigsqcup K = \Omega$ iff $\exists \mathcal{G} \in K : A \in \text{up } \mathcal{G}$ for every infinite set A .

PROOF.

\Rightarrow . Suppose $\bigsqcup K = \Omega$ and let A be a set such that $\nexists \mathcal{G} \in K : A \in \text{up } \mathcal{G}$. Let's prove A is finite.

Really, $\forall \mathcal{G} \in K : \mathcal{U} \setminus A \in \text{up } \mathcal{G}; \mathcal{U} \setminus A \in \text{up } \Omega; A$ is finite.

\Leftarrow . Let $\exists \mathcal{G} \in K : A \in \text{up } \mathcal{G}$. Suppose A is a set in $\text{up } \bigsqcup K$.

To finish the proof it's enough to show that $\mathcal{U} \setminus A$ is finite.

Suppose $\mathcal{U} \setminus A$ is infinite. Then $\exists \mathcal{G} \in K : \mathcal{U} \setminus A \in \text{up } \mathcal{G}; \exists \mathcal{G} \in K : A \notin \text{up } \mathcal{G}; A \notin \text{up } \bigsqcup K$, contradiction. \square

LEMMA 700. (by NIELS DIEPEVEEN) If K is a non-empty set of ultrafilters such that $\bigsqcup K = \Omega$, then for every $\mathcal{G} \in K$ we have $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$.

PROOF. $\exists \mathcal{F} \in K : A \in \text{up } \mathcal{F}$ for every infinite set A .

The set A can be partitioned into two infinite sets A_1, A_2 .

Take $\mathcal{F}_1, \mathcal{F}_2 \in K$ such that $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$.

$\mathcal{F}_1 \neq \mathcal{F}_2$ because otherwise A_1 and A_2 are not disjoint.

Obviously $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$.

So there exist two different $\mathcal{F} \in K$ such that $A \in \text{up } \mathcal{F}$. Consequently $\exists \mathcal{F} \in K \setminus \{\mathcal{G}\} : A \in \text{up } \mathcal{F}$ that is $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$. \square

EXAMPLE 701. There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

PROOF. Consider cofinite filter Ω on any infinite set.

Suppose K is its weak partition into ultrafilters. Then $x \asymp \bigsqcup(K \setminus \{x\})$ for some ultrafilter $x \in K$.

We have $\bigsqcup(K \setminus \{x\}) \sqsubset \bigsqcup K$ (otherwise $x \sqsubseteq \bigsqcup(K \setminus \{x\})$) what is impossible due the last lemma. \square

COROLLARY 702. There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

5.37. Open problems about filters

Under which conditions $a \setminus^* b$ and $a \# b$ are complementive to a ?
Generalize straight maps for arbitrary posets.

5.38. Further notation

Below to define functors and relocks we need a fixed powerset filtrator.

Let $(\mathcal{F}A, \mathcal{T}A)$ be an arbitrary but fixed powerset filtrator. This filtrator exists by the theorem 462.

I will call elements of \mathcal{F} *filter objects*.

For brevity we will denote lattice operations on $\mathcal{F}A$ without indexes (for example, take $\prod S = \prod^{\mathcal{F}A} S$ for $S \in \mathcal{P}\mathcal{F}A$).

Note that above we also took operations on $\mathcal{T}A$ without indexes (for example, take $\prod S = \prod^{\mathcal{T}A} S$ for $S \in \mathcal{P}\mathcal{T}A$).

Because we identify $\mathcal{T}A$ with principal elements of $\mathcal{F}A$, the notation like $\prod S$ for $S \in \mathcal{P}\mathcal{T}A$ would be inconsistent (it can mean both $\prod^{\mathcal{T}A} S$ or $\prod^{\mathcal{F}A} S$). We explicitly state that $\prod S$ in this case does *not* mean $\prod^{\mathcal{F}A} S$.

For $\mathcal{X} \in \mathcal{F}$ we will denote $\text{GR } \mathcal{X}$ the corresponding filter on $\mathcal{P}A$. It is a convenient notation to describe relations between filters and sets, consider for example the formula: $\{x\} \subseteq \bigcap \text{GR } \mathcal{X}$.

We will denote lattice operations without pointing a specific set like $\prod^{\mathcal{F}} S = \prod^{\mathcal{F}(A)} S$ for a set $S \in \mathcal{P}\mathcal{F}(A)$.

5.39. Equivalent filters and rebase of filters

Throughout this section we will assume that \mathfrak{Z} is a lattice.

An important example: \mathfrak{Z} is the lattice of all small (regarding some Grothendieck universe) sets. (This \mathfrak{Z} is not a powerset, and even not a complete lattice.)

Throughout this section I will use the word *filter* to denote a filter on a sublattice DA where $A \in \mathfrak{Z}$ (if not told explicitly to be a filter on some other set).

The following is an embedding from filters \mathcal{A} on a lattice DA into the lattice of filters on \mathfrak{Z} : $\mathcal{S}\mathcal{A} = \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A}: X \sqsubseteq K} \right\}$.

PROPOSITION 703. Values of this embedding are filters on the lattice \mathfrak{Z} .

PROOF. That $\mathcal{S}\mathcal{A}$ is an upper set is obvious.

Let $P, Q \in \mathcal{S}\mathcal{A}$. Then $P, Q \in \mathfrak{Z}$ and there is an $X \in \mathcal{A}$ such that $X \sqsubseteq P$ and $Y \in \mathcal{A}$ such that $Y \sqsubseteq Q$. So $X \sqcap Y \in \mathcal{A}$ and $P \sqcap Q \sqsupseteq X \sqcap Y \in \mathcal{A}$, so $P \sqcap Q \in \mathcal{S}\mathcal{A}$. \square

5.39.1. Rebase of filters.

DEFINITION 704. *Rebase* for every filter \mathcal{A} and every $A \in \mathfrak{Z}$ is $\mathcal{A} \div A = \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{A}} \right\}$.

OBVIOUS 705. $\langle A \sqcap \rangle^* \mathcal{S}\mathcal{A}$ is a filter on A .

PROPOSITION 706. The rebase conforms to the formula

$$\mathcal{A} \div A = \langle A \sqcap \rangle^* \mathcal{S}\mathcal{A}.$$

PROOF. We know that $\langle A \sqcap \rangle^* \mathcal{S}\mathcal{A}$ is a filter.

If $P \in \langle A \sqcap \rangle^* \mathcal{S}\mathcal{A}$ then $P \in \mathcal{P}A$ and $Y \sqcap A \sqsubseteq P$ for some $Y \in \mathcal{A}$. Thus $P \sqsupseteq Y \sqcap A \in \prod \left\{ \frac{\uparrow^A(Y \sqcap A)}{Y \in \mathcal{A}} \right\}$.

If $P \in \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{A}} \right\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \sqsupseteq X \sqcap A$. Also $P \in \mathcal{P}A$. Thus $P \in \langle A \sqcap \rangle^* \mathcal{S}\mathcal{A}$. \square

PROPOSITION 707. $\mathcal{X} \div \text{Base}(\mathcal{X}) = \mathcal{X}$.

PROOF. Because $X \sqcap \text{Base}(\mathcal{X}) = X$ for $X \in \mathcal{X}$. \square

PROPOSITION 708. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$.

PROOF.

$$\begin{aligned} (\mathcal{X} \div A) \div B &= \prod \left\{ \frac{\uparrow^B(Y \sqcap B)}{Y \in \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{X}} \right\}} \right\} = \prod \left\{ \frac{\uparrow^B(X \sqcap A)}{X \in \mathcal{X}} \right\} \sqcap \uparrow^B B = \\ &= \prod \left\{ \frac{\uparrow^B(X \sqcap A \sqcap B)}{X \in \mathcal{X}} \right\} = \prod \left\{ \frac{\uparrow^B(X \sqcap B)}{X \in \mathcal{X}} \right\} = \mathcal{X} \div B. \end{aligned}$$

\square

PROPOSITION 709. If $A \in \mathcal{A}$ then $\mathcal{A} \div A = \mathcal{A} \cap \mathcal{P}A$.

PROOF. $\mathcal{A} \div A = \langle A \sqcap \rangle^* \mathcal{S}\mathcal{A} = \langle A \sqcap \rangle^* \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A}: X \sqsubseteq K} \right\} = \left\{ \frac{K \in \mathfrak{Z}}{K \in \mathcal{A} \wedge K \in \mathcal{P}A} \right\} = \mathcal{A} \cap \mathcal{P}A$. \square

PROPOSITION 710. Let filters \mathcal{X} and \mathcal{Y} be such that $\text{Base}(\mathcal{X}) = \text{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$ for every $\mathfrak{Z} \ni C \sqsupseteq B$.

PROOF. $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \cup \left\{ \frac{K \in \mathcal{P}C}{K \sqsupseteq B} \right\} = \mathcal{Y} \cup \left\{ \frac{K \in \mathcal{P}C}{K \sqsupseteq B} \right\} \Leftrightarrow \mathcal{X} = \mathcal{Y}$. \square

5.39.2. Equivalence of filters.

DEFINITION 711. Two filters \mathcal{A} and \mathcal{B} (with possibly different base sets) are equivalent ($\mathcal{A} \sim \mathcal{B}$) iff there exists an $X \in \mathfrak{Z}$ such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathcal{P}X \cap \mathcal{A} = \mathcal{P}X \cap \mathcal{B}$.

PROPOSITION 712. \mathcal{X} and \mathcal{Y} are equivalent iff $(\mathcal{X} \sim \mathcal{Y})$ iff $\mathcal{Y} = \mathcal{X} \div \text{Base}(\mathcal{Y})$ and $\mathcal{X} = \mathcal{Y} \div \text{Base}(\mathcal{X})$.

PROOF.

\Rightarrow . Suppose $\mathcal{X} \sim \mathcal{Y}$ that is there exists a set P such that $\mathcal{P}P \cap \mathcal{X} = \mathcal{P}P \cap \mathcal{Y}$ and $P \in \mathcal{X}$, $P \in \mathcal{Y}$. Then $\mathcal{X} \div \text{Base}(\mathcal{Y}) = (\mathcal{P}P \cap \mathcal{X}) \cup \left\{ \frac{K \in \mathcal{P} \text{Base}(\mathcal{Y})}{K \sqsupseteq P} \right\} = (\mathcal{P}P \cap \mathcal{Y}) \cup \left\{ \frac{K \in \mathcal{P} \text{Base}(\mathcal{Y})}{K \sqsupseteq P} \right\} = \mathcal{Y}$. So $\mathcal{X} \div \text{Base}(\mathcal{Y}) = \mathcal{Y}$, $\mathcal{Y} \div \text{Base}(\mathcal{X}) = \mathcal{X}$ is similar.

\Leftarrow . We have $\mathcal{Y} = (\mathcal{Y} \div \text{Base}(\mathcal{X})) \div \text{Base}(\mathcal{Y})$.

Thus as easy to show $\text{Base}(\mathcal{X}) \sqcap \text{Base}(\mathcal{Y}) \in \mathcal{Y}$ and similarly $\text{Base}(\mathcal{X}) \sqcap \text{Base}(\mathcal{Y}) \in \mathcal{X}$.

It's enough to show $\mathcal{X} \div (\text{Base}(\mathcal{X}) \sqcap \text{Base}(\mathcal{Y})) = \mathcal{Y} \div (\text{Base}(\mathcal{X}) \sqcap \text{Base}(\mathcal{Y}))$ because for every $P \in \mathcal{X}, \mathcal{Y}$ we have $\mathcal{X} \cap \mathcal{P}P = \mathcal{X} \div P = (\mathcal{X} \div (\text{Base}(\mathcal{X}) \sqcap \text{Base}(\mathcal{Y}))) \div P$ and similarly $\mathcal{Y} \cap \mathcal{P}P = (\mathcal{Y} \div (\text{Base}(\mathcal{X}) \sqcap \text{Base}(\mathcal{Y}))) \div P$. But it follows from the conditions and proposition 708. \square

PROPOSITION 713. If two filters with the same base are equivalent they are equal.

PROOF. Let \mathcal{A} and \mathcal{B} be two filters and $\mathcal{P}X \cap \mathcal{A} = \mathcal{P}X \cap \mathcal{B}$ for some set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$, and $\text{Base}(\mathcal{A}) = \text{Base}(\mathcal{B})$. Then

$$\begin{aligned} \mathcal{A} &= (\mathcal{P}X \cap \mathcal{A}) \cup \left\{ \frac{Y \in D \text{Base}(\mathcal{A})}{Y \supseteq X} \right\} = \\ &= (\mathcal{P}X \cap \mathcal{B}) \cup \left\{ \frac{Y \in D \text{Base}(\mathcal{B})}{Y \supseteq X} \right\} = \mathcal{B}. \end{aligned}$$

\square

PROPOSITION 714. If $A \in \mathcal{S}\mathcal{A}$ then $\mathcal{A} \div A \sim \mathcal{A}$.

PROOF.

$$\begin{aligned} (\mathcal{A} \div A) \cap \mathcal{P}(A \sqcap \text{Base}(\mathcal{A})) &= \\ \mathcal{S}\mathcal{A} \cap \mathcal{P}A \cap \mathcal{P}(A \sqcap \text{Base}(\mathcal{A})) &= \\ \mathcal{S}\mathcal{A} \cap \mathcal{P}(A \sqcap \text{Base}(\mathcal{A})) &= \mathcal{A} \cap \mathcal{P}(A \sqcap \text{Base}(\mathcal{A})). \end{aligned}$$

Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \sqcap \text{Base}(\mathcal{A}) \supseteq X \in \mathcal{A}$ for some $X \in \mathcal{A}$ and

$$A \sqcap \text{Base}(\mathcal{A}) \supseteq X \sqcap \text{Base}(\mathcal{A}) \in \mathcal{A} \div A.$$

\square

PROPOSITION 715. \sim is an equivalence relation.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some filters \mathcal{A} , \mathcal{B} , and \mathcal{C} . Then there exist a set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathcal{P}X \cap \mathcal{A} = \mathcal{P}X \cap \mathcal{B}$ and a set Y such that $Y \in \mathcal{B}$ and $Y \in \mathcal{C}$ and $\mathcal{P}Y \cap \mathcal{B} = \mathcal{P}Y \cap \mathcal{C}$. So $X \sqcap Y \in \mathcal{A}$ because

$$\mathcal{P}Y \cap \mathcal{P}X \cap \mathcal{A} = \mathcal{P}Y \cap \mathcal{P}X \cap \mathcal{B} = \mathcal{P}(X \sqcap Y) \cap \mathcal{B} \supseteq \{X \sqcap Y\} \cap \mathcal{B} \ni X \sqcap Y.$$

Similarly we have $X \sqcap Y \in \mathcal{C}$. Finally

$$\begin{aligned} \mathcal{P}(X \sqcap Y) \cap \mathcal{A} &= \mathcal{P}Y \cap \mathcal{P}X \cap \mathcal{A} = \mathcal{P}Y \cap \mathcal{P}X \cap \mathcal{B} = \\ &= \mathcal{P}X \cap \mathcal{P}Y \cap \mathcal{B} = \mathcal{P}X \cap \mathcal{P}Y \cap \mathcal{C} = \mathcal{P}(X \sqcap Y) \cap \mathcal{C}. \end{aligned}$$

\square

DEFINITION 716. I will call equivalence classes as *unfixed filters*.

REMARK 717. The word “unfixed” is meant to negate “fixed” (having a particular base) filters.

PROPOSITION 718. $\mathcal{A} \sim \mathcal{B}$ iff $\mathcal{S}\mathcal{A} = \mathcal{S}\mathcal{B}$ for every filters \mathcal{A}, \mathcal{B} on sets.³

PROOF. Let $\mathcal{A} \sim \mathcal{B}$. Then there is a set P such that $P \in \mathcal{A}, P \in \mathcal{B}$ and $\mathcal{A} \cap \mathcal{P}P = \mathcal{B} \cap \mathcal{P}P$. So $\mathcal{S}\mathcal{A} = (\mathcal{A} \cap \mathcal{P}P) \cup \left\{ \frac{K \in \mathfrak{J}}{K \supseteq P} \right\}$. Similarly $\mathcal{S}\mathcal{B} = (\mathcal{B} \cap \mathcal{P}P) \cup \left\{ \frac{K \in \mathfrak{J}}{K \supseteq P} \right\}$. Combining, we have $\mathcal{S}\mathcal{A} = \mathcal{S}\mathcal{B}$.

Let now $\mathcal{S}\mathcal{A} = \mathcal{S}\mathcal{B}$. Take $K \in \mathcal{S}\mathcal{A} = \mathcal{S}\mathcal{B}$. Then $\mathcal{A} \div K = \mathcal{B} \div K$ and thus (proposition 714) $\mathcal{A} \sim \mathcal{A} \div K = \mathcal{B} \div K \sim \mathcal{B}$, so having $\mathcal{A} \sim \mathcal{B}$. \square

PROPOSITION 719. $\mathcal{A} \sim \mathcal{B} \Rightarrow \mathcal{A} \div B = \mathcal{B} \div B$ for every filters \mathcal{A} and \mathcal{B} and set B .

PROOF. $\mathcal{A} \div B = \langle B \cap \rangle^* \mathcal{S}\mathcal{A} = \langle B \cap \rangle^* \mathcal{S}\mathcal{B} = \mathcal{B} \div B$. \square

5.39.3. Poset of unfixed filters.

LEMMA 720. Let filters \mathcal{X} and \mathcal{Y} be such that $\text{Base}(\mathcal{X}) = \text{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$ for every set $C \supseteq B$.

PROOF.

$$\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \div C \supseteq \mathcal{Y} \div C \Leftrightarrow$$

$$\mathcal{X} \cup \left\{ \frac{K \in \mathcal{P}C}{K \supseteq B} \right\} \supseteq \mathcal{Y} \cup \left\{ \frac{K \in \mathcal{P}C}{K \supseteq B} \right\} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}.$$

\square

PROPOSITION 721. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$ for every filters \mathcal{X}, \mathcal{Y} with the same base and set B .

PROOF. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \supseteq \mathcal{Y} \div B \Leftrightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$. \square

Define order of unfixed filters using already defined order of filters of a fixed base:

DEFINITION 722. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \exists x \in \mathcal{X}, y \in \mathcal{Y} : (\text{Base}(x) = \text{Base}(y) \wedge x \sqsubseteq y)$ for unfixed filters \mathcal{X}, \mathcal{Y} .

Proposition 718 allows to define:

DEFINITION 723. $\mathcal{S}\mathcal{A} = \mathcal{S}a$ for every $a \in \mathcal{A}$ for every unfixed filter \mathcal{A} .

THEOREM 724. \mathcal{S} is an order-isomorphism from the poset of unfixed filters to the poset of filters on \mathfrak{J} .

PROOF. We already know that \mathcal{S} is an order embedding. It remains to prove that it is a surjection.

Let \mathcal{Y} be a filter on \mathfrak{J} . Take $\mathfrak{J} \ni X \in \mathcal{Y}$. Then $\langle X \cap \rangle^* \mathcal{Y}$ is a filter on X and $\mathcal{S}[\langle X \cap \rangle^* \mathcal{Y}] = \mathcal{S}\langle X \cap \rangle^* \mathcal{Y} = \mathcal{Y}$. We have proved that it is a surjection. \square

LEMMA 725. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{S}\mathcal{X} \sqsubseteq \mathcal{S}\mathcal{Y}$ for every unfixed filters \mathcal{X}, \mathcal{Y} .

PROOF.

\Rightarrow . Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there exist $x \in \mathcal{X}, y \in \mathcal{Y}$ such that $\text{Base}(x) = \text{Base}(y)$ and $x \sqsubseteq y$. Then $\mathcal{S}x \sqsubseteq \mathcal{S}y = \mathcal{S}y$.

\Leftarrow . Suppose $\mathcal{S}\mathcal{X} \sqsubseteq \mathcal{S}\mathcal{Y}$. Then there are $x \in \mathcal{X}, y \in \mathcal{Y}$ such that $\mathcal{S}x \sqsubseteq \mathcal{S}y$. Consequently $\mathcal{S}x' \sqsubseteq \mathcal{S}y'$ for $x' = x \div (\text{Base}(x) \sqcup \text{Base}(y))$, $y' = y \div (\text{Base}(x) \sqcup \text{Base}(y))$. So we have $x' \in \mathcal{X}, y' \in \mathcal{Y}$, $\text{Base}(x') = \text{Base}(y')$ and $x' \sqsubseteq y'$, thus $\mathcal{X} \sqsubseteq \mathcal{Y}$.

³Use this proposition to shorten proofs of other theorem about equivalence of filters? (Our proof uses transitivity of equivalence of filters. So we can't use it to prove that it is an equivalence relation, to avoid circular proof.)

□

THEOREM 726. \sqsubseteq on the set of unfixed filters is a poset.

PROOF.

Reflexivity. From the previous theorem.

Transitivity. From the previous theorem.

Antisymmetry. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$ and $\mathcal{Y} \sqsubseteq \mathcal{X}$. Then $\mathcal{S}\mathcal{X} \sqsubseteq \mathcal{S}\mathcal{Y}$ and $\mathcal{S}\mathcal{Y} \sqsubseteq \mathcal{S}\mathcal{X}$.

Thus $\mathcal{S}\mathcal{X} = \mathcal{S}\mathcal{Y}$ and so $\mathcal{S}x = \mathcal{S}y$ for some $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Consequently $\mathcal{S}(x \div B) = \mathcal{S}(y \div B)$ for $B = \text{Base}(x) \sqcup \text{Base}(y)$. Thus $x \div B = y \div B$ and so $x \sim y$, thus $\mathcal{X} = \mathcal{Y}$.

□

THEOREM 727. $[x] \sqsubseteq [y] \Leftrightarrow x \sqsubseteq y$ for filters x and y with the same base set.

PROOF.

\Leftarrow . Obvious.

\Rightarrow . Let $\text{Base}(x) = \text{Base}(y) = B$. Suppose $[x] \sqsubseteq [y]$. Then there exist $x' \sim x$ and $y' \sim y$ such that $C = \text{Base}(x') = \text{Base}(y')$ (for some set C) and $x' \sqsubseteq y'$.

We have by the lemma $x' \div (B \sqcup C) \sqsubseteq y' \div (B \sqcup C)$.

But $x' \div (B \sqcup C) = x \div (B \sqcup C)$ and $y' \div (B \sqcup C) = y \div (B \sqcup C)$. So $x \div (B \sqcup C) \sqsubseteq y \div (B \sqcup C)$ and thus again applying the lemma $x \sqsubseteq y$.

□

5.39.4. Rebase of unfixed filters. Proposition 719 allows to define:

DEFINITION 728. $\mathcal{A} \div B = a \div B$ for an unfixed filter \mathcal{A} and arbitrary $a \in \mathcal{A}$.

PROPOSITION 729. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$ for every unfixed filters \mathcal{X} , \mathcal{Y} and set C .

PROOF. Let $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\text{Base}(x) = \text{Base}(y)$ and $x \sqsubseteq y$. Then by proved above $x \div C \sqsubseteq y \div C$ what is equivalent to $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$.

□

PROPOSITION 730. If $C \in \mathcal{S}\mathcal{X}$ and $C \in \mathcal{S}\mathcal{Y}$ for unfixed filters \mathcal{X} and \mathcal{Y} then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$.

PROOF.

\Leftarrow . Previous proposition.

\Rightarrow . Let $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. We have some $x \in \mathcal{X}$, $y \in \mathcal{Y}$, such that $\text{Base}(x) = \text{Base}(y)$ and $x \div C \sqsubseteq y \div C$. So $\mathcal{S}(x \div C) \sqsubseteq \mathcal{S}(y \div C)$. But $\mathcal{S}(x \div C) \sim x$ and $\mathcal{S}(y \div C) \sim y$. Thus $\mathcal{S}x \sqsubseteq \mathcal{S}y$ that is $x \sqsubseteq y$ and so $\mathcal{X} \sqsubseteq \mathcal{Y}$.

□

OBVIOUS 731. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$ for every unfixed filter \mathcal{X} and sets A , B .

OBVIOUS 732. $\mathcal{A} \div B = \langle B \cap \rangle^* \mathcal{S}\mathcal{A}$ for every unfixed filter \mathcal{A} .

OBVIOUS 733. If $A \in \mathcal{S}\mathcal{A}$ then $\mathcal{A} \div A \in \mathcal{A}$ for every unfixed filter \mathcal{A} .

PROPOSITION 734. If $C \in \mathcal{S}\mathcal{X}$ and $C \in \mathcal{S}\mathcal{Y}$ for unfixed filters \mathcal{X} and \mathcal{Y} then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$.

PROOF. The backward implication is obvious. Let now $\mathcal{X} \div C = \mathcal{Y} \div C$. Take $x \in \mathcal{X}$, $y \in \mathcal{Y}$. We have $\mathcal{X} \div C = x \div C = (x \div B) \div C$ for $B = C \sqcup \text{Base}(x) \sqcup \text{Base}(y)$. Similary $\mathcal{Y} \div C = (y \div B) \div C$. Thus $(x \div B) \div C = (y \div B) \div C$ and thus $x \div B = y \div B$, so $x \sim y$ that is $\mathcal{X} = \mathcal{Y}$.

□

PROPOSITION 735. $\mathcal{A} \div A = \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{S}\mathcal{A}} \right\}$ for every unfixed filter \mathcal{A} .

PROOF. Take $a \in \mathcal{A}$.

$$\begin{aligned} \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{S}\mathcal{A}} \right\} &= \prod \left\{ \frac{\uparrow^A(X \sqcap A \sqcap \text{Base}(a))}{X \in \mathcal{S}\mathcal{A}} \right\} = \\ &= \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{S}\mathcal{A} \cap \mathcal{P}\text{Base}(a)} \right\} = \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in \mathcal{S}a \cap \mathcal{P}\text{Base}(a)} \right\} = \\ &= \prod \left\{ \frac{\uparrow^A(X \sqcap A)}{X \in a} \right\} = a \div A = \mathcal{A} \div A. \end{aligned}$$

□

5.39.5. The diagram for unfixed filters. Fix a set B .

LEMMA 736. $\mathcal{X} \mapsto \mathcal{X} \div B$ and $x \mapsto [x]$ are mutually inverse order isomorphisms between $\left\{ \frac{\text{unfixed filter } \mathcal{X}}{B \in \mathcal{S}\mathcal{X}} \right\}$ and $\mathfrak{F}(DB)$.

PROOF. First, $\mathcal{X} \div B \in \mathfrak{F}(DB)$ for $\mathcal{X} \in \left\{ \frac{\text{unfixed filter } \mathcal{X}}{B \in \mathcal{S}\mathcal{X}} \right\}$ and $[x] \in \left\{ \frac{\text{unfixed filter } \mathcal{X}}{B \in \mathcal{S}\mathcal{X}} \right\}$ for $x \in \mathfrak{F}(DB)$.

Suppose $\mathcal{X}_0 \in \left\{ \frac{\text{unfixed filter } \mathcal{X}}{B \in \mathcal{S}\mathcal{X}} \right\}$, $x = \mathcal{X}_0 \div B$, and $\mathcal{X}_1 = [x]$. We will prove $\mathcal{X}_0 = \mathcal{X}_1$. Really, $x \in \mathcal{X}_1$, $x = k \div B$ for $k \in \mathcal{X}_0$, $x \sim k$, thus $x \in \mathcal{X}_0$. So $\mathcal{X}_0 = \mathcal{X}_1$.

Suppose $x_0 \in \mathfrak{F}(DB)$, $\mathcal{X} = [x_0]$, $x_1 = \mathcal{X} \div B$. We will prove $x_0 = x_1$. Really, $x_1 = x_0 \div B$. So $x_1 = x_0$ because $\text{Base}(x_0) = \text{Base}(x_1) = B$.

So we proved that they are mutually inverse bijections. That they are order preserving is obvious. □

LEMMA 737. \mathcal{S} and $\mathcal{X} \mapsto \langle B \sqcap \rangle^* \mathcal{X} = \mathcal{X} \cap \mathcal{P}B$ are mutually inverse order isomorphisms between $\mathfrak{F}(DB)$ and $\left\{ \frac{\mathcal{X} \in \mathfrak{F}(3)}{B \in \mathcal{X}} \right\}$.

PROOF. First, $\mathcal{S}x \in \left\{ \frac{\mathcal{X} \in \mathfrak{F}(3)}{B \in \mathcal{X}} \right\}$ for $x \in \mathfrak{F}(DB)$ because of theorem 724 and $\langle B \sqcap \rangle^* \mathcal{X} \in \mathfrak{F}(DB)$ obviously.

Let's prove $\langle B \sqcap \rangle^* \mathcal{X} = \mathcal{X} \cap \mathcal{P}B$. If $X \in \langle B \sqcap \rangle^* \mathcal{X}$ then $X \in \mathcal{X}$ (because $B \in \mathcal{X}$) and $X \in \mathcal{P}B$. So $X \in \mathcal{X} \cap \mathcal{P}B$. If $X \in \mathcal{X} \cap \mathcal{P}B$ then $X = B \sqcap X \in \langle B \sqcap \rangle^* \mathcal{X}$.

Let $x_0 \in \mathfrak{F}(DB)$, $\mathcal{X} = \mathcal{S}x$, and $x_1 = \langle B \sqcap \rangle^* \mathcal{X}$. Then obviously $x_0 = x_1$.

Let now $\mathcal{X}_0 \in \left\{ \frac{\mathcal{X} \in \mathfrak{F}(3)}{B \in \mathcal{X}} \right\}$, $x = \langle B \sqcap \rangle^* \mathcal{X}_0$, and $\mathcal{X}_1 = \mathcal{S}x$. Then $\mathcal{X}_1 = \mathcal{X}_0 \cup \left\{ \frac{K \in 3}{K \sqsupseteq B} \right\} = \mathcal{X}_0$.

So we proved that they are mutually inverse bijections. That they are order preserving is obvious. □

THEOREM 738. The diagram at the figure 7 (with the horizontal “unnamed” arrow *defined* as the inverse isomorphism of its opposite arrow) is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore “parallel” arrows are mutually inverse). The arrows preserve order.

PROOF. It's proved above, that all morphisms (except the “unnamed” arrow, which is the inverse morphism by definition) depicted on the diagram are bijections and the depicted “opposite” morphisms are mutually inverse.

That arrows preserve order is obvious.

It remains to apply lemma 196 (taking into account the proof of theorem 724). □

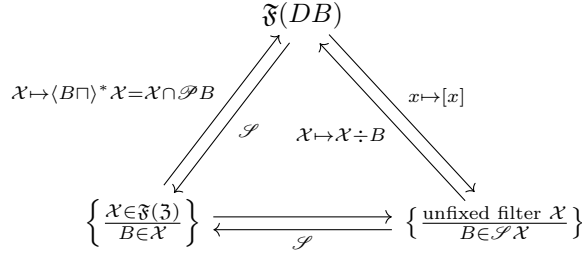


FIGURE 7

5.39.6. The lattice of unfixed filters.

THEOREM 739. Every nonempty set of unfixed filters has an infimum, provided that the lattice \mathfrak{Z} is distributive.

PROOF. Theorem 520. □

THEOREM 740. Every bounded above set of unfixed filters has a supremum.

PROOF. Theorem 515 for nonempty sets of unfixed filters. The join $\bigsqcup \emptyset = [\perp]$ for the least filter $\perp \in \mathfrak{Z}(DA)$ for arbitrary $A \in \mathfrak{Z}$. □

COROLLARY 741. If \mathfrak{Z} is the set of small sets, then every small set of unfixed filters has a supremum.

PROOF. Let S be a set of filters on \mathfrak{Z} . Then $T_{\mathcal{X}} \in \mathcal{X}$ is a small set for every $\mathcal{X} \in S$. Thus $\{\frac{T_{\mathcal{X}}}{\mathcal{X} \in S}\}$ is small set and thus $T = \bigcup \{\frac{T_{\mathcal{X}}}{\mathcal{X} \in S}\}$ is small set. Take the filter $\mathcal{T} = \uparrow T$. Then \mathcal{T} is an upper bound of S and we can apply the theorem. □

OBVIOUS 742. The poset of unfixed filters for the lattice of small sets is bounded below (but not above).

PROPOSITION 743. The set of unfixed filters forms a co-brouwerian (and thus distributive) lattice, provided that \mathfrak{Z} is distributive lattice which is an ideal base.

PROOF. Corollary 531. □

5.39.7. Principal unfixed filters and filtrator of unfixed filters.

DEFINITION 744. *Principal* unfixed filter is an unfixed filter corresponding to a principal filter on the poset \mathfrak{Z} .

DEFINITION 745. The *filtrator of unfixed filters* is the filtrator whose base are unfixed filters and whose core are principal unfixed filters.

We will equate principal unfixed filters with corresponding sets.

THEOREM 746. If we add principal filters on DB , principal filters on \mathfrak{Z} containing B , and above defined principal unfixed filters corresponding to them to appropriate nodes of the diagram 7, then the diagram turns into a commutative diagram of isomorphisms between filtrators. (I will not draw the modified diagram for brevity.)

Every arrow of this diagram is an isomorphism between filtrators, every cycle in the diagram is identity.

PROOF. We need to prove only that principal filters on B and principal filters on \mathfrak{Z} containing B correspond to each other by the isomorphisms of the diagram. But that's obvious. □

OBVIOUS 747. The filtrator of unfixed filters is a primary filtrator.

OBVIOUS 748. The filtrator of unfixed filters is down-aligned.

PROPOSITION 749. The filtrator of unfixed filters is

- 1°. filtered;
- 2°. with join-closed core.

PROOF. Theorem 534. □

PROPOSITION 750. The filtrator of unfixed filters is with binarily meet-closed core.

PROOF. Corollary 536. □

PROPOSITION 751. The filtrator of unfixed filters is with separable core.

PROOF. Theorem 537. □

PROPOSITION 752. $\text{Cor } \mathcal{X}$ and $\text{Cor}' \mathcal{X}$ are defined for every unfixed filter \mathcal{X} and $\text{Cor } \mathcal{X} = \text{Cor}' \mathcal{X}$, provided that every DA is a complete lattice.

PROOF. $\text{Cor } \mathcal{X}$ and $\text{Cor}' \mathcal{X}$ exists because of the above isomorphism.

$\text{Cor}' \mathcal{X} = \text{Cor } \mathcal{X}$ by theorem 545. □

OBVIOUS 753. $\text{Cor } \mathcal{X} = \text{Cor}' \mathcal{X} = \bigcap \mathcal{X}$ for every filter $\mathcal{X} \in \mathfrak{F}(\text{small sets})$.

PROPOSITION 754. $\text{atoms } \prod S = \bigcap \langle \text{atoms} \rangle^* S$ whenever $\prod S$ is defined.

PROOF. Theorem 108. □

PROPOSITION 755. $\text{atoms}(\mathcal{A} \sqcup \mathcal{B}) = \text{atoms } \mathcal{A} \cup \text{atoms } \mathcal{B}$ for unfixed filters \mathcal{A}, \mathcal{B} , whenever \mathfrak{J} is a distributive lattice which is an ideal base.

PROOF. Proposition 557. □

PROPOSITION 756. $\partial \mathcal{X}$ is a free star for every unfixed filter \mathcal{X} , whenever \mathfrak{J} is a distributive lattice which is an ideal base which has a least element.

PROOF. Theorem 566. □

PROPOSITION 757. The poset of unfixed filters is an atomistic lattice if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Easily follows from 738 by isomorphism. □

PROPOSITION 758. The poset of unfixed filters is a strongly separable lattice if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Theorem 234. □

PROPOSITION 759. $\text{Cor } \mathcal{X} = \bigsqcup (\mathfrak{J} \cap \text{atoms}^{\text{unfixed filters}})$ for every unfixed filter \mathcal{X} if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Theorem 599. □

PROPOSITION 760. $\text{Cor}(\mathcal{A} \sqcap \mathcal{B}) = \text{Cor } \mathcal{A} \sqcap \text{Cor } \mathcal{B}$ for every unfixed filters \mathcal{A}, \mathcal{B} , provided every DA (for $A \in \mathfrak{A}$) is a complete lattice.

PROOF. Theorem 601. □

PROPOSITION 761. $\text{Cor } \prod^{\mathfrak{A}} S = \prod^{\mathfrak{J}} \langle \text{Cor} \rangle^* S$ for the filtrator of unfixed filters for every nonempty set S of unfixed filters, provided every DA (for $A \in \mathfrak{A}$) is a complete lattice.

PROOF. Theorem 602. □

PROPOSITION 762. $\text{Cor}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{B}) = \text{Cor} \mathcal{A} \sqcup^3 \text{Cor} \mathcal{B}$ for the filtrator of unfixed filters for every unfixed filters \mathcal{A} , and \mathcal{B} , provided every DA (for $A \in \mathfrak{A}$) is a complete atomistic distributive lattice.

PROOF. Can be easily deduced from theorem 603 and the triangular diagram (above) of isomorphic filtrators. \square

CONJECTURE 763. The theorem 614 holds for unfixed filters, too.

It is expected to be easily provable using isomorphisms from the triangular diagram.

Common knowledge, part 2 (topology)

In this chapter I describe basics of the theory known as *general topology*. Starting with the next chapter after this one I will describe generalizations of customary objects of general topology described in this chapter.

The reason why I've written this chapter is to show to the reader kinds of objects which I generalize below in this book. For example, funcoids and a generalization of proximity spaces, and funcoids are a generalization of pretopologies. To understand the intuitive meaning of funcoids one needs first know what are proximities and what are pretopologies.

Having said that, customary topology is *not* used in my definitions and proofs below. It is just to feed your intuition.

6.1. Metric spaces

The theory of topological spaces started immediately with the definition would be completely non-intuitive for the reader. It is the reason why I first describe metric spaces and show that metric spaces give rise for a topology (see below). Topological spaces are understandable as a generalization of topologies induced by metric spaces.

Metric spaces is a formal way to express the notion of *distance*. For example, there are distance $|x - y|$ between real numbers x and y , distance between points of a plane, etc.

DEFINITION 764. A *metric space* is a set U together with a function $d : U \times U \rightarrow \mathbb{R}$ (*distance* or *metric*) such that for every $x, y, z \in U$:

- 1°. $d(x, y) \geq 0$;
- 2°. $d(x, y) = 0 \Leftrightarrow x = y$;
- 3°. $d(x, y) = d(y, x)$ (*symmetry*);
- 4°. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

EXERCISE 765. Show that the Euclid space \mathbb{R}^n (with the standard distance) is a metric space for every $n \in \mathbb{N}$.

DEFINITION 766. *Open ball* of *radius* $r > 0$ centered at point $a \in U$ is the set

$$B_r(a) = \left\{ \frac{x \in U}{d(a, x) < r} \right\}.$$

DEFINITION 767. *Closed ball* of *radius* $r > 0$ centered at point $a \in U$ is the set

$$B_r[a] = \left\{ \frac{x \in U}{d(a, x) \leq r} \right\}.$$

One example of use of metric spaces: *Limit* of a sequence x in a metric space can be defined as a point y in this space such that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N : d(x_n, y) < \epsilon.$$

6.1.1. Open and closed sets.

DEFINITION 768. A set A in a metric space is called *open* when $\forall a \in A \exists r > 0 : B_r(a) \subseteq A$.

DEFINITION 769. A set A in a metric space is closed when its complement $U \setminus A$ is open.

EXERCISE 770. Show that: closed intervals on real line are closed sets, open intervals are open sets.

EXERCISE 771. Show that open balls are open and closed balls are closed.

DEFINITION 772. Closure $\text{cl}(A)$ of a set A in a metric space is the set of points y such that

$$\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon.$$

PROPOSITION 773. $\text{cl}(A) \supseteq A$.

PROOF. It follows from $d(a, a) = 0 < \epsilon$. \square

EXERCISE 774. Prove $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ for every subsets A and B of a metric space.

6.2. Pretopological spaces

Pretopological space can be defined in two equivalent ways: a *neighborhood system* or a *preclosure operator*. To be more clear I will call *pretopological space* only the first (neighborhood system) and the second call a *preclosure space*.

DEFINITION 775. *Pretopological space* is a set U together with a filter $\Delta(x)$ on U for every $x \in U$, such that $\uparrow^U \{x\} \subseteq \Delta(x)$. Δ is called a *pretopology* on U . Elements of $\text{up } \Delta(x)$ are called *neighborhoods* of point x .

DEFINITION 776. *Preclosure* on a set U is a unary operation cl on $\mathcal{P}U$ such that for every $A, B \in \mathcal{P}U$:

- 1°. $\text{cl}(\emptyset) = \emptyset$;
- 2°. $\text{cl}(A) \supseteq A$;
- 3°. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

I call a preclosure together with a set U as *preclosure space*.

THEOREM 777. Small pretopological spaces and small preclosure spaces bijectively correspond to each other by the formulas:

$$\text{cl}(A) = \left\{ \frac{x \in U}{A \in \partial \Delta(x)} \right\}; \quad (3)$$

$$\text{up } \Delta(x) = \left\{ \frac{A \in \mathcal{P}U}{x \notin \text{cl}(U \setminus A)} \right\}. \quad (4)$$

PROOF. First let's prove that cl defined by formula (3) is really a preclosure.

$\text{cl}(\emptyset) = \emptyset$ is obvious. If $x \in A$ then $A \in \partial \Delta(x)$ and so $\text{cl}(A) \supseteq A$. $\text{cl}(A \cup B) = \left\{ \frac{x \in U}{A \cup B \in \partial \Delta(x)} \right\} = \left\{ \frac{x \in U}{A \in \partial \Delta(x) \vee B \in \partial \Delta(x)} \right\} = \text{cl}(A) \cup \text{cl}(B)$. So, it is really a preclosure.

Next let's prove that Δ defined by formula (4) is a pretopology. That $\text{up } \Delta(x)$ is an upper set is obvious. Let $A, B \in \text{up } \Delta(x)$. Then $x \notin \text{cl}(U \setminus A) \wedge x \notin \text{cl}(U \setminus B)$; $x \notin \text{cl}(U \setminus A) \cup \text{cl}(U \setminus B) = \text{cl}((U \setminus A) \cup (U \setminus B)) = \text{cl}(U \setminus (A \cap B))$; $A \cap B \in \text{up } \Delta(x)$. We have proved that $\Delta(x)$ is a filter object.

Let's prove $\uparrow^U \{x\} \subseteq \Delta(x)$. If $A \in \text{up } \Delta(x)$ then $x \notin \text{cl}(U \setminus A)$ and consequently $x \notin U \setminus A$; $x \in A$; $A \in \text{up } \uparrow^U \{x\}$. So $\uparrow^U \{x\} \subseteq \Delta(x)$ and thus Δ is a pretopology.

It is left to prove that the functions defined by the above formulas are mutually inverse.

Let cl_0 be a preclosure, let Δ be the pretopology induced by cl_0 by the formula (4), let cl_1 be the preclosure induced by Δ by the formula (3). Let's prove $\text{cl}_1 = \text{cl}_0$. Really,

$$\begin{aligned}
x \in \text{cl}_1(A) &\Leftrightarrow \\
\Delta(x) \not\prec^U A &\Leftrightarrow \\
\forall X \in \text{up } \Delta(x) : X \cap A \neq \emptyset &\Leftrightarrow \\
\forall X \in \mathcal{P}U : (x \notin \text{cl}_0(U \setminus X) \Rightarrow X \cap A \neq \emptyset) &\Leftrightarrow \\
\forall X' \in \mathcal{P}U : (x \notin \text{cl}_0(X') \Rightarrow A \setminus X' \neq \emptyset) &\Leftrightarrow \\
\forall X' \in \mathcal{P}U : (A \setminus X' = \emptyset \Rightarrow x \in \text{cl}_0(X')) &\Leftrightarrow \\
\forall X' \in \mathcal{P}U : (A \subseteq X' \Rightarrow x \in \text{cl}_0(X')) &\Leftrightarrow \\
x \in \text{cl}_0(A). &
\end{aligned}$$

So $\text{cl}_1(A) = \text{cl}_0(A)$.

Let now Δ_0 be a pretopology, let cl be the closure induced by Δ_0 by the formula (3), let Δ_1 be the pretopology induced by cl by the formula (4). Really

$$\begin{aligned}
A \in \text{up } \Delta_1(x) &\Leftrightarrow \\
x \notin \text{cl}(U \setminus A) &\Leftrightarrow \\
\Delta_0(x) \not\prec^U (U \setminus A) &\Leftrightarrow \text{(proposition 551)} \\
\uparrow^U A \supseteq \Delta_0(x) &\Leftrightarrow \\
A \in \text{up } \Delta_0(x). &
\end{aligned}$$

So $\Delta_1(x) = \Delta_0(x)$.

That these functions are mutually inverse, is now proved. \square

6.2.1. Pretopology induced by a metric. Every metric space induces a pretopology by the formula:

$$\Delta(x) = \prod^{\mathcal{F}U} \left\{ \frac{B_r(x)}{r \in \mathbb{R}, r > 0} \right\}.$$

EXERCISE 778. Show that it is a pretopology.

PROPOSITION 779. The preclosure corresponding to this pretopology is the same as the preclosure of the metric space.

PROOF. I denote the preclosure of the metric space as cl_M and the preclosure corresponding to our pretopology as cl_P . We need to show $\text{cl}_P = \text{cl}_M$. Really:

$$\begin{aligned}
\text{cl}_P(A) &= \\
\left\{ \frac{x \in U}{A \in \partial \Delta(x)} \right\} &= \\
\left\{ \frac{x \in U}{\forall \epsilon > 0 : B_\epsilon(x) \not\prec A} \right\} &= \\
\left\{ \frac{y \in U}{\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon} \right\} &= \\
\text{cl}_M(A) &
\end{aligned}$$

for every set $A \in \mathcal{P}U$. \square

6.3. Topological spaces

PROPOSITION 780. For the set of open sets of a metric space (U, d) it holds:

- 1°. Union of any (possibly infinite) number of open sets is an open set.
- 2°. Intersection of a finite number of open sets is an open set.
- 3°. U is an open set.

PROOF. Let S be a set of open sets. Let $a \in \bigcup S$. Then there exists $A \in S$ such that $a \in A$. Because A is open we have $B_r(a) \subseteq A$ for some $r > 0$. Consequently $B_r(a) \subseteq \bigcup S$ that is $\bigcup S$ is open.

Let A_0, \dots, A_n be open sets. Let $a \in A_0 \cap \dots \cap A_n$ for some $n \in \mathbb{N}$. Then there exist r_i such that $B_{r_i}(a) \subseteq A_i$. So $B_r(a) \subseteq A_0 \cap \dots \cap A_n$ for $r = \min\{r_0, \dots, r_n\}$ that is $A_0 \cap \dots \cap A_n$ is open.

That U is an open set is obvious. \square

The above proposition suggests the following definition:

DEFINITION 781. A *topology* on a set U is a collection \mathcal{O} (called the set of *open sets*) of subsets of U such that:

- 1°. Union of any (possibly infinite) number of open sets is an open set.
- 2°. Intersection of a finite number of open sets is an open set.
- 3°. U is an open set.

The pair (U, \mathcal{O}) is called a *topological space*.

REMARK 782. From the above it is clear that every metric induces a topology.

PROPOSITION 783. Empty set is always open.

PROOF. Empty set is union of an empty set. \square

DEFINITION 784. A *closed set* is a complement of an open set.

Topology can be equivalently expressed in terms of closed sets:

A *topology* on a set U is a collection (called the set of *closed sets*) of subsets of U such that:

- 1°. Intersection of any (possibly infinite) number of closed sets is a closed set.
- 2°. Union of a finite number of closed sets is a closed set.
- 3°. \emptyset is a closed set.

EXERCISE 785. Show that the definitions using open and closed sets are equivalent.

6.3.1. Relationships between pretopologies and topologies.

6.3.1.1. *Topological space induced by preclosure space.* Having a preclosure space (U, cl) we define a topological space whose closed sets are such sets $A \in \mathcal{P}U$ that $\text{cl}(A) = A$.

PROPOSITION 786. This really defines a topology.

PROOF. Let S be a set of closed sets. First, we need to prove that $\bigcap S$ is a closed set. We have $\text{cl}(\bigcap S) \subseteq A$ for every $A \in S$. Thus $\text{cl}(\bigcap S) \subseteq \bigcap S$ and consequently $\text{cl}(\bigcap S) = \bigcap S$. So $\bigcap S$ is a closed set.

Let now A_0, \dots, A_n be closed sets, then

$$\text{cl}(A_0 \cup \dots \cup A_n) = \text{cl}(A_0) \cup \dots \cup \text{cl}(A_n) = A_0 \cup \dots \cup A_n$$

that is $A_0 \cup \dots \cup A_n$ is a closed set.

That \emptyset is a closed set is obvious. \square

Having a pretopological space (U, Δ) we define a topological space whose open sets are

$$\left\{ \frac{X \in \mathcal{P}U}{\forall x \in X : X \in \text{up } \Delta(x)} \right\}.$$

PROPOSITION 787. This really defines a topology.

PROOF. Let set $S \subseteq \left\{ \frac{X \in \mathcal{P}U}{\forall x \in X : X \in \text{up } \Delta(x)} \right\}$. Then $\forall X \in S \forall x \in X : X \in \text{up } \Delta(x)$. Thus

$$\forall x \in \bigcup S \exists X \in S : X \in \text{up } \Delta(x)$$

and so $\forall x \in \bigcup S : \bigcup S \in \text{up } \Delta(x)$. So $\bigcup S$ is an open set.

Let now $A_0, \dots, A_n \in \left\{ \frac{X \in \mathcal{P}U}{\forall x \in X : X \in \text{up } \Delta(x)} \right\}$ for $n \in \mathbb{N}$. Then $\forall x \in A_i : A_i \in \text{up } \Delta(x)$ and so

$$\forall x \in A_0 \cap \dots \cap A_n : A_i \in \text{up } \Delta(x);$$

thus $\forall x \in A_0 \cap \dots \cap A_n : A_0 \cap \dots \cap A_n \in \text{up } \Delta(x)$. So $A_0 \cap \dots \cap A_n \in \left\{ \frac{X \in \mathcal{P}U}{\forall x \in X : X \in \text{up } \Delta(x)} \right\}$.

That U is an open set is obvious. \square

PROPOSITION 788. Topology τ defined by a pretopology and topology ρ defined by the corresponding preclosure, are the same.

PROOF. Let $A \in \mathcal{P}U$.

A is ρ -closed $\Leftrightarrow \text{cl}(A) = A \Leftrightarrow \text{cl}(A) \subseteq A \Leftrightarrow \forall x \in U : (A \in \partial \Delta(x) \Rightarrow x \in A)$;

A is τ -open \Leftrightarrow

$$\forall x \in A : A \in \text{up } \Delta(x) \Leftrightarrow$$

$$\forall x \in U : (x \in A \Rightarrow A \in \text{up } \Delta(x)) \Leftrightarrow$$

$$\forall x \in U : (x \notin U \setminus A \Rightarrow U \setminus A \notin \partial \Delta(x)).$$

So ρ -closed and τ -open sets are complements of each other. It follows $\rho = \tau$. \square

6.3.1.2. *Preclosure space induced by topological space.* We define a preclosure and a pretopology induced by a topology and then show these two are equivalent. Having a topological space we define a preclosure space by the formula

$$\text{cl}(A) = \bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set, } X \supseteq A} \right\}.$$

PROPOSITION 789. It is really a preclosure.

PROOF. $\text{cl}(\emptyset) = \emptyset$ because \emptyset is a closed set. $\text{cl}(A) \supseteq A$ is obvious.

$$\begin{aligned} \text{cl}(A \cup B) &= \\ &= \bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set, } X \supseteq A \cup B} \right\} = \\ &= \bigcap \left\{ \frac{X_1 \cup X_2}{X_1, X_2 \in \mathcal{P}U \text{ are closed sets, } X_1 \supseteq A, X_2 \supseteq B} \right\} = \\ &= \bigcap \left\{ \frac{X_1 \in \mathcal{P}U}{X_1 \text{ is a closed set, } X_1 \supseteq A} \right\} \cup \bigcap \left\{ \frac{X_2 \in \mathcal{P}U}{X_2 \text{ is a closed set, } X_2 \supseteq B} \right\} = \\ &= \text{cl}(A) \cup \text{cl}(B). \end{aligned}$$

Thus cl is a preclosure. \square

Or: $\Delta(x) = \bigcap_{x \in X} \left\{ \frac{X \in \mathcal{O}}{x \in X} \right\}$.

It is trivially a pretopology (used the fact that $U \in \mathcal{O}$).

PROPOSITION 790. The preclosure and the pretopology defined in this section above correspond to each other (by the formulas from theorem 777).

PROOF. We need to prove $\text{cl}(A) = \left\{ \frac{x \in U}{\Delta(x) \not\uparrow^U A} \right\}$, that is

$$\bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set, } X \supseteq A} \right\} = \left\{ \frac{x \in U}{\prod^{\mathcal{P}U} \left\{ \frac{X \in \mathcal{O}}{x \in X} \right\} \not\uparrow^U A} \right\}.$$

Equivalently transforming it, we get:

$$\begin{aligned} \bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set, } X \supseteq A} \right\} &= \left\{ \frac{x \in U}{\forall X \in \mathcal{O} : (x \in X \Rightarrow \uparrow^U X \not\uparrow^U A)} \right\}; \\ \bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set, } X \supseteq A} \right\} &= \left\{ \frac{x \in U}{\forall X \in \mathcal{O} : (x \in X \Rightarrow X \not\uparrow A)} \right\}. \end{aligned}$$

We have

$$\begin{aligned} x \in \bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set, } X \supseteq A} \right\} &\Leftrightarrow \\ \forall X \in \mathcal{P}U : (X \text{ is a closed set} \wedge X \supseteq A \Rightarrow x \in X) &\Leftrightarrow \\ \forall X' \in \mathcal{O} : (U \setminus X' \supseteq A \Rightarrow x \in U \setminus X') &\Leftrightarrow \\ \forall X' \in \mathcal{O} : (X' \asymp A \Rightarrow x \notin X') &\Leftrightarrow \\ \forall X \in \mathcal{O} : (x \in X \Rightarrow X \not\uparrow A). & \end{aligned}$$

So our equivalence holds. \square

PROPOSITION 791. If τ is the topology induced by pretopology π , in turn induced by topology ρ , then $\tau = \rho$.

PROOF. The set of closed sets of τ is

$$\begin{aligned} \left\{ \frac{A \in \mathcal{P}U}{\text{cl}_\pi(A) = A} \right\} &= \\ \left\{ \frac{A \in \mathcal{P}U}{\bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set in } \rho, X \supseteq A} \right\} = A} \right\} &= \\ \left\{ \frac{A \in \mathcal{P}U}{A \text{ is a closed set in } \rho} \right\} & \end{aligned}$$

(taken into account that intersecting closed sets is a closed set). \square

DEFINITION 792. Idempotent closures are called *Kuratowski closures*.

THEOREM 793. The above defined correspondences between topologies and pretopologies, restricted to Kuratowski closures, is a bijection.

PROOF. Taking into account the above proposition, it's enough to prove that:

If τ is the pretopology induced by topology π , in turn induced by a Kuratowski closure ρ , then $\tau = \rho$.

$$\begin{aligned}
 \text{cl}_\tau(A) &= \\
 \bigcap \left\{ \frac{X \in \mathcal{P}U}{X \text{ is a closed set in } \pi, X \supseteq A} \right\} &= \\
 \bigcap \left\{ \frac{X \in \mathcal{P}U}{\text{cl}_\rho(X) = X, X \supseteq A} \right\} &= \\
 \bigcap \left\{ \frac{\text{cl}_\rho(X)}{X \in \mathcal{P}U, \text{cl}_\rho(X) = X, X \supseteq \text{cl}_\rho(A)} \right\} &= \\
 \bigcap \left\{ \frac{\text{cl}_\rho(\text{cl}_\rho(X))}{X = A} \right\} &= \\
 \text{cl}_\rho(\text{cl}_\rho(A)) &= \\
 \text{cl}_\rho(A). &
 \end{aligned}$$

□

6.3.1.3. Topology induced by a metric.

DEFINITION 794. Every metric space induces a topology in this way: A set X is open iff

$$\forall x \in X \exists \epsilon > 0 : B_r(x) \subseteq X.$$

EXERCISE 795. Prove it is really a topology and this topology is the same as the topology, induced by the pretopology, in turn induced by our metric space.

6.4. Proximity spaces

Let (U, d) be metric space. We will define *distance* between sets $A, B \in \mathcal{P}U$ by the formula

$$d(A, B) = \inf \left\{ \frac{d(a, b)}{a \in A, b \in B} \right\}.$$

(Here “inf” denotes infimum on the real line.)

DEFINITION 796. Sets $A, B \in \mathcal{P}U$ are *near* (denoted $A \delta B$) iff $d(A, B) = 0$.

δ defined in this way (for a metric space) is an example of proximity as defined below.

DEFINITION 797. A *proximity space* is a set (U, δ) conforming to the following axioms (for every $A, B, C \in \mathcal{P}U$):

- 1°. $A \cap B \neq \emptyset \Rightarrow A \delta B$;
- 2°. if $A \delta B$ then $A \neq \emptyset$ and $B \neq \emptyset$;
- 3°. $A \delta B \Rightarrow B \delta A$ (*symmetry*);
- 4°. $(A \cup B) \delta C \Leftrightarrow A \delta C \vee B \delta C$;
- 5°. $C \delta (A \cup B) \Leftrightarrow C \delta A \vee C \delta B$;
- 6°. $A \bar{\delta} B$ implies existence of $P, Q \in \mathcal{P}U$ with $A \bar{\delta} P$, $B \bar{\delta} Q$ and $P \cup Q = U$.

EXERCISE 798. Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

DEFINITION 799. *Quasi-proximity* is defined as the above but without the symmetry axiom.

DEFINITION 800. Closure is generated by a proximity by the following formula:

$$\text{cl}(A) = \left\{ \frac{a \in U}{\{a\} \delta A} \right\}.$$

PROPOSITION 801. Every closure generated by a proximity is a Kuratowski closure.

PROOF. First prove it is a preclosure. $\text{cl}(\emptyset) = \emptyset$ is obvious. $\text{cl}(A) \supseteq A$ is obvious.

$$\begin{aligned} \text{cl}(A \cup B) &= \\ \left\{ \frac{a \in U}{\{a\} \delta A \cup B} \right\} &= \\ \left\{ \frac{a \in U}{\{a\} \delta A \vee \{a\} \delta B} \right\} &= \\ \left\{ \frac{a \in U}{\{a\} \delta A} \right\} \cup \left\{ \frac{a \in U}{\{a\} \delta B} \right\} &= \\ \text{cl}(A) \cup \text{cl}(B). & \end{aligned}$$

It is remained to prove that cl is idempotent, that is $\text{cl}(\text{cl}(A)) = \text{cl}(A)$. It is enough to show $\text{cl}(\text{cl}(A)) \subseteq \text{cl}(A)$ that is if $x \notin \text{cl}(A)$ then $x \notin \text{cl}(\text{cl}(A))$.

If $x \notin \text{cl}(A)$ then $\{x\} \bar{\delta} A$. So there are $P, Q \in \mathcal{P}U$ such that $\{x\} \bar{\delta} P$, $A \bar{\delta} Q$, $P \cup Q = U$. Then $U \setminus Q \subseteq P$, so $\{x\} \bar{\delta} U \setminus Q$ and hence $x \in Q$. Hence $U \setminus \text{cl}(A) \subseteq Q$, and so $\text{cl}(A) \subseteq U \setminus Q \subseteq P$. Consequently $\{x\} \bar{\delta} \text{cl}(A)$ and hence $x \notin \text{cl}(\text{cl}(A))$. \square

6.5. Definition of uniform spaces

Here I will present the traditional definition of uniform spaces. Below in the chapter about reloids I will present a shortened and more algebraic (however a little less elementary) definition of uniform spaces.

DEFINITION 802. *Uniform space* is a pair (U, D) of a set U and filter $D \in \mathfrak{F}(U \times U)$ (called *uniformity* or the set of *entourages*) such that:

- 1°. If $F \in D$ then $\text{id}_U \subseteq F$.
- 2°. If $F \in D$ then there exists $G \in D$ such that $G \circ G \subseteq F$.
- 3°. If $F \in D$ then $F^{-1} \in D$.

Part 2

Funcoids and reloids

CHAPTER 7

Functoids

In this chapter (and several following chapters) the word *filter* will refer to a filter (or equivalently any filter object) on a set (rather than a filter on an arbitrary poset).

7.1. Informal introduction into functoids

Functoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also functoids are a generalization of binary relations.

That functoids are a common generalization of “spaces” (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement “ f is a continuous function from a space μ to a space ν ” can be described in terms of functoids as the formula $f \circ \mu \sqsubseteq \nu \circ f$ (see below for details).

Most naturally functoids appear as a generalization of proximity spaces.¹

Let δ be a proximity. We will extend the relation δ from sets to filters by the formula:

$$\mathcal{A} \delta' \mathcal{B} \Leftrightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : A \delta B.$$

Then (as it will be proved below) there exist two functions $\alpha, \beta \in \mathcal{F}^{\mathcal{F}}$ such that

$$\mathcal{A} \delta' \mathcal{B} \Leftrightarrow \mathcal{B} \sqcap \alpha \mathcal{A} \neq \perp^{\mathcal{F}} \Leftrightarrow \mathcal{A} \sqcap \beta \mathcal{B} \neq \perp^{\mathcal{F}}.$$

The pair (α, β) is called *functoid* when $\mathcal{B} \sqcap \alpha \mathcal{A} \neq \perp^{\mathcal{F}} \Leftrightarrow \mathcal{A} \sqcap \beta \mathcal{B} \neq \perp^{\mathcal{F}}$. So functoids are a generalization of proximity spaces.

Functoids consist of two components the first α and the second β . The first component of a functoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of principal functoids (see below) these coincide.)

One of the most important properties of a functoid is that it is uniquely determined by just one of its components. That is a functoid f is uniquely determined by the function $\langle f \rangle$. Moreover a functoid f is uniquely determined by values of $\langle f \rangle$ on principal filters.

Next we will consider some examples of functoids determined by specified values of the first component on sets.

Functoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathcal{F}^{\mathcal{U}}$ for some set \mathcal{U} . Then we define $\alpha' X = \bigsqcup_{x \in X} \alpha x$ for every set $X \in \mathcal{P}\mathcal{U}$. We will prove that there exists a unique functoid f such that $\alpha' = \langle f \rangle|_{\mathfrak{P}} \circ \uparrow$ where \mathfrak{P} is the set of principal filters on \mathcal{U} . So functoids are a generalization of pretopological spaces. Functoids are also a generalization of preclosure operators: For every preclosure operator p on a set \mathcal{U} it exists a unique functoid f such that $\langle f \rangle|_{\mathfrak{P}} \circ \uparrow = \uparrow \circ p$.

¹In fact I discovered functoids pondering on topological spaces, not on proximity spaces, but this is only of a historic interest.

For every binary relation p on a set \mathcal{U} there exists unique funcoïd f such that

$$\forall X \in \mathcal{P}\mathcal{U} : \langle f \rangle \uparrow X = \uparrow \langle p \rangle^* X$$

(where $\langle p \rangle^*$ is defined in the introduction), recall that a funcoïd is uniquely determined by the values of its first component on sets. I will call such funcoïds *principal*. So funcoïds are a generalization of binary relations.

Composition of binary relations (i.e. of principal funcoïds) complies with the formulas:

$$\langle g \circ f \rangle^* = \langle g \rangle^* \circ \langle f \rangle^* \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle^* = \langle f^{-1} \rangle^* \circ \langle g^{-1} \rangle^*.$$

By similar formulas we can define composition of every two funcoïds. Funcoïds with this composition form a category (*the category of funcoïds*).

Also funcoïds can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha, \beta)^{-1} = (\beta, \alpha)$. In the particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoïds behave similarly to (multivalued) functions but acting on filters instead of acting on sets. Below there will be defined domain and image of a funcoïd (the domain and the image of a funcoïd are filters).

7.2. Basic definitions

DEFINITION 803. Let us call a *funcoïd* from a set A to a set B a quadruple (A, B, α, β) where $\alpha \in \mathcal{F}(B)^{\mathcal{F}(A)}$, $\beta \in \mathcal{F}(A)^{\mathcal{F}(B)}$ such that

$$\forall \mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B) : (\mathcal{Y} \neq \alpha \mathcal{X} \Leftrightarrow \mathcal{X} \neq \beta \mathcal{Y}).$$

DEFINITION 804. *Source* and *destination* of every funcoïd (A, B, α, β) are defined as:

$$\text{Src}(A, B, \alpha, \beta) = A \quad \text{and} \quad \text{Dst}(A, B, \alpha, \beta) = B.$$

I will denote $\text{FCD}(A, B)$ the set of funcoïds from A to B .

I will denote FCD the set of all funcoïds (for small sets).

DEFINITION 805. I will call an *endofuncoïd* a funcoïd whose source is the same as it's destination.

DEFINITION 806. $\langle (A, B, \alpha, \beta) \rangle \stackrel{\text{def}}{=} \alpha$ for a funcoïd (A, B, α, β) .

DEFINITION 807. The *reverse* funcoïd $(A, B, \alpha, \beta)^{-1} = (B, A, \beta, \alpha)$ for a funcoïd (A, B, α, β) .

NOTE 808. The reverse funcoïd is *not* an inverse in the sense of group theory or category theory.

PROPOSITION 809. If f is a funcoïd then f^{-1} is also a funcoïd.

PROOF. It follows from symmetry in the definition of funcoïd. \square

OBVIOUS 810. $(f^{-1})^{-1} = f$ for a funcoïd f .

DEFINITION 811. The relation $[f] \in \mathcal{P}(\mathcal{F}(\text{Src } f) \times \mathcal{F}(\text{Dst } f))$ is defined (for every funcoïd f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$) by the formula $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \neq \langle f \rangle \mathcal{X}$.

OBVIOUS 812. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \neq \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \neq \langle f^{-1} \rangle \mathcal{Y}$ for every funcoïd f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$.

OBVIOUS 813. $[f^{-1}] = [f]^{-1}$ for a funcoïd f .

THEOREM 814. Let A, B be sets.

- 1°. For given value of $\langle f \rangle \in \mathcal{F}(B)^{\mathcal{F}(A)}$ there exists no more than one funcoïd $f \in \text{FCD}(A, B)$.
- 2°. For given value of $[f] \in \mathcal{P}(\mathcal{F}(A) \times \mathcal{F}(B))$ there exists no more than one funcoïd $f \in \text{FCD}(A, B)$.

PROOF. Let $f, g \in \text{FCD}(A, B)$.

Obviously, $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\langle f^{-1} \rangle = \langle g^{-1} \rangle \Rightarrow [f] = [g]$. So it's enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.

Provided that $[f] = [g]$ we have

$$\mathcal{Y} \neq \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{X} [g] \mathcal{Y} \Leftrightarrow \mathcal{Y} \neq \langle g \rangle \mathcal{X}$$

and consequently $\langle f \rangle \mathcal{X} = \langle g \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Y} \in \mathcal{F}(B)$ because a set of filters is separable, thus $\langle f \rangle = \langle g \rangle$. \square

PROPOSITION 815. $\langle f \rangle \perp = \perp$ for every funcoïd f .

PROOF. $\mathcal{Y} \neq \langle f \rangle \perp \Leftrightarrow \perp \neq \langle f^{-1} \rangle \mathcal{Y} \Leftrightarrow 0 \Leftrightarrow \mathcal{Y} \neq \perp$. Thus $\langle f \rangle \perp = \perp$ by separability of filters. \square

PROPOSITION 816. $\langle f \rangle (\mathcal{I} \sqcup \mathcal{J}) = \langle f \rangle \mathcal{I} \sqcup \langle f \rangle \mathcal{J}$ for every funcoïd f and $\mathcal{I}, \mathcal{J} \in \mathcal{F}(\text{Src } f)$.

PROOF.

$$\begin{aligned} \star \langle f \rangle (\mathcal{I} \sqcup \mathcal{J}) &= \\ \left\{ \frac{\mathcal{Y} \in \mathcal{F}}{\mathcal{Y} \neq \langle f \rangle (\mathcal{I} \sqcup \mathcal{J})} \right\} &= \\ \left\{ \frac{\mathcal{Y} \in \mathcal{F}}{\mathcal{I} \sqcup \mathcal{J} \neq \langle f^{-1} \rangle \mathcal{Y}} \right\} &= \\ \left\{ \frac{\mathcal{Y} \in \mathcal{F}}{\mathcal{I} \neq \langle f^{-1} \rangle \mathcal{Y} \vee \mathcal{J} \neq \langle f^{-1} \rangle \mathcal{Y}} \right\} &= \\ \left\{ \frac{\mathcal{Y} \in \mathcal{F}}{\mathcal{Y} \neq \langle f \rangle \mathcal{I} \vee \mathcal{Y} \neq \langle f \rangle \mathcal{J}} \right\} &= \\ \left\{ \frac{\mathcal{Y} \in \mathcal{F}}{\mathcal{Y} \neq \langle f \rangle \mathcal{I} \sqcup \langle f \rangle \mathcal{J}} \right\} &= \\ \star \langle f \rangle \mathcal{I} \sqcup \langle f \rangle \mathcal{J}. & \end{aligned}$$

Thus $\langle f \rangle (\mathcal{I} \sqcup \mathcal{J}) = \langle f \rangle \mathcal{I} \sqcup \langle f \rangle \mathcal{J}$ because $\mathcal{F}(\text{Dst } f)$ is separable. \square

PROPOSITION 817. For every $f \in \text{FCD}(A, B)$ for every sets A and B we have:

- 1°. $\mathcal{K} [f] \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} [f] \mathcal{I} \vee \mathcal{K} [f] \mathcal{J}$ for every $\mathcal{I}, \mathcal{J} \in \mathcal{F}(B)$, $\mathcal{K} \in \mathcal{F}(A)$.
- 2°. $\mathcal{I} \sqcup \mathcal{J} [f] \mathcal{K} \Leftrightarrow \mathcal{I} [f] \mathcal{K} \vee \mathcal{J} [f] \mathcal{K}$ for every $\mathcal{I}, \mathcal{J} \in \mathcal{F}(A)$, $\mathcal{K} \in \mathcal{F}(B)$.

PROOF.

1°.

$$\begin{aligned} \mathcal{K} [f] \mathcal{I} \sqcup \mathcal{J} &\Leftrightarrow \\ (\mathcal{I} \sqcup \mathcal{J}) \cap \langle f \rangle \mathcal{K} \neq \perp^{\mathcal{F}(B)} &\Leftrightarrow \\ \mathcal{I} \cap \langle f \rangle \mathcal{K} \neq \perp^{\mathcal{F}(B)} \vee \mathcal{J} \cap \langle f \rangle \mathcal{K} \neq \perp^{\mathcal{F}(B)} &\Leftrightarrow \\ \mathcal{K} [f] \mathcal{I} \vee \mathcal{K} [f] \mathcal{J}. & \end{aligned}$$

2°. Similar. \square

7.2.1. Composition of functors.

DEFINITION 818. Functors f and g are *composable* when $\text{Dst } f = \text{Src } g$.

DEFINITION 819. *Composition* of composable functors is defined by the formula

$$(B, C, \alpha_2, \beta_2) \circ (A, B, \alpha_1, \beta_1) = (A, C, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2).$$

PROPOSITION 820. If f, g are composable functors then $g \circ f$ is a functor.

PROOF. Let $f = (A, B, \alpha_1, \beta_1)$, $g = (B, C, \alpha_2, \beta_2)$. For every $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Y} \in \mathcal{F}(C)$ we have

$$\mathcal{Y} \neq (\alpha_2 \circ \alpha_1)\mathcal{X} \Leftrightarrow \mathcal{Y} \neq \alpha_2\alpha_1\mathcal{X} \Leftrightarrow \alpha_1\mathcal{X} \neq \beta_2\mathcal{Y} \Leftrightarrow \mathcal{X} \neq \beta_1\beta_2\mathcal{Y} \Leftrightarrow \mathcal{X} \neq (\beta_1 \circ \beta_2)\mathcal{Y}.$$

So $(A, C, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$ is a functor. \square

OBVIOUS 821. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable functors f and g .

PROPOSITION 822. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable functors f, g, h .

PROOF.

$$\begin{aligned} \langle (h \circ g) \circ f \rangle &= \\ \langle h \circ g \rangle \circ \langle f \rangle &= \\ (\langle h \rangle \circ \langle g \rangle) \circ \langle f \rangle &= \\ \langle h \rangle \circ (\langle g \rangle \circ \langle f \rangle) &= \\ \langle h \rangle \circ \langle g \circ f \rangle &= \\ \langle h \circ (g \circ f) \rangle. & \end{aligned}$$

\square

THEOREM 823. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable functors f and g .

PROOF. $\langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle = \langle f^{-1} \circ g^{-1} \rangle$. \square

7.3. Functor as continuation

Let f be a functor.

DEFINITION 824. $\langle f \rangle^*$ is the function $\mathcal{T}(\text{Src } f) \rightarrow \mathcal{F}(\text{Dst } f)$ defined by the formula

$$\langle f \rangle^* X = \langle f \rangle \uparrow X.$$

DEFINITION 825. $[f]^*$ is the relation between $\mathcal{T}(\text{Src } f)$ and $\mathcal{T}(\text{Dst } f)$ defined by the formula

$$X [f]^* Y \Leftrightarrow \uparrow X [f] \uparrow Y.$$

OBVIOUS 826.

- 1°. $\langle f \rangle^* = \langle f \rangle \circ \uparrow$;
- 2°. $[f]^* = \uparrow^{-1} \circ [f] \circ \uparrow$.

OBVIOUS 827. $\langle g \rangle \langle f \rangle^* X = \langle g \circ f \rangle^* X$ for every $X \in \mathcal{T}(\text{Src } f)$.

THEOREM 828. For every functor f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$

- 1°. $\langle f \rangle \mathcal{X} = \prod \langle \langle f \rangle^* \rangle^* \text{ up } \mathcal{X}$;
- 2°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X [f]^* Y$.

PROOF.

2°.

$$\begin{aligned}
\mathcal{X} [f] \mathcal{Y} &\Leftrightarrow \\
\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} &\neq \perp \Leftrightarrow \\
\forall Y \in \text{up } \mathcal{Y} : \uparrow Y \sqcap \langle f \rangle \mathcal{X} &\neq \perp \Leftrightarrow \\
\forall Y \in \text{up } \mathcal{Y} : \mathcal{X} [f] \uparrow Y &.
\end{aligned}$$

Analogously $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X} : \uparrow X [f] \mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : \uparrow X [f] \uparrow Y \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X [f]^* Y.$$

1°.

$$\begin{aligned}
\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} &\neq \perp \Leftrightarrow \\
\mathcal{X} [f] \mathcal{Y} &\Leftrightarrow \\
\forall X \in \text{up } \mathcal{X} : \uparrow X [f] \mathcal{Y} &\Leftrightarrow \\
\forall X \in \text{up } \mathcal{X} : \mathcal{Y} \sqcap \langle f \rangle^* X &\neq \perp.
\end{aligned}$$

Let's denote $W = \left\{ \frac{\mathcal{Y} \sqcap \langle f \rangle^* X}{X \in \text{up } \mathcal{X}} \right\}$. We will prove that W is a generalized filter base. To prove this it is enough to show that $V = \left\{ \frac{\langle f \rangle^* X}{X \in \text{up } \mathcal{X}} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f \rangle^* A$, $\mathcal{Q} = \langle f \rangle^* B$ where $A, B \in \text{up } \mathcal{X}$; $A \sqcap B \in \text{up } \mathcal{X}$ and $\mathcal{R} \sqsubseteq \mathcal{P} \sqcap \mathcal{Q}$ for $\mathcal{R} = \langle f \rangle^* (A \sqcap B) \in V$. So V is a generalized filter base and thus W is a generalized filter base.

$\perp \notin W \Leftrightarrow \prod W \neq \perp$ by properties of generalized filter bases. That is

$$\forall X \in \text{up } \mathcal{X} : \mathcal{Y} \sqcap \langle f \rangle^* X \neq \perp \stackrel{\mathcal{F}(\text{Dst } f)}{\Leftrightarrow} \mathcal{Y} \sqcap \prod \langle \langle f \rangle^* \rangle^* \text{up } \mathcal{X} \neq \perp.$$

Comparing with the above, $\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \perp \stackrel{\mathcal{F}(\text{Dst } f)}{\Leftrightarrow} \mathcal{Y} \sqcap \prod \langle \langle f \rangle^* \rangle^* \text{up } \mathcal{X} \neq \perp$. So $\langle f \rangle \mathcal{X} = \prod \langle \langle f \rangle^* \rangle^* \text{up } \mathcal{X}$ because the lattice of filters is separable. □

COROLLARY 829. Let f be a funcoid.

- 1°. The value of f can be restored from the value of $\langle f \rangle^*$.
- 2°. The value of f can be restored from the value of $[f]^*$.

PROPOSITION 830. For every $f \in \text{FCD}(A, B)$ we have (for every $I, J \in \mathcal{T}A$)

$$\langle f \rangle^* \perp = \perp, \quad \langle f \rangle^* (I \sqcup J) = \langle f \rangle^* I \sqcup \langle f \rangle^* J$$

and

$$\neg(I [f]^* \perp), I \sqcup J [f]^* K \Leftrightarrow I [f]^* K \vee J [f]^* K$$

(for every $I, J \in \mathcal{T}A, K \in \mathcal{T}B$),

$$\neg(\perp [f]^* I), K [f]^* I \sqcup J \Leftrightarrow K [f]^* I \vee K [f]^* J$$

(for every $I, J \in \mathcal{T}B, K \in \mathcal{T}A$).

PROOF. $\langle f \rangle^* \perp = \langle f \rangle \perp = \langle f \rangle \perp = \perp$;

$$\langle f \rangle^* (I \sqcup J) = \langle f \rangle \uparrow (I \sqcup J) = \langle f \rangle \uparrow I \sqcup \langle f \rangle \uparrow J = \langle f \rangle^* I \sqcup \langle f \rangle^* J.$$

$$I [f]^* \perp \Leftrightarrow \perp \neq \langle f \rangle \uparrow I \Leftrightarrow 0;$$

$$\begin{aligned} I \sqcup J [f]^* K &\Leftrightarrow \\ \uparrow (I \sqcup J) [f] \uparrow K &\Leftrightarrow \\ \uparrow K \neq \langle f \rangle \uparrow (I \sqcup J) &\Leftrightarrow \\ \uparrow K \neq \langle f \rangle^* (I \sqcup J) &\Leftrightarrow \\ \uparrow K \neq \langle f \rangle^* I \sqcup \langle f \rangle^* J &\Leftrightarrow \\ \uparrow K \neq \langle f \rangle^* I \vee \uparrow K \neq \langle f \rangle^* J &\Leftrightarrow \\ I [f]^* K \vee J [f]^* K. & \end{aligned}$$

The rest follows from symmetry. \square

THEOREM 831. (fundamental theorem of theory of funcoids) Fix sets A and B . Let $L_F = \lambda f \in \text{FCD}(A, B) : \langle f \rangle^*$ and $L_R = \lambda f \in \text{FCD}(A, B) : [f]^*$.

1°. L_F is a bijection from the set $\text{FCD}(A, B)$ to the set of functions $\alpha \in \mathcal{F}(B)^{\mathcal{F}A}$ that obey the conditions (for every $I, J \in \mathcal{T}A$)

$$\alpha \perp = \perp, \quad \alpha(I \sqcup J) = \alpha I \sqcup \alpha J. \quad (5)$$

For such α it holds (for every $\mathcal{X} \in \mathcal{F}(A)$)

$$\langle L_F^{-1} \alpha \rangle \mathcal{X} = \prod \langle \alpha \rangle^* \text{ up } \mathcal{X}. \quad (6)$$

2°. L_R is a bijection from the set $\text{FCD}(A, B)$ to the set of binary relations $\delta \in \mathcal{P}(\mathcal{T}A \times \mathcal{T}B)$ that obey the conditions

$$\begin{aligned} \neg(I \delta \perp), \quad I \sqcup J \delta K &\Leftrightarrow I \delta K \vee J \delta K \quad (\text{for every } I, J \in \mathcal{T}A, K \in \mathcal{T}B), \\ \neg(\perp \delta I), \quad K \delta I \sqcup J &\Leftrightarrow K \delta I \vee K \delta J \quad (\text{for every } I, J \in \mathcal{T}B, K \in \mathcal{T}A). \end{aligned} \quad (7)$$

For such δ it holds (for every $\mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B)$)

$$\mathcal{X} [L_R^{-1} \delta] \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \delta Y. \quad (8)$$

PROOF. Injectivity of L_F and L_R , formulas (6) (for $\alpha \in \text{im } L_F$) and (8) (for $\delta \in \text{im } L_R$), formulas (5) and (7) follow from two previous theorems. The only thing remaining to prove is that for every α and δ that obey the above conditions a corresponding funcoid f exists.

2°. Let define $\alpha \in \mathcal{F}(B)^{\mathcal{F}A}$ by the formula $\partial(\alpha X) = \left\{ \frac{Y \in \mathcal{T}B}{X \delta Y} \right\}$ for every $X \in \mathcal{T}A$. (It is obvious that $\left\{ \frac{Y \in \mathcal{T}B}{X \delta Y} \right\}$ is a free star.) Analogously it can be defined $\beta \in \mathcal{F}(A)^{\mathcal{F}B}$ by the formula $\partial(\beta Y) = \left\{ \frac{X \in \mathcal{T}A}{X \delta Y} \right\}$. Let's continue α and β to $\alpha' \in \mathcal{F}(B)^{\mathcal{F}(A)}$ and $\beta' \in \mathcal{F}(A)^{\mathcal{F}(B)}$ by the formulas

$$\alpha' \mathcal{X} = \prod \langle \alpha \rangle^* \text{ up } \mathcal{X} \quad \text{and} \quad \beta' \mathcal{Y} = \prod \langle \beta \rangle^* \text{ up } \mathcal{Y}$$

and δ to δ' by the formula

$$\mathcal{X} \delta' \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \delta Y.$$

$\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp \Leftrightarrow \mathcal{Y} \sqcap \prod \langle \alpha \rangle^* \text{ up } \mathcal{X} \neq \perp \Leftrightarrow \prod \langle \mathcal{Y} \sqcap \rangle^* \langle \alpha \rangle^* \text{ up } \mathcal{X} \neq \perp$. Let's prove that

$$W = \langle \mathcal{Y} \sqcap \rangle^* \langle \alpha \rangle^* \text{ up } \mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle \alpha \rangle^* \text{ up } \mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle \alpha \rangle^* \text{ up } \mathcal{X}$ then exist $X_1, X_2 \in \text{up } \mathcal{X}$ such that $\mathcal{A} = \alpha X_1, \mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \sqcap X_2) \in \langle \alpha \rangle^* \text{ up } \mathcal{X}$. So $\langle \alpha \rangle^* \text{ up } \mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\prod \langle \mathcal{Y} \sqcap \rangle^* \langle \alpha \rangle^* \mathcal{X} \neq \perp$ is equivalent to

$$\forall X \in \text{up } \mathcal{X} : \mathcal{Y} \sqcap \alpha X \neq \perp,$$

what is equivalent to

$$\begin{aligned} \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : \uparrow Y \sqcap \alpha X \neq \perp &\Leftrightarrow \\ \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : Y \in \partial(\alpha X) &\Leftrightarrow \\ \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \delta Y. & \end{aligned}$$

Combining the equivalencies we get $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp \Leftrightarrow \mathcal{X} \delta' \mathcal{Y}$. Analogously $\mathcal{X} \sqcap \beta' \mathcal{Y} \neq \perp \Leftrightarrow \mathcal{X} \delta' \mathcal{Y}$. So $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp \Leftrightarrow \mathcal{X} \sqcap \beta' \mathcal{Y} \neq \perp$, that is (A, B, α', β') is a funcoid. From the formula $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp \stackrel{\mathcal{F}(B)}{\Leftrightarrow} \mathcal{X} \delta' \mathcal{Y}$ it follows that

$$X [(A, B, \alpha', \beta')]^* Y \Leftrightarrow \uparrow Y \sqcap \alpha' \uparrow X \neq \perp \Leftrightarrow \uparrow X \delta' \uparrow Y \Leftrightarrow X \delta Y.$$

1°. Let define the relation $\delta \in \mathcal{P}(\mathcal{T}A \times \mathcal{T}B)$ by the formula $X \delta Y \Leftrightarrow \uparrow Y \sqcap \alpha X \neq \perp$.

That $\neg(I \delta \perp)$ and $\neg(\perp \delta I)$ is obvious. We have

$$\begin{aligned} I \sqcup J \delta K &\Leftrightarrow \\ \uparrow K \sqcap \alpha(I \sqcup J) \neq \perp &\Leftrightarrow \\ \uparrow K \sqcap (\alpha I \sqcup \alpha J) \neq \perp &\Leftrightarrow \\ \uparrow K \sqcap \alpha I \neq \perp \vee \uparrow K \sqcap \alpha J \neq \perp &\Leftrightarrow \\ I \delta K \vee J \delta K & \end{aligned}$$

and

$$\begin{aligned} K \delta I \sqcup J &\Leftrightarrow \\ \uparrow(I \sqcup J) \sqcap \alpha K \neq \perp &\Leftrightarrow \\ (\uparrow I \sqcup \uparrow J) \sqcap \alpha K \neq \perp &\Leftrightarrow \\ \uparrow I \sqcap \alpha K \neq \perp \vee \uparrow J \sqcap \alpha K \neq \perp &\Leftrightarrow \\ K \delta I \vee K \delta J. & \end{aligned}$$

That is the formulas (7) are true.

Accordingly to the above there exists a funcoid f such that

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \delta Y.$$

For every $X \in \mathcal{T}A, Y \in \mathcal{T}B$ we have:

$$\uparrow Y \sqcap \langle f \rangle \uparrow X \neq \perp \Leftrightarrow \uparrow X [f] \uparrow Y \Leftrightarrow X \delta Y \Leftrightarrow \uparrow Y \sqcap \alpha X \neq \perp,$$

consequently $\forall X \in \mathcal{T}A : \alpha X = \langle f \rangle \uparrow X = \langle f \rangle^* X$.

□

Note that by the last theorem to every (quasi-)proximity δ corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures. Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

COROLLARY 832. If $\alpha \in \mathcal{F}(B)^{\mathcal{T}A}, \beta \in \mathcal{F}(A)^{\mathcal{T}B}$ are functions such that $Y \not\prec \alpha X \Leftrightarrow X \not\prec \beta Y$ for every $X \in \mathcal{T}A, Y \in \mathcal{T}B$, then there exists exactly one funcoid f such that $\langle f \rangle^* = \alpha, \langle f^{-1} \rangle^* = \beta$.

PROOF. Prove $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$. Really,

$$\begin{aligned} Y \not\prec \alpha(I \sqcup J) &\Leftrightarrow I \sqcup J \not\prec \beta Y \Leftrightarrow I \not\prec \beta Y \vee J \not\prec \beta Y \Leftrightarrow \\ Y \not\prec \alpha I \vee Y \not\prec \alpha J &\Leftrightarrow Y \not\prec \alpha I \sqcup \alpha J. \end{aligned}$$

So $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$ by star-separability. Similarly $\beta(I \sqcup J) = \beta I \sqcup \beta J$.

Thus by the theorem there exists a funcoid f such that $\langle f \rangle^* = \alpha, \langle f^{-1} \rangle^* = \beta$.

That this funcoid is unique, follows from the above. □

DEFINITION 833. Any **Rel**-morphism $F : A \rightarrow B$ corresponds to a funcoid $\uparrow^{\text{FCD}} F \in \text{FCD}(A, B)$, where by definition $\langle \uparrow^{\text{FCD}} F \rangle \mathcal{X} = \prod_{\mathcal{F}} \langle \langle F \rangle^* \rangle^* \text{up } \mathcal{X}$ for every $\mathcal{X} \in \widehat{\mathcal{F}}(A)$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff. (Take $\alpha = \uparrow \circ \langle F \rangle^*$.)

PROPOSITION 834. $\langle \uparrow^{\text{FCD}} f \rangle^* X = \langle f \rangle^* X$ for a **Rel**-morphism f and $X \in \mathcal{T} \text{Src } f$.

PROOF. $\langle \uparrow^{\text{FCD}} f \rangle^* X = \min \langle \uparrow \rangle^* \langle \langle f \rangle^* \rangle^* \text{up } X = \uparrow \langle f \rangle^* X = \langle f \rangle^* X$. \square

COROLLARY 835. $[\uparrow^{\text{FCD}} f]^* = [f]^*$ for every **Rel**-morphism f .

PROOF. $X [\uparrow^{\text{FCD}} f]^* Y \Leftrightarrow Y \neq \langle \uparrow^{\text{FCD}} f \rangle^* X \Leftrightarrow Y \neq \langle f \rangle^* X \Leftrightarrow X [f]^* Y$ for $X \in \mathcal{T} \text{Src } f, Y \in \mathcal{T} \text{Dst } f$. \square

DEFINITION 836. $\uparrow^{\text{FCD}(A,B)} f = \uparrow^{\text{FCD}}(A, B, f)$ for every binary relation f between sets A and B .

DEFINITION 837. Funcoids corresponding to a binary relation (= multivalued function) are called *principal funcoids*.

PROPOSITION 838. $\uparrow^{\text{FCD}} g \circ \uparrow^{\text{FCD}} f = \uparrow^{\text{FCD}}(g \circ f)$ for composable morphisms f, g of category **Rel**.

PROOF. For every $X \in \mathcal{T} \text{Src } f$

$$\begin{aligned} \langle \uparrow^{\text{FCD}} g \circ \uparrow^{\text{FCD}} f \rangle^* X &= \langle \uparrow^{\text{FCD}} g \rangle^* \langle \uparrow^{\text{FCD}} f \rangle^* X = \\ &= \langle g \rangle^* \langle f \rangle^* X = \langle g \circ f \rangle^* X = \langle \uparrow^{\text{FCD}}(g \circ f) \rangle^* X. \end{aligned} \quad \square$$

We may equate principal funcoids with corresponding binary relations by the method of appendix A. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below).

Thus $(\text{FCD}(A, B), \mathbf{Rel}(A, B))$ is a filtrator. I call it *filtrator of funcoids*.

THEOREM 839. If S is a generalized filter base on $\text{Src } f$ then $\langle f \rangle \prod S = \prod \langle \langle f \rangle^* \rangle^* S$ for every funcoid f .

PROOF. $\langle f \rangle \prod S \subseteq \langle f \rangle X$ for every $X \in S$ and thus $\langle f \rangle \prod S \subseteq \prod \langle \langle f \rangle^* \rangle^* S$.
By properties of generalized filter bases:

$$\begin{aligned} \langle f \rangle \prod S &= \\ \prod \langle \langle f \rangle^* \rangle^* \text{up } \prod S &= \\ \prod \langle \langle f \rangle^* \rangle^* \left\{ \frac{X}{\exists \mathcal{P} \in S : X \in \text{up } \mathcal{P}} \right\} &= \\ \prod \left\{ \frac{\langle f \rangle^* X}{\exists \mathcal{P} \in S : X \in \text{up } \mathcal{P}} \right\} &\supseteq \\ \prod_{\mathcal{P} \in S} \langle f \rangle \mathcal{P} &= \\ \prod \langle \langle f \rangle^* \rangle^* S. & \end{aligned} \quad \square$$

PROPOSITION 840. $\mathcal{X} [f] \sqcap S \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X} [f] \mathcal{Y}$ if f is a funcoid and S is a generalized filter base on $\text{Dst } f$.

PROOF.

$$\begin{aligned} \mathcal{X} [f] \sqcap S &\Leftrightarrow \sqcap S \sqcap \langle f \rangle \mathcal{X} \neq \perp \Leftrightarrow \sqcap \langle \langle f \rangle \mathcal{X} \sqcap \rangle^* S \neq \perp \Leftrightarrow \\ &\text{(by properties of generalized filter bases)} \Leftrightarrow \\ &\exists \mathcal{Y} \in \langle \langle f \rangle \mathcal{X} \sqcap \rangle^* S : \mathcal{Y} \neq \perp \Leftrightarrow \exists \mathcal{Y} \in S : \langle f \rangle \mathcal{X} \sqcap \mathcal{Y} \neq \perp \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X} [f] \mathcal{Y}. \end{aligned}$$

□

DEFINITION 841. A function f between two posets is said to *preserve filtered meets*, when $f \sqcap S = \sqcap \langle f \rangle^* S$ whenever $\sqcap S$ is defined for a filter base S on the first of the two posets.

THEOREM 842. (discovered by TODD TRIMBLE) A function $\varphi : \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ preserves finite joins (including nullary joins) and filtered meets iff there exists a funcoid f such that $\langle f \rangle = \varphi$.

PROOF. Backward implication follows from above.

Let $\psi = \varphi|_{\mathcal{F}A}$. Then ψ preserves bottom element and binary joins. Thus there exists a funcoid f such that $\langle f \rangle^* = \psi$.

It remains to prove that $\langle f \rangle = \varphi$.

Really, $\langle f \rangle \mathcal{X} = \sqcap \langle \langle f \rangle^* \rangle^* \text{up } \mathcal{X} = \sqcap \langle \psi \rangle^* \text{up } \mathcal{X} = \sqcap \langle \varphi \rangle^* \text{up } \mathcal{X} = \varphi \sqcap \text{up } \mathcal{X} = \varphi \mathcal{X}$ for every $\mathcal{X} \in \mathcal{F}(A)$. □

COROLLARY 843. Funcoids f from A to B bijectively correspond by the formula $\langle f \rangle = \varphi$ to functions $\varphi : \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ preserving finite joins and filtered meets.

7.4. Another way to represent funcoids as binary relations

This is based on a TODD TRIMBLE's idea.

DEFINITION 844. The binary relation $\xi^{\circledast} \in \mathcal{P}(\mathcal{F}(\text{Src } \xi) \times \mathcal{F}(\text{Dst } \xi))$ for a funcoid ξ is defined by the formula $\mathcal{A} \xi^{\circledast} \mathcal{B} \Leftrightarrow \mathcal{B} \sqsupseteq \langle \xi \rangle \mathcal{A}$.

DEFINITION 845. The binary relation $\xi^* \in \mathcal{P}(\mathcal{F} \text{Src } \xi \times \mathcal{F} \text{Dst } \xi)$ for a funcoid ξ is defined by the formula

$$\mathcal{A} \xi^* \mathcal{B} \Leftrightarrow \mathcal{B} \sqsupseteq \langle \xi \rangle \mathcal{A} \Leftrightarrow \mathcal{B} \in \text{up} \langle \xi \rangle \mathcal{A}.$$

PROPOSITION 846. Funcoid ξ can be restored from

- 1°. the value of ξ^{\circledast} ;
- 2°. the value of ξ^* .

PROOF.

- 1°. The value of $\langle \xi \rangle$ can be restored from ξ^{\circledast} .
- 2°. The value of $\langle \xi \rangle^*$ can be restored from ξ^* .

□

THEOREM 847. Let ν and ξ be composable funcoids. Then:

- 1°. $\xi^{\circledast} \circ \nu^{\circledast} = (\xi \circ \nu)^{\circledast}$;
- 2°. $\xi^* \circ \nu^* = (\xi \circ \nu)^*$.

PROOF.

1°.

$$\begin{aligned} \mathcal{A} (\xi^{\circledast} \circ \nu^{\circledast}) \mathcal{C} &\Leftrightarrow \exists \mathcal{B} : (\mathcal{A} \nu^{\circledast} \mathcal{B} \wedge \mathcal{B} \xi^{\circledast} \mathcal{C}) \Leftrightarrow \\ &\exists \mathcal{B} \in \mathcal{F}(\text{Dst } \nu) : (\mathcal{B} \sqsupseteq \langle \nu \rangle \mathcal{A} \wedge \mathcal{C} \sqsupseteq \langle \xi \rangle \mathcal{B}) \Leftrightarrow \\ &\mathcal{C} \sqsupseteq \langle \xi \rangle \langle \nu \rangle \mathcal{A} \Leftrightarrow \mathcal{C} \sqsupseteq \langle \xi \circ \nu \rangle \mathcal{A} \Leftrightarrow \mathcal{A} (\xi \circ \nu)^{\circledast} \mathcal{C}. \end{aligned}$$

2°.

$$\begin{aligned} A (\xi^* \circ \nu^*) C &\Leftrightarrow \exists B : (A \nu^* B \wedge B \xi^* C) \Leftrightarrow \\ &\exists B : (B \in \text{up}\langle \nu \rangle A \wedge C \in \text{up}\langle \xi \rangle B) \Leftrightarrow \exists B \in \text{up}\langle \nu \rangle A : C \in \text{up}\langle \xi \rangle B. \end{aligned}$$

$$A (\xi \circ \nu)^* C \Leftrightarrow C \in \text{up}\langle \xi \circ \nu \rangle B \Leftrightarrow C \in \text{up}\langle \xi \rangle \langle \nu \rangle B.$$

It remains to prove

$$\exists B \in \text{up}\langle \nu \rangle A : C \in \text{up}\langle \xi \rangle B \Leftrightarrow C \in \text{up}\langle \xi \rangle \langle \nu \rangle A.$$

$\exists B \in \text{up}\langle \nu \rangle A : C \in \text{up}\langle \xi \rangle B \Rightarrow C \in \text{up}\langle \xi \rangle \langle \nu \rangle A$ is obvious.

Let $C \in \text{up}\langle \xi \rangle \langle \nu \rangle A$. Then $C \in \text{up}\prod \langle \langle \xi \rangle \rangle^* \text{up}\langle \nu \rangle A$; so by properties of generalized filter bases, $\exists P \in \langle \langle \xi \rangle \rangle^* \text{up}\langle \nu \rangle A : C \in \text{up} P$; $\exists B \in \text{up}\langle \nu \rangle A : C \in \text{up}\langle \xi \rangle B$. \square

REMARK 848. The above theorem is interesting by the fact that composition of funcoids is represented as relational composition of binary relations.

7.5. Lattices of funcoids

DEFINITION 849. $f \sqsubseteq g \stackrel{\text{def}}{=} [f] \subseteq [g]$ for $f, g \in \text{FCD}(A, B)$ for every sets A, B .

Thus every $\text{FCD}(A, B)$ is a poset. (It's taken into account that $[f] \neq [g]$ when $f \neq g$.)

We will consider filtrators (*filtrators of funcoids*) whose base is $\text{FCD}(A, B)$ and whose core are principal funcoids from A to B .

LEMMA 850. $\langle f \rangle^* X = \prod_{F \in \text{up}_f} \langle F \rangle^* X$ for every funcoid f and typed set $X \in \mathcal{T}(\text{Src } f)$.

PROOF. Obviously $\langle f \rangle^* X \subseteq \prod_{F \in \text{up}_f} \langle F \rangle^* X$.

Let $B \in \text{up}\langle f \rangle^* X$. Let $F_B = X \times B \sqcup \bar{X} \times \top$.

$\langle F_B \rangle^* X = B$.

Let $P \in \mathcal{T}(\text{Src } f)$. We have

$$\perp \neq P \subseteq X \Rightarrow \langle F_B \rangle^* P = B \supseteq \langle f \rangle^* P$$

and

$$P \not\subseteq X \Rightarrow \langle F_B \rangle^* P = \top \supseteq \langle f \rangle^* P.$$

Thus $\langle F_B \rangle^* P \supseteq \langle f \rangle^* P$ for every P and so $F_B \supseteq f$ that is $F_B \in \text{up } f$.

Thus $\forall B \in \text{up}\langle f \rangle^* X : B \in \text{up}\prod_{F \in \text{up}_f} \langle F \rangle^* X$ because $B \in \text{up}\langle F_B \rangle^* X$.

So $\prod_{F \in \text{up}_f} \langle F \rangle^* X \subseteq \langle f \rangle^* X$. \square

THEOREM 851. $\langle f \rangle \mathcal{X} = \prod_{F \in \text{up}_f} \langle F \rangle \mathcal{X}$ for every funcoid f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

PROOF.

$$\begin{aligned}
& \prod_{F \in \text{up } f}^{\mathcal{F}} \langle F \rangle \mathcal{X} = \\
& \prod_{F \in \text{up } f}^{\mathcal{F}} \prod_{X \in \text{up } \mathcal{X}}^{\mathcal{F}} \langle \langle F \rangle^* \rangle^* X = \\
& \prod_{F \in \text{up } f}^{\mathcal{F}} \prod_{X \in \text{up } \mathcal{X}}^{\mathcal{F}} \langle F \rangle^* X = \\
& \prod_{X \in \text{up } \mathcal{X}}^{\mathcal{F}} \prod_{F \in \text{up } f}^{\mathcal{F}} \langle F \rangle^* X = \\
& \prod_{X \in \text{up } \mathcal{X}}^{\mathcal{F}} \langle f \rangle^* X = \\
& \langle f \rangle \mathcal{X}
\end{aligned}$$

(the lemma used). □

Below it is shown that $\text{FCD}(A, B)$ are complete lattices for every sets A and B . We will apply lattice operations to subsets of such sets without explicitly mentioning $\text{FCD}(A, B)$.

THEOREM 852. $\text{FCD}(A, B)$ is a complete lattice (for every sets A and B). For every $R \in \mathcal{P}\text{FCD}(A, B)$ and $X \in \mathcal{T}A, Y \in \mathcal{T}B$

- 1°. $X \llbracket \sqcup R \rrbracket^* Y \Leftrightarrow \exists f \in R : X \llbracket f \rrbracket^* Y$;
- 2°. $\langle \sqcup R \rangle^* X = \sqcup_{f \in R} \langle f \rangle^* X$.

PROOF. Accordingly [27] to prove that it is a complete lattice it's enough to prove existence of all joins.

- 2°. $\alpha X \stackrel{\text{def}}{=} \sqcup_{f \in R} \langle f \rangle^* X$. We have $\alpha \perp = \perp$;

$$\begin{aligned}
& \alpha(I \sqcup J) = \\
& \sqcup_{f \in R} \langle f \rangle^* (I \sqcup J) = \\
& \sqcup_{f \in R} (\langle f \rangle^* I \sqcup \langle f \rangle^* J) = \\
& \sqcup_{f \in R} \langle f \rangle^* I \sqcup \sqcup_{f \in R} \langle f \rangle^* J = \\
& \alpha I \sqcup \alpha J.
\end{aligned}$$

So $\langle h \rangle^* = \alpha$ for some funcoid h . Obviously

$$\forall f \in R : h \sqsupseteq f. \tag{9}$$

And h is the least funcoid for which holds the condition (9). So $h = \sqcup R$.

1°.

$$\begin{aligned}
X \left[\bigsqcup R \right]^* Y &\Leftrightarrow \\
\uparrow Y \sqcap \left\langle \bigsqcup R \right\rangle^* X \neq \perp &\Leftrightarrow \\
\uparrow Y \sqcap \bigsqcup_{f \in R} \langle f \rangle^* X \neq \perp &\Leftrightarrow \\
\exists f \in R : \uparrow Y \sqcap \langle f \rangle^* X \neq \perp &\Leftrightarrow \\
\exists f \in R : X [f]^* Y &
\end{aligned}$$

(used proposition 610).

□

In the next theorem, compared to the previous one, the class of infinite joins is replaced with lesser class of binary joins and simultaneously class of sets is changed to more wide class of filters.

THEOREM 853. For every $f, g \in \text{FCD}(A, B)$ and $\mathcal{X} \in \mathcal{F}(A)$ (for every sets A, B)

- 1°. $\langle f \sqcup g \rangle \mathcal{X} = \langle f \rangle \mathcal{X} \sqcup \langle g \rangle \mathcal{X}$;
- 2°. $[f \sqcup g] = [f] \cup [g]$.

PROOF.

1°. Let $\alpha \mathcal{X} \stackrel{\text{def}}{=} \langle f \rangle \mathcal{X} \sqcup \langle g \rangle \mathcal{X}$; $\beta \mathcal{Y} \stackrel{\text{def}}{=} \langle f^{-1} \rangle \mathcal{Y} \sqcup \langle g^{-1} \rangle \mathcal{Y}$ for every $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Y} \in \mathcal{F}(B)$. Then

$$\begin{aligned}
\mathcal{Y} \sqcap \alpha \mathcal{X} \neq \perp &\Leftrightarrow \\
\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \perp \vee \mathcal{Y} \sqcap \langle g \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\
\mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \perp \vee \mathcal{X} \sqcap \langle g^{-1} \rangle \mathcal{Y} \neq \perp &\Leftrightarrow \\
\mathcal{X} \sqcap \beta \mathcal{Y} \neq \perp. &
\end{aligned}$$

So $h = (A, B, \alpha, \beta)$ is a functor. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some functor p then $\langle p \rangle \mathcal{X} \sqsupseteq \langle f \rangle \mathcal{X} \sqcup \langle g \rangle \mathcal{X} = \langle h \rangle \mathcal{X}$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.

2°. For every $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Y} \in \mathcal{F}(B)$ we have

$$\begin{aligned}
\mathcal{X} [f \sqcup g] \mathcal{Y} &\Leftrightarrow \\
\mathcal{Y} \sqcap \langle f \sqcup g \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\
\mathcal{Y} \sqcap (\langle f \rangle \mathcal{X} \sqcup \langle g \rangle \mathcal{X}) \neq \perp &\Leftrightarrow \\
\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \perp \vee \mathcal{Y} \sqcap \langle g \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\
\mathcal{X} [f] \mathcal{Y} \vee \mathcal{X} [g] \mathcal{Y}. &
\end{aligned}$$

□

7.6. More on composition of functors

PROPOSITION 854. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for every composable functors f and g .

PROOF. For every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } g)$ we have

$$\begin{aligned} \mathcal{X} [g \circ f] \mathcal{Y} &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle g \circ f \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle g \rangle \langle f \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\ \langle f \rangle \mathcal{X} [g] \mathcal{Y} &\Leftrightarrow \\ \mathcal{X} ([g] \circ \langle f \rangle) \mathcal{Y} & \end{aligned}$$

and

$$\begin{aligned} [g \circ f] &= \\ [(f^{-1} \circ g^{-1})^{-1}] &= \\ [f^{-1} \circ g^{-1}]^{-1} &= \\ ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} &= \\ \langle g^{-1} \rangle^{-1} \circ [f]. & \end{aligned}$$

□

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x (g \circ f) z \Leftrightarrow \exists y : (x f y \wedge y g z)$ for every x and z and every binary relations f and g .

THEOREM 855. For every sets A, B, C and $f \in \text{FCD}(A, B)$, $g \in \text{FCD}(B, C)$ and $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Z} \in \mathcal{F}(C)$

$$\mathcal{X} [g \circ f] \mathcal{Z} \Leftrightarrow \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}).$$

PROOF.

$$\begin{aligned} \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{Z} \sqcap \langle g \rangle y \neq \perp \wedge y \sqcap \langle f \rangle \mathcal{X} \neq \perp) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{Z} \sqcap \langle g \rangle y \neq \perp \wedge y \sqsubseteq \langle f \rangle \mathcal{X}) &\Rightarrow \\ \mathcal{Z} \sqcap \langle g \rangle \langle f \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\ \mathcal{X} [g \circ f] \mathcal{Z}. & \end{aligned}$$

Reversely, if $\mathcal{X} [g \circ f] \mathcal{Z}$ then $\langle f \rangle \mathcal{X} [g] \mathcal{Z}$, consequently there exists $y \in \text{atoms} \langle f \rangle \mathcal{X}$ such that $y [g] \mathcal{Z}$; we have $\mathcal{X} [f] y$. □

THEOREM 856. For every sets A, B, C

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \text{FCD}(A, B)$, $f \in \text{FCD}(B, C)$;

2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $g, h \in \text{FCD}(B, C)$, $f \in \text{FCD}(A, B)$.

PROOF. I will prove only the first equality because the other is analogous.

For every $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Z} \in \mathcal{F}(C)$

$$\begin{aligned} \mathcal{X} [f \circ (g \sqcup h)] \mathcal{Z} &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{X} [g \sqcup h] y \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathcal{F}(B)} : ((\mathcal{X} [g] y \vee \mathcal{X} [h] y) \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathcal{F}(B)} : ((\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee (\mathcal{X} [h] y \wedge y [f] \mathcal{Z})) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee \exists y \in \text{atoms}^{\mathcal{F}(B)} : (\mathcal{X} [h] y \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \mathcal{X} [f \circ g] \mathcal{Z} \vee \mathcal{X} [f \circ h] \mathcal{Z} &\Leftrightarrow \\ \mathcal{X} [f \circ g \sqcup f \circ h] \mathcal{Z}. & \end{aligned}$$

□

Another proof of the above theorem (without atomic filters):

PROOF.

$$\begin{aligned}
\langle f \circ (g \sqcup h) \rangle \mathcal{X} &= \\
\langle f \rangle \langle g \sqcup h \rangle \mathcal{X} &= \\
\langle f \rangle (\langle g \rangle \mathcal{X} \sqcup \langle h \rangle \mathcal{X}) &= \\
\langle f \rangle \langle g \rangle \mathcal{X} \sqcup \langle f \rangle \langle h \rangle \mathcal{X} &= \\
\langle f \circ g \rangle \mathcal{X} \sqcup \langle f \circ h \rangle \mathcal{X} &= \\
\langle f \circ g \sqcup f \circ h \rangle \mathcal{X}. &
\end{aligned}$$

□

7.7. Domain and range of a funcoid

DEFINITION 857. Let A be a set. The *identity funcoid* $1_A^{\text{FCD}} = (A, A, \text{id}_{\mathcal{F}(A)}, \text{id}_{\mathcal{F}(A)})$.

OBVIOUS 858. The identity funcoid is a funcoid.

PROPOSITION 859. $[f] = [1_{\text{Dst } f}] \circ \langle f \rangle$ for every funcoid f .

PROOF. From proposition 854. □

DEFINITION 860. Let A be a set, $\mathcal{A} \in \mathcal{F}(A)$. The *restricted identity funcoid*

$$\text{id}_{\mathcal{A}}^{\text{FCD}} = (A, A, \mathcal{A} \sqcap, \mathcal{A} \sqcap).$$

PROPOSITION 861. The restricted identity funcoid is a funcoid.

PROOF. We need to prove that $(\mathcal{A} \sqcap \mathcal{X}) \sqcap \mathcal{Y} \neq \perp \Leftrightarrow (\mathcal{A} \sqcap \mathcal{Y}) \sqcap \mathcal{X} \neq \perp$ what is obvious. □

OBVIOUS 862.

- 1°. $(1_A^{\text{FCD}})^{-1} = 1_A^{\text{FCD}}$;
- 2°. $(\text{id}_{\mathcal{A}}^{\text{FCD}})^{-1} = \text{id}_{\mathcal{A}}^{\text{FCD}}$.

OBVIOUS 863. For every $\mathcal{X}, \mathcal{Y} \in \mathcal{F}(A)$

- 1°. $\mathcal{X} [1_A^{\text{FCD}}] \mathcal{Y} \Leftrightarrow \mathcal{X} \sqcap \mathcal{Y} \neq \perp$;
- 2°. $\mathcal{X} [\text{id}_{\mathcal{A}}^{\text{FCD}}] \mathcal{Y} \Leftrightarrow \mathcal{A} \sqcap \mathcal{X} \sqcap \mathcal{Y} \neq \perp$.

DEFINITION 864. I will define *restricting* of a funcoid f to a filter $\mathcal{A} \in \mathcal{F}(\text{Src } f)$ by the formula

$$f|_{\mathcal{A}} = f \circ \text{id}_{\mathcal{A}}^{\text{FCD}}.$$

DEFINITION 865. *Image* of a funcoid f will be defined by the formula $\text{im } f = \langle f \rangle \sqcap^{\mathcal{F}(\text{Src } f)}$.

Domain of a funcoid f is defined by the formula $\text{dom } f = \text{im } f^{-1}$.

OBVIOUS 866. For every morphism $f \in \mathbf{Rel}(A, B)$ for sets A and B

- 1°. $\text{im } \uparrow^{\text{FCD}} f = \uparrow \text{im } f$;
- 2°. $\text{dom } \uparrow^{\text{FCD}} f = \uparrow \text{dom } f$.

PROPOSITION 867. $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \sqcap \text{dom } f)$ for every funcoid f , $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

PROOF. For every $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$ we have

$$\begin{aligned} \mathcal{Y} \sqcap \langle f \rangle (\mathcal{X} \sqcap \text{dom } f) &\neq \perp \Leftrightarrow \\ \mathcal{X} \sqcap \text{dom } f \sqcap \langle f^{-1} \rangle \mathcal{Y} &\neq \perp \Leftrightarrow \\ \mathcal{X} \sqcap \text{im } f^{-1} \sqcap \langle f^{-1} \rangle \mathcal{Y} &\neq \perp \Leftrightarrow \\ \mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} &\neq \perp \Leftrightarrow \\ \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} &\neq \perp. \end{aligned}$$

Thus $\langle f \rangle (\mathcal{X} \sqcap \text{dom } f) = \langle f \rangle \mathcal{X}$ because the lattice of filters is separable. \square

PROPOSITION 868. $\langle f \rangle \mathcal{X} = \text{im}(f|_{\mathcal{X}})$ for every funcoid f , $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

PROOF.

$$\begin{aligned} \text{im}(f|_{\mathcal{X}}) &= \\ \langle f \circ \text{id}_{\mathcal{X}}^{\text{FCD}} \rangle_{\top} &= \\ \langle f \rangle \langle \text{id}_{\mathcal{X}}^{\text{FCD}} \rangle_{\top} &= \\ \langle f \rangle (\mathcal{X} \sqcap \top) &= \\ \langle f \rangle \mathcal{X}. & \end{aligned}$$

\square

PROPOSITION 869. $\mathcal{X} \sqcap \text{dom } f \neq \perp \Leftrightarrow \langle f \rangle \mathcal{X} \neq \perp$ for every funcoid f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

PROOF.

$$\begin{aligned} \mathcal{X} \sqcap \text{dom } f \neq \perp &\Leftrightarrow \\ \mathcal{X} \sqcap \langle f^{-1} \rangle_{\top} \mathcal{F}(\text{Dst } f) \neq \perp &\Leftrightarrow \\ \top \sqcap \langle f \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\ \langle f \rangle \mathcal{X} \neq \perp. & \end{aligned}$$

\square

COROLLARY 870. $\text{dom } f = \bigsqcup \left\{ \frac{a \in \text{atoms}_{\mathcal{F}(\text{Src } f)}}{\langle f \rangle a \neq \perp} \right\}$.

PROOF. This follows from the fact that $\mathcal{F}(\text{Src } f)$ is an atomistic lattice. \square

PROPOSITION 871. $\text{dom}(f|_{\mathcal{A}}) = \mathcal{A} \sqcap \text{dom } f$ for every funcoid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$.

PROOF.

$$\begin{aligned} \text{dom}(f|_{\mathcal{A}}) &= \\ \text{im}(\text{id}_{\mathcal{A}}^{\text{FCD}} \circ f^{-1}) &= \\ \langle \text{id}_{\mathcal{A}}^{\text{FCD}} \rangle \langle f^{-1} \rangle_{\top} &= \\ \mathcal{A} \sqcap \langle f^{-1} \rangle_{\top} &= \\ \mathcal{A} \sqcap \text{dom } f. & \end{aligned}$$

\square

THEOREM 872. $\text{im } f = \prod^{\mathcal{F}} \langle \text{im} \rangle^* \text{ up } f$ and $\text{dom } f = \prod^{\mathcal{F}} \langle \text{dom} \rangle^* \text{ up } f$ for every funcoid f .

PROOF.

$$\begin{aligned}
\text{im } f &= \\
\langle f \rangle \top &= \\
\prod_{F \in \text{up } f}^{\mathcal{F}} \langle F \rangle \top &= \\
\prod_{F \in \text{up } f}^{\mathcal{F}} \text{im } F &= \\
\prod_{F \in \text{up } f}^{\mathcal{F}} \langle \text{im} \rangle^* \text{ up } f. &
\end{aligned}$$

The second formula follows from symmetry. \square

PROPOSITION 873. For every composable funcoids f, g :

- 1°. If $\text{im } f \sqsupseteq \text{dom } g$ then $\text{im}(g \circ f) = \text{im } g$.
- 2°. If $\text{im } f \sqsubseteq \text{dom } g$ then $\text{dom}(g \circ f) = \text{dom } f$.

PROOF.

1°.

$$\begin{aligned}
\text{im}(g \circ f) &= \\
\langle g \circ f \rangle \top &= \\
\langle g \rangle \langle f \rangle \top &= \\
\langle g \rangle \text{im } f &= \\
\langle g \rangle (\text{im } f \sqcap \text{dom } g) &= \\
\langle g \rangle \text{dom } g &= \\
\langle g \rangle \top &= \\
\text{im } g. &
\end{aligned}$$

2°. $\text{dom}(g \circ f) = \text{im}(f^{-1} \circ g^{-1})$ what by proved above is equal to $\text{im } f^{-1}$ that is $\text{dom } f$. \square

7.8. Categories of funcoids

I will define two categories, the *category of funcoids* and the *category of funcoid triples*.

The *category of funcoids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\text{FCD}(A, B)$.
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.

To show it is really a category is trivial.

The *category of funcoid triples* is defined as follows:

- Objects are filters on small sets.
- The morphisms from a filter \mathcal{A} to a filter \mathcal{B} are triples $(\mathcal{A}, \mathcal{B}, f)$ where $f \in \text{FCD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B}))$ and $\text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$.
- The composition is defined by the formula $(\mathcal{B}, \mathcal{C}, g) \circ (\mathcal{A}, \mathcal{B}, f) = (\mathcal{A}, \mathcal{C}, g \circ f)$.
- Identity morphism for a filter \mathcal{A} is $\text{id}_{\mathcal{A}}^{\text{FCD}}$.

To prove that it is really a category is trivial.

PROPOSITION 874. \uparrow^{FCD} is a functor from **Rel** to **FCD**.

PROOF. $\uparrow^{\text{FCD}}(g \circ f) = \uparrow^{\text{FCD}} g \circ \uparrow^{\text{FCD}} f$ was proved above. $\uparrow^{\text{FCD}} 1_{\mathbf{A}}^{\text{Rel}} = 1_{\mathbf{A}}^{\text{FCD}}$ is obvious. \square

7.9. Specifying funcoids by functions or relations on atomic filters

THEOREM 875. For every funcoid f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$

- 1°. $\langle f \rangle \mathcal{X} = \bigsqcup \langle \langle f \rangle \rangle^* \text{atoms } \mathcal{X}$;
- 2°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \text{atoms } \mathcal{X}, y \in \text{atoms } \mathcal{Y} : x [f] y$.

PROOF.

1°.

$$\begin{aligned} \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\ \mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \perp &\Leftrightarrow \\ \exists x \in \text{atoms } \mathcal{X} : x \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \perp &\Leftrightarrow \\ \exists x \in \text{atoms } \mathcal{X} : \mathcal{Y} \sqcap \langle f \rangle x \neq \perp. & \end{aligned}$$

$\partial \langle f \rangle \mathcal{X} = \bigsqcup \langle \partial \rangle^* \langle \langle f \rangle \rangle^* \text{atoms } \mathcal{X} = \partial \bigsqcup \langle \langle f \rangle \rangle^* \text{atoms } \mathcal{X}$. So $\langle f \rangle \mathcal{X} = \bigsqcup \langle \langle f \rangle \rangle^* \text{atoms } \mathcal{X}$ by corollary 568.

2°. If $\mathcal{X} [f] \mathcal{Y}$, then $\mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \perp$, consequently there exists $y \in \text{atoms } \mathcal{Y}$ such that $y \sqcap \langle f \rangle \mathcal{X} \neq \perp$, $\mathcal{X} [f] y$. Repeating this second time we get that there exists $x \in \text{atoms } \mathcal{X}$ such that $x [f] y$. From this it follows

$$\exists x \in \text{atoms } \mathcal{X}, y \in \text{atoms } \mathcal{Y} : x [f] y.$$

The reverse is obvious. \square

COROLLARY 876. Let f be a funcoid.

- The value of f can be restored from the value of $\langle f \rangle|_{\text{atoms } \mathcal{F}(\text{Src } f)}$.
- The value of f can be restored from the value of $[f]|_{\text{atoms } \mathcal{F}(\text{Src } f) \times \text{atoms } \mathcal{F}(\text{Dst } f)}$.

THEOREM 877. Let A and B be sets.

- 1°. A function $\alpha \in \mathcal{F}(B)^{\text{atoms } \mathcal{F}(A)}$ such that (for every $a \in \text{atoms } \mathcal{F}(A)$)

$$\alpha a \sqsubseteq \bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \text{atoms} \circ \uparrow \right\rangle^* \text{up } a \quad (10)$$

can be continued to the function $\langle f \rangle$ for a unique $f \in \text{FCD}(A, B)$;

$$\langle f \rangle \mathcal{X} = \bigsqcup \langle \alpha \rangle^* \text{atoms } \mathcal{X} \quad (11)$$

for every $\mathcal{X} \in \mathcal{F}(A)$.

- 2°. A relation $\delta \in \mathcal{P}(\text{atoms } \mathcal{F}(A) \times \text{atoms } \mathcal{F}(B))$ such that (for every $a \in \text{atoms } \mathcal{F}(A)$, $b \in \text{atoms } \mathcal{F}(B)$)

$$\forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms } \uparrow X, y \in \text{atoms } \uparrow Y : x \delta y \Rightarrow a \delta b \quad (12)$$

can be continued to the relation $[f]$ for a unique $f \in \text{FCD}(A, B)$;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \text{atoms } \mathcal{X}, y \in \text{atoms } \mathcal{Y} : x \delta y \quad (13)$$

for every $\mathcal{X} \in \mathcal{F}(A)$, $\mathcal{Y} \in \mathcal{F}(B)$.

PROOF. Existence of no more than one such funcoids and formulas (11) and (13) follow from the previous theorem.

1°. Consider the function $\alpha' \in \mathcal{F}(B)^{\mathcal{T}A}$ defined by the formula (for every $X \in \mathcal{T}A$)

$$\alpha' X = \bigsqcup \langle \alpha \rangle^* \text{atoms} \uparrow X.$$

Obviously $\alpha' \perp_{\mathcal{T}A} = \perp_{\mathcal{F}(B)}$. For every $I, J \in \mathcal{T}A$

$$\begin{aligned} \alpha'(I \sqcup J) &= \\ \bigsqcup \langle \alpha \rangle^* \text{atoms} \uparrow (I \sqcup J) &= \\ \bigsqcup \langle \alpha \rangle^* (\text{atoms} \uparrow \cup \text{atoms} \uparrow J) &= \\ \bigsqcup (\langle \alpha \rangle^* \text{atoms} \uparrow I \cup \langle \alpha \rangle^* \text{atoms} \uparrow J) &= \\ \bigsqcup \langle \alpha \rangle^* \text{atoms} \uparrow I \sqcup \bigsqcup \langle \alpha \rangle^* \text{atoms} \uparrow J &= \\ \alpha' I \sqcup \alpha' J. & \end{aligned}$$

Let continue α' till a funcoid f (by the theorem 831): $\langle f \rangle \mathcal{X} = \prod \langle \alpha' \rangle^* \text{up } \mathcal{X}$.
Let's prove the reverse of (10):

$$\begin{aligned} \prod \langle \bigsqcup \langle \alpha \rangle^* \circ \text{atoms} \circ \uparrow \rangle^* \text{up } a &= \\ \prod \langle \bigsqcup \langle \alpha \rangle^* \rangle^* \langle \text{atoms} \rangle^* \langle \uparrow \rangle^* \text{up } a &\sqsubseteq \\ \prod \langle \bigsqcup \langle \alpha \rangle^* \rangle^* \{ \{ a \} \} &= \\ \prod \{ (\bigsqcup \langle \alpha \rangle^*) \{ a \} \} &= \\ \prod \{ \bigsqcup \langle \alpha \rangle^* \{ a \} \} &= \\ \prod \{ \bigsqcup \{ \alpha a \} \} &= \\ \prod \{ \alpha a \} &= \\ \alpha a. & \end{aligned}$$

Finally,

$$\alpha a = \prod \langle \bigsqcup \langle \alpha \rangle^* \circ \text{atoms} \circ \uparrow \rangle^* \text{up } a = \prod \langle \alpha' \rangle^* \text{up } a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α .

2°. Consider the relation $\delta' \in \mathcal{P}(\mathcal{T}A \times \mathcal{T}B)$ defined by the formula (for every $X \in \mathcal{T}A, Y \in \mathcal{T}B$)

$$X \delta' Y \Leftrightarrow \exists x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y : x \delta y.$$

Obviously $\neg(X \delta' \perp_{\mathcal{T}B})$ and $\neg(\perp_{\mathcal{T}A} \delta' Y)$.

For suitable I and J we have:

$$\begin{aligned} I \sqcup J \delta' Y &\Leftrightarrow \\ \exists x \in \text{atoms} \uparrow (I \sqcup J), y \in \text{atoms} \uparrow Y : x \delta y &\Leftrightarrow \\ \exists x \in \text{atoms} \uparrow I \cup \text{atoms} \uparrow J, y \in \text{atoms} \uparrow Y : x \delta y &\Leftrightarrow \\ \exists x \in \text{atoms} \uparrow I, y \in \text{atoms} \uparrow Y : x \delta y \vee \exists x \in \text{atoms} \uparrow J, y \in \text{atoms} \uparrow Y : x \delta y &\Leftrightarrow \\ I \delta' Y \vee J \delta' Y; & \end{aligned}$$

similarly $X \delta' I \sqcup J \Leftrightarrow X \delta' I \vee X \delta' J$ for suitable I and J . Let's continue δ' till a funcoid f (by the theorem 831):

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \delta' Y.$$

The reverse of (12) implication is trivial, so

$$\forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y : x \delta y \Leftrightarrow a \delta b.$$

Also

$$\begin{aligned} \forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y : x \delta y &\Leftrightarrow \\ \forall X \in \text{up } a, Y \in \text{up } b : X \delta' Y &\Leftrightarrow \\ a [f] b. & \end{aligned}$$

So $a \delta b \Leftrightarrow a [f] b$, that is $[f]$ is a continuation of δ . □

One of uses of the previous theorem is the proof of the following theorem:

THEOREM 878. If A and B are sets, $R \in \mathcal{PFCD}(A, B)$, $x \in \text{atoms}^{\mathcal{F}(A)}$, $y \in \text{atoms}^{\mathcal{F}(B)}$, then

- 1°. $\langle \sqcap R \rangle x = \prod_{f \in R} \langle f \rangle x$;
- 2°. $x [\sqcap R] y \Leftrightarrow \forall f \in R : x [f] y$.

PROOF.

2°. Let denote $x \delta y \Leftrightarrow \forall f \in R : x [f] y$. For every $a \in \text{atoms}^{\mathcal{F}(A)}$, $b \in \text{atoms}^{\mathcal{F}(B)}$

$$\begin{aligned} \forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y : x \delta y &\Rightarrow \\ \forall f \in R, X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y : x [f] y &\Rightarrow \\ \forall f \in R, X \in \text{up } a, Y \in \text{up } b : X [f]^* Y &\Rightarrow \\ \forall f \in R : a [f] b &\Leftrightarrow \\ a \delta b. & \end{aligned}$$

So by theorem 877, δ can be continued till $[p]$ for some funcoid $p \in \text{FCD}(A, B)$. For every funcoid $q \in \text{FCD}(A, B)$ such that $\forall f \in R : q \sqsubseteq f$ we have

$$x [q] y \Rightarrow \forall f \in R : x [f] y \Leftrightarrow x \delta y \Leftrightarrow x [p] y,$$

so $q \sqsubseteq p$. Consequently $p = \sqcap R$.

From this $x [\sqcap R] y \Leftrightarrow \forall f \in R : x [f] y$.

1°. From the former

$$\begin{aligned} y \in \text{atoms} \langle \sqcap R \rangle x &\Leftrightarrow \\ y \sqcap \langle \sqcap R \rangle x \neq \perp &\Leftrightarrow \\ \forall f \in R : y \sqcap \langle f \rangle x \neq \perp &\Leftrightarrow \\ y \in \prod \langle \text{atoms} \rangle^* \left\{ \frac{\langle f \rangle x}{f \in R} \right\} &\Leftrightarrow \\ y \in \text{atoms} \prod_{f \in R} \langle f \rangle x & \end{aligned}$$

for every $y \in \text{atoms}^{\mathcal{F}(A)}$. From this it follows $\langle \sqcap R \rangle x = \prod_{f \in R} \langle f \rangle x$. □

THEOREM 879. $g \circ f = \prod^{\text{FCD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$ for every composable funcoids f and g .

PROOF. Let $x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}$. Then

$$\begin{aligned}
\langle g \circ f \rangle x &= \\
\langle g \rangle \langle f \rangle x &= \text{(theorem 851)} \\
\prod_{G \in \text{up } g}^{\mathcal{F}} \langle G \rangle \langle f \rangle x &= \text{(theorem 851)} \\
\prod_{G \in \text{up } g}^{\mathcal{F}} \langle G \rangle \prod_{F \in \text{up } f}^{\mathcal{F}} \langle F \rangle x &= \text{(theorem 839)} \\
\prod_{G \in \text{up } g}^{\mathcal{F}} \prod_{F \in \text{up } f}^{\mathcal{F}} \langle G \rangle \langle F \rangle x &= \\
\prod_{F \in \text{up } f, G \in \text{up } g}^{\mathcal{F}} \left\{ \frac{\langle G \rangle \langle F \rangle x}{F \in \text{up } f, G \in \text{up } g} \right\} &= \\
\prod_{F \in \text{up } f, G \in \text{up } g}^{\mathcal{F}} \left\{ \frac{\langle G \circ F \rangle x}{F \in \text{up } f, G \in \text{up } g} \right\} &= \text{(theorem 878)} \\
\left\langle \prod_{F \in \text{up } f, G \in \text{up } g}^{\text{FCD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\} \right\rangle x.
\end{aligned}$$

Thus $g \circ f = \prod^{\text{FCD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$. □

PROPOSITION 880. For $f \in \text{FCD}(A, B)$, a finite set $X \in \mathcal{P}A$ and a function $t \in \mathcal{F}(B)^X$ there exists (obviously unique) $g \in \text{FCD}(A, B)$ such that $\langle g \rangle p = \langle f \rangle p$ for $p \in \text{atoms}^{\mathcal{F}(A)} \setminus \text{atoms } X$ and $\langle g \rangle @\{x\} = t(x)$ for $x \in X$.

This funcoid g is determined by the formula

$$g = (f \setminus (@X \times^{\text{FCD}} \top)) \sqcup \bigsqcup_{x \in X} (@\{x\} \times^{\text{FCD}} t(x)).$$

PROOF. Take $g = (f \setminus (@X \times^{\text{FCD}} \top)) \sqcup \bigsqcup_{q \in X} (@\{q\} \times^{\text{FCD}} t(x))$ that is

$$\begin{aligned}
g &= (f \sqcap \overline{X} \times \top) \sqcup \bigsqcup_{q \in X} (@\{q\} \times^{\text{FCD}} t(x)) = \\
&= (f \sqcap (\overline{X} \times \top)) \sqcup \bigsqcup_{q \in X} (@\{q\} \times^{\text{FCD}} t(x)).
\end{aligned}$$

$$\langle g \rangle p = \text{(theorem 853)} =$$

$$\langle f \sqcap (\overline{X} \times \top) \rangle p \sqcup \bigsqcup_{q \in X} \langle @\{q\} \times^{\text{FCD}} t(x) \rangle p =$$

$$\text{(theorem 878)} = (\langle f \rangle p \sqcap \langle \overline{X} \times \top \rangle p) \sqcup \bigsqcup_{q \in X} \langle @\{q\} \times^{\text{FCD}} t(x) \rangle p.$$

So $\langle g \rangle @\{x\} = (\langle f \rangle^* @\{x\} \sqcap \perp) \sqcup t(x) = t(x)$ for $x \in X$.

If $p \in \text{atoms}^{\mathcal{F}(A)} \setminus \text{atoms } X$ then we have $\langle g \rangle p = (\langle f \rangle p \sqcap \top) \sqcup \perp = \langle f \rangle p$. □

COROLLARY 881. If $f \in \text{FCD}(A, B)$, $x \in A$, and $\mathcal{Y} \in \mathcal{F}(B)$, then there exists an (obviously unique) $g \in \text{FCD}(A, B)$ such that $\langle g \rangle p = \langle f \rangle p$ for all ultrafilters p except of $p = @\{x\}$ and $\langle g \rangle @\{x\} = \mathcal{Y}$.

This funcoid g is determined by the formula

$$g = (f \setminus (@\{x\} \times^{\text{FCD}} \top)) \sqcup (@\{x\} \times^{\text{FCD}} \mathcal{Y}).$$

THEOREM 882. Let A, B, C be sets, $f \in \text{FCD}(A, B)$, $g \in \text{FCD}(B, C)$, $h \in \text{FCD}(A, C)$. Then

$$g \circ f \not\prec h \Leftrightarrow g \not\prec h \circ f^{-1}.$$

PROOF.

$$\begin{aligned} g \circ f \not\prec h &\Leftrightarrow \\ \exists a \in \text{atoms}^{\mathcal{F}(A)}, c \in \text{atoms}^{\mathcal{F}(C)} : a [(g \circ f) \sqcap h] c &\Leftrightarrow \\ \exists a \in \text{atoms}^{\mathcal{F}(A)}, c \in \text{atoms}^{\mathcal{F}(C)} : (a [g \circ f] c \wedge a [h] c) &\Leftrightarrow \\ \exists a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, c \in \text{atoms}^{\mathcal{F}(C)} : (a [f] b \wedge b [g] c \wedge a [h] c) &\Leftrightarrow \\ \exists b \in \text{atoms}^{\mathcal{F}(B)}, c \in \text{atoms}^{\mathcal{F}(C)} : (b [g] c \wedge b [h \circ f^{-1}] c) &\Leftrightarrow \\ \exists b \in \text{atoms}^{\mathcal{F}(B)}, c \in \text{atoms}^{\mathcal{F}(C)} : b [g \sqcap (h \circ f^{-1})] c &\Leftrightarrow \\ g \not\prec h \circ f^{-1}. & \end{aligned}$$

□

7.10. Functorial product of filters

A generalization of Cartesian product of two sets is functorial product of two filters:

DEFINITION 883. *Functorial product* of filters \mathcal{A} and \mathcal{B} is such a functor $\mathcal{A} \times^{\text{FCD}} \mathcal{B} \in \text{FCD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B}))$ that for every $\mathcal{X} \in \text{Base}(\mathcal{A})$, $\mathcal{Y} \in \text{Base}(\mathcal{B})$

$$\mathcal{X} [\mathcal{A} \times^{\text{FCD}} \mathcal{B}] \mathcal{Y} \Leftrightarrow \mathcal{X} \not\prec \mathcal{A} \wedge \mathcal{Y} \not\prec \mathcal{B}.$$

PROPOSITION 884. $\mathcal{A} \times^{\text{FCD}} \mathcal{B}$ is really a functor and

$$\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\prec \mathcal{A} \\ \perp_{\mathcal{F}(\text{Base}(\mathcal{B}))} & \text{if } \mathcal{X} \prec \mathcal{A}. \end{cases}$$

PROOF. Obvious. □

OBVIOUS 885.

- $\uparrow^{\text{FCD}(U, V)} (A \times B) = \uparrow^U A \times \uparrow^V B$ for sets $A \subseteq U$ and $B \subseteq V$.
- $\uparrow^{\text{FCD}} (A \times B) = \uparrow A \times \uparrow B$ for typed sets A and B .

PROPOSITION 886. $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Leftrightarrow \text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$ for every $f \in \text{FCD}(A, B)$ and $\mathcal{A} \in \mathcal{F}(A)$, $\mathcal{B} \in \mathcal{F}(B)$.

PROOF. If $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ then $\text{dom } f \sqsubseteq \text{dom}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$, $\text{im } f \sqsubseteq \text{im}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \sqsubseteq \mathcal{B}$. If $\text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$ then

$$\forall \mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B) : (\mathcal{X} [f] \mathcal{Y} \Rightarrow \mathcal{X} \sqcap \mathcal{A} \neq \perp \wedge \mathcal{Y} \sqcap \mathcal{B} \neq \perp);$$

consequently $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of a meet on the lattice of functors:

THEOREM 887. $f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \text{id}_{\mathcal{B}}^{\text{FCD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{FCD}}$ for every functor f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$, $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

PROOF. $h \stackrel{\text{def}}{=} \text{id}_{\mathcal{B}}^{\text{FCD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{FCD}}$. For every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$

$$\langle h \rangle \mathcal{X} = \langle \text{id}_{\mathcal{B}}^{\text{FCD}} \rangle \langle f \rangle \langle \text{id}_{\mathcal{A}}^{\text{FCD}} \rangle \mathcal{X} = \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X}).$$

From this, as easy to show, $h \sqsubseteq f$ and $h \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$. If $g \sqsubseteq f \wedge g \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ for a $g \in \text{FCD}(\text{Src } f, \text{Dst } f)$ then $\text{dom } g \sqsubseteq \mathcal{A}$, $\text{im } g \sqsubseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \sqcap \langle g \rangle (\mathcal{A} \sqcap \mathcal{X}) \sqsubseteq \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X}) = \langle \text{id}_{\mathcal{B}}^{\text{FCD}} \rangle \langle f \rangle \langle \text{id}_{\mathcal{A}}^{\text{FCD}} \rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \sqsubseteq h$. So $h = f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B})$. \square

COROLLARY 888. $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\text{FCD}} \top^{\mathcal{F}(\text{Dst } f)})$ for every funcooid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$.

PROOF. $f \sqcap (\mathcal{A} \times^{\text{FCD}} \top^{\mathcal{F}(\text{Dst } f)}) = \text{id}_{\top^{\mathcal{F}(\text{Dst } f)}}^{\text{FCD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{FCD}} = f \circ \text{id}_{\mathcal{A}}^{\text{FCD}} = f|_{\mathcal{A}}$. \square

COROLLARY 889. $f \not\prec \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Leftrightarrow \mathcal{A} [f] \mathcal{B}$ for every funcooid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$, $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

PROOF.

$$\begin{aligned} f \not\prec \mathcal{A} \times^{\text{FCD}} \mathcal{B} &\Leftrightarrow \\ \langle f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \rangle^* \top &\neq \perp \Leftrightarrow \\ \langle \text{id}_{\mathcal{B}}^{\text{FCD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{FCD}} \rangle^* \top &\neq \perp \Leftrightarrow \\ \langle \text{id}_{\mathcal{B}}^{\text{FCD}} \rangle \langle f \rangle \langle \text{id}_{\mathcal{A}}^{\text{FCD}} \rangle^* \top &\neq \perp \Leftrightarrow \\ \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \top) &\neq \perp \Leftrightarrow \\ \mathcal{B} \sqcap \langle f \rangle \mathcal{A} &\neq \perp \Leftrightarrow \\ \mathcal{A} [f] \mathcal{B}. & \end{aligned}$$

\square

COROLLARY 890. Every filtrator of funcoids is star-separable.

PROOF. The set of functorial products of principal filters is a separation subset of the lattice of funcoids. \square

THEOREM 891. Let A, B be sets. If $S \in \mathcal{P}(\mathcal{F}(A) \times \mathcal{F}(B))$ then

$$\bigsqcap_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \bigsqcap \text{dom } S \times^{\text{FCD}} \bigsqcap \text{im } S.$$

PROOF. If $x \in \text{atoms}^{\mathcal{F}(A)}$ then by theorem 878

$$\left\langle \bigsqcap_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \right\rangle x = \bigsqcap_{(\mathcal{A}, \mathcal{B}) \in S} \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x.$$

If $x \not\prec \bigsqcap \text{dom } S$ then

$$\forall (\mathcal{A}, \mathcal{B}) \in S : (x \sqcap \mathcal{A} \neq \perp \wedge \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$

$$\left\{ \frac{\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\} = \text{im } S;$$

if $x \asymp \bigsqcap \text{dom } S$ then

$$\exists (\mathcal{A}, \mathcal{B}) \in S : (x \sqcap \mathcal{A} = \perp \wedge \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x = \perp);$$

$$\left\{ \frac{\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\} \ni \perp.$$

So

$$\left\langle \bigsqcap_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \right\rangle x = \begin{cases} \bigsqcap \text{im } S & \text{if } x \not\prec \bigsqcap \text{dom } S \\ \perp^{\mathcal{F}(B)} & \text{if } x \asymp \bigsqcap \text{dom } S. \end{cases}$$

From this the statement of the theorem follows. \square

COROLLARY 892. For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathcal{F}(B)$ (for every sets A, B)

$$(\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\text{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

PROOF. $(\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1) = \sqcap \{ \mathcal{A} \times^{\text{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 \}$ what is by the last theorem equal to $(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\text{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1)$. \square

THEOREM 893. If A, B are sets and $\mathcal{A} \in \mathcal{F}(A)$ then $\mathcal{A} \times^{\text{FCD}}$ is a complete homomorphism from the lattice $\mathcal{F}(B)$ to the lattice $\text{FCD}(A, B)$, if also $\mathcal{A} \neq \perp^{\mathcal{F}(A)}$ then it is an order embedding.

PROOF. Let $S \in \mathcal{P}\mathcal{F}(B)$, $X \in \mathcal{T}A$, $x \in \text{atoms}^{\mathcal{F}(A)}$.

$$\begin{aligned} & \left\langle \bigsqcup \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S \right\rangle^* X = \\ & \bigsqcup_{B \in S} \langle \mathcal{A} \times^{\text{FCD}} B \rangle^* X = \\ & \begin{cases} \bigsqcup S & \text{if } X \in \partial \mathcal{A} \\ \perp^{\mathcal{F}(B)} & \text{if } X \notin \partial \mathcal{A} \end{cases} = \\ & \langle \mathcal{A} \times^{\text{FCD}} \bigsqcup S \rangle^* X; \\ & \left\langle \bigsqcap \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S \right\rangle x = \\ & \bigsqcap_{B \in S} \langle \mathcal{A} \times^{\text{FCD}} B \rangle x = \\ & \begin{cases} \bigsqcap S & \text{if } x \not\asymp \mathcal{A} \\ \perp^{\mathcal{F}(B)} & \text{if } x \asymp \mathcal{A}. \end{cases} \end{aligned}$$

Thus $\bigsqcup \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S = \mathcal{A} \times^{\text{FCD}} \bigsqcup S$ and $\bigsqcap \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S = \mathcal{A} \times^{\text{FCD}} \bigsqcap S$.

If $\mathcal{A} \neq \perp$ then obviously $\mathcal{A} \times^{\text{FCD}} \mathcal{X} \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$. \square

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a funcoidal product of filters) funcoid (of atomic width).

PROPOSITION 894. If f is a funcoid and a is an atomic filter on $\text{Src } f$ then

$$f|_a = a \times^{\text{FCD}} \langle f \rangle a.$$

PROOF. Let $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

$$\mathcal{X} \not\asymp a \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \asymp a \Rightarrow \langle f|_a \rangle \mathcal{X} = \perp^{\mathcal{F}(\text{Dst } f)}.$$

\square

LEMMA 895. $\lambda \mathcal{B} \in \mathcal{F}(B) : \top^{\mathcal{F}} \times^{\text{FCD}} \mathcal{B}$ is an upper adjoint of $\lambda f \in \text{FCD}(A, B) : \text{im } f$ (for every sets A, B).

PROOF. We need to prove $\text{im } f \sqsubseteq \mathcal{B} \Leftrightarrow f \sqsubseteq \top \times^{\text{FCD}} \mathcal{B}$ what is obvious. \square

COROLLARY 896. Image and domain of funcoids preserve joins.

PROOF. By properties of Galois connections and duality. \square

PROPOSITION 897. $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Leftrightarrow \text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$ for every funcoid f and filters $\mathcal{A} \in \mathfrak{F}(\text{Src } f)$, $\mathcal{B} \in \mathfrak{F}(\text{Dst } f)$.

PROOF. $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Rightarrow \text{dom } f \sqsubseteq \mathcal{A}$ because $\text{dom}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$.

Let now $\text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$. Then $\langle f \rangle \mathcal{X} \neq \perp \Rightarrow \mathcal{X} \neq \mathcal{A}$ that is $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \top$. Similarly $f \sqsubseteq \top \times^{\text{FCD}} \mathcal{B}$. Thus $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$. \square

7.11. Atomic functors

THEOREM 898. An $f \in \text{FCD}(A, B)$ is an atom of the lattice $\text{FCD}(A, B)$ (for some sets A, B) iff it is a functorial product of two atomic filter objects.

PROOF.

\Rightarrow . Let $f \in \text{FCD}(A, B)$ be an atom of the lattice $\text{FCD}(A, B)$. Let's get elements $a \in \text{atoms } \text{dom } f$ and $b \in \text{atoms } \langle f \rangle a$. Then for every $\mathcal{X} \in \mathcal{F}(A)$

$$\mathcal{X} \asymp a \Rightarrow \langle a \times^{\text{FCD}} b \rangle \mathcal{X} = \perp \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \langle a \times^{\text{FCD}} b \rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}.$$

So $a \times^{\text{FCD}} b \sqsubseteq f$; because f is atomic we have $f = a \times^{\text{FCD}} b$.

\Leftarrow . Let $a \in \text{atoms}^{\mathcal{F}(A)}$, $b \in \text{atoms}^{\mathcal{F}(B)}$, $f \in \text{FCD}(A, B)$. If $b \asymp \langle f \rangle a$ then $\neg(a [f] b)$, $f \asymp a \times^{\text{FCD}} b$; if $b \sqsubseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathcal{F}(A) : (\mathcal{X} \not\asymp a \Rightarrow \langle f \rangle \mathcal{X} \sqsupseteq b)$, $f \sqsupseteq a \times^{\text{FCD}} b$. Consequently $f \asymp a \times^{\text{FCD}} b \vee f \sqsupseteq a \times^{\text{FCD}} b$; that is $a \times^{\text{FCD}} b$ is an atom. \square

THEOREM 899. The lattice $\text{FCD}(A, B)$ is atomic (for every fixed sets A, B).

PROOF. Let f be a non-empty functor from A to B . Then $\text{dom } f \neq \perp$, thus by theorem 576 there exists $a \in \text{atoms } \text{dom } f$. So $\langle f \rangle a \neq \perp$ thus it exists $b \in \text{atoms } \langle f \rangle a$. Finally the atomic functor $a \times^{\text{FCD}} b \sqsubseteq f$. \square

THEOREM 900. The lattice $\text{FCD}(A, B)$ is separable (for every fixed sets A, B).

PROOF. Let $f, g \in \text{FCD}(A, B)$, $f \sqsubset g$. Then there exists $a \in \text{atoms}^{\mathcal{F}(A)}$ such that $\langle f \rangle a \sqsubset \langle g \rangle a$. So because the lattice $\mathcal{F}(B)$ is atomically separable, there exists $b \in \text{atoms}$ such that $\langle f \rangle a \sqcap b = \perp$ and $b \sqsubseteq \langle g \rangle a$. For every $x \in \text{atoms}^{\mathcal{F}(A)}$

$$\langle f \rangle a \sqcap \langle a \times^{\text{FCD}} b \rangle a = \langle f \rangle a \sqcap b = \perp,$$

$$x \neq a \Rightarrow \langle f \rangle x \sqcap \langle a \times^{\text{FCD}} b \rangle x = \langle f \rangle x \sqcap \perp = \perp.$$

Thus $\langle f \rangle x \sqcap \langle a \times^{\text{FCD}} b \rangle x = \perp$ and consequently $f \asymp a \times^{\text{FCD}} b$.

$$\langle a \times^{\text{FCD}} b \rangle a = b \sqsubseteq \langle g \rangle a,$$

$$x \neq a \Rightarrow \langle a \times^{\text{FCD}} b \rangle x = \perp \sqsubseteq \langle g \rangle x.$$

Thus $\langle a \times^{\text{FCD}} b \rangle x \sqsubseteq \langle g \rangle x$ and consequently $a \times^{\text{FCD}} b \sqsubseteq g$.

So the lattice $\text{FCD}(A, B)$ is separable by theorem 225. \square

COROLLARY 901. The lattice $\text{FCD}(A, B)$ is:

- 1°. separable;
- 2°. strongly separable;
- 3°. atomically separable;
- 4°. conforming to Wallman's disjunction property.

PROOF. By theorem 233. \square

REMARK 902. For more ways to characterize (atomic) separability of the lattice of functors see subsections "Separation subsets and full stars" and "Atomically separable lattices".

COROLLARY 903. The lattice $\text{FCD}(A, B)$ is an atomistic lattice.

PROOF. By theorem 231. \square

PROPOSITION 904. $\text{atoms}(f \sqcup g) = \text{atoms } f \cup \text{atoms } g$ for every funcoids $f, g \in \text{FCD}(A, B)$ (for every sets A, B).

PROOF. $a \times^{\text{FCD}} b \not\approx f \sqcup g \Leftrightarrow a [f \sqcup g] b \Leftrightarrow a [f] b \vee a [g] b \Leftrightarrow a \times^{\text{FCD}} b \not\approx f \vee a \times^{\text{FCD}} b \not\approx g$ for every atomic filters a and b . \square

THEOREM 905. The set of funcoids between sets A and B is a co-frame.

PROOF. Theorems 831 and 533. \square

REMARK 906. The above proof does not use axiom of choice (unlike the below proof).

See also an older proof of the set of funcoids being co-brouwerian:

THEOREM 907. For every $f, g, h \in \text{FCD}(A, B)$, $R \in \mathcal{P}\text{FCD}(A, B)$ (for every sets A and B)

- 1°. $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$;
- 2°. $f \sqcup \bigcap R = \bigcap \langle f \sqcup \rangle^* R$.

PROOF. We will take into account that the lattice of funcoids is an atomistic lattice.

1°.

$$\begin{aligned}
 \text{atoms}(f \sqcap (g \sqcup h)) &= \\
 \text{atoms } f \cap \text{atoms}(g \sqcup h) &= \\
 \text{atoms } f \cap (\text{atoms } g \cup \text{atoms } h) &= \\
 (\text{atoms } f \cap \text{atoms } g) \cup (\text{atoms } f \cap \text{atoms } h) &= \\
 \text{atoms}(f \sqcap g) \cup \text{atoms}(f \sqcap h) &= \\
 \text{atoms}((f \sqcap g) \sqcup (f \sqcap h)). &
 \end{aligned}$$

2°.

$$\begin{aligned}
& \text{atoms}(f \sqcup \prod R) = \\
& \text{atoms } f \cup \text{atoms } \prod R = \\
& \text{atoms } f \cup \bigcap \langle \text{atoms} \rangle^* R = \\
& \bigcap \langle (\text{atoms } f) \cup \rangle^* \langle \text{atoms} \rangle^* R = \text{ (use the following equality)} \\
& \bigcap \langle \text{atoms} \rangle^* \langle f \sqcup \rangle^* R = \\
& \text{atoms } \prod \langle f \sqcup \rangle^* R. \\
& \langle (\text{atoms } f) \cup \rangle^* \langle \text{atoms} \rangle^* R = \\
& \left\{ \frac{(\text{atoms } f) \cup A}{A \in \langle \text{atoms} \rangle^* R} \right\} = \\
& \left\{ \frac{(\text{atoms } f) \cup A}{\exists C \in R : A = \text{atoms } C} \right\} = \\
& \left\{ \frac{(\text{atoms } f) \cup (\text{atoms } C)}{C \in R} \right\} = \\
& \left\{ \frac{\text{atoms}(f \sqcup C)}{C \in R} \right\} = \\
& \left\{ \frac{\text{atoms } B}{\exists C \in R : B = f \sqcup C} \right\} = \\
& \left\{ \frac{\text{atoms } B}{B \in \langle f \sqcup \rangle^* C} \right\} = \\
& \langle \text{atoms} \rangle^* \langle f \sqcup \rangle^* R.
\end{aligned}$$

□

CONJECTURE 908. $f \sqcap \sqcup S = \sqcup \langle f \sqcap \rangle^* S$ for principal funcoid f and a set S of funcoids of appropriate sources and destinations.

REMARK 909. See also example 1334 below.

The next proposition is one more (among the theorem 855) generalization for funcoids of composition of relations.

PROPOSITION 910. For every composable funcoids f, g

$$\text{atoms}(g \circ f) = \left\{ \frac{x \times^{\text{FCD}} z}{\begin{array}{l} x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, z \in \text{atoms}^{\mathcal{F}(\text{Dst } g)}, \\ \exists y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)} : (x \times^{\text{FCD}} y \in \text{atoms } f \wedge y \times^{\text{FCD}} z \in \text{atoms } g) \end{array}} \right\}.$$

PROOF. Using the theorem 855,

$$x \times^{\text{FCD}} z \not\in g \circ f \Leftrightarrow x [g \circ f] z \Leftrightarrow \exists y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)} : (x \times^{\text{FCD}} y \not\in f \wedge y \times^{\text{FCD}} z \not\in g).$$

□

COROLLARY 911. $g \circ f = \sqcup \left\{ \frac{G \circ F}{F \in \text{atoms } f, G \in \text{atoms } g} \right\}$ for every composable funcoids f, g .

THEOREM 912. Let f be a funcoid.

- 1°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists F \in \text{atoms } f : \mathcal{X} [F] \mathcal{Y}$ for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$;
 2°. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

PROOF.

1°.

$$\begin{aligned} & \exists F \in \text{atoms } f : \mathcal{X} [F] \mathcal{Y} \Leftrightarrow \\ & \exists a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)} : (a \times^{\text{FCD}} b \neq f \wedge \mathcal{X} [a \times^{\text{FCD}} b] \mathcal{Y}) \Leftrightarrow \\ & \exists a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)} : (a \times^{\text{FCD}} b \neq f \wedge a \times^{\text{FCD}} b \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \Leftrightarrow \\ & \exists F \in \text{atoms } f : (F \neq f \wedge F \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \Leftrightarrow \\ & f \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y} \Leftrightarrow \\ & \mathcal{X} [f] \mathcal{Y}. \end{aligned}$$

2°. Let $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$. Suppose $\mathcal{Y} \neq \langle f \rangle \mathcal{X}$. Then $\mathcal{X} [f] \mathcal{Y}$; $\exists F \in \text{atoms } f : \mathcal{X} [F] \mathcal{Y}$; $\exists F \in \text{atoms } f : \mathcal{Y} \neq \langle F \rangle \mathcal{X}$; $\mathcal{Y} \neq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubset \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$. The contrary $\langle f \rangle \mathcal{X} \supseteq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ is obvious. \square

7.12. Complete functors

DEFINITION 913. I will call *co-complete* such a functor f that $\langle f \rangle^* X$ is a principal filter for every $X \in \mathcal{T}(\text{Src } f)$.

OBVIOUS 914. Functor f is co-complete iff $\langle f \rangle \mathcal{X} \in \mathfrak{P}(\text{Dst } f)$ for every $\mathcal{X} \in \mathfrak{P}(\text{Src } f)$.

DEFINITION 915. I will call *generalized closure* such a function $\alpha \in (\mathcal{T}B)^{\mathcal{T}A}$ (for some sets A, B) that

- 1°. $\alpha \perp = \perp$;
 2°. $\forall I, J \in \mathcal{T}A : \alpha(I \sqcup J) = \alpha I \sqcup \alpha J$.

OBVIOUS 916. A functor f is co-complete iff $\langle f \rangle^* = \uparrow \circ \alpha$ for a generalized closure α .

REMARK 917. Thus functors can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of functors.

DEFINITION 918. I will call a *complete functor* a functor whose reverse is co-complete.

THEOREM 919. The following conditions are equivalent for every functor f :

- 1°. functor f is complete;
 2°. $\forall S \in \mathcal{P}\mathcal{T}(\text{Src } f), J \in \mathcal{T}(\text{Dst } f) : (\bigsqcup S [f] J \Leftrightarrow \exists I \in S : I [f] J)$;
 3°. $\forall S \in \mathcal{P}\mathcal{T}(\text{Src } f), J \in \mathcal{T}(\text{Dst } f) : (\bigsqcup S [f]^* J \Leftrightarrow \exists I \in S : I [f]^* J)$;
 4°. $\forall S \in \mathcal{P}\mathcal{T}(\text{Src } f) : \langle f \rangle \sqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S$;
 5°. $\forall S \in \mathcal{P}\mathcal{T}(\text{Src } f) : \langle f \rangle^* \sqcup S = \bigsqcup \langle \langle f \rangle^* \rangle^* S$;
 6°. $\forall A \in \mathcal{T}(\text{Src } f) : \langle f \rangle^* A = \bigsqcup_{a \in \text{atoms } A} \langle f \rangle^* a$.

PROOF.

3° \Rightarrow 1°. For every $S \in \mathcal{P}\mathcal{T}(\text{Src } f), J \in \mathcal{T}(\text{Dst } f)$

$$\bigsqcup S \cap \langle f^{-1} \rangle^* J \neq \perp \Leftrightarrow \exists I \in S : I \cap \langle f^{-1} \rangle^* J \neq \perp,$$

consequently by theorem 583 we have that $\langle f^{-1} \rangle^* J$ is a principal filter.

$1^\circ \Rightarrow 2^\circ$. For every $S \in \mathcal{PF}(\text{Src } f)$, $J \in \mathcal{T}(\text{Dst } f)$ we have that $\langle f^{-1} \rangle^* J$ is a principal filter, consequently

$$\bigsqcup S \sqcap \langle f^{-1} \rangle^* J \neq \perp \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} \sqcap \langle f^{-1} \rangle^* J \neq \perp.$$

From this follows 2° .

$6^\circ \Rightarrow 5^\circ$.

$$\begin{aligned} \langle f \rangle^* \bigsqcup S &= \\ \bigsqcup_{a \in \text{atoms } \bigsqcup S} \langle f \rangle^* a &= \\ \bigsqcup_{A \in S} \bigcup_{a \in \text{atoms } A} \left\{ \frac{\langle f \rangle^* a}{a \in \text{atoms } A} \right\} &= \\ \bigsqcup_{A \in S} \bigsqcup_{a \in \text{atoms } A} \langle f \rangle^* a &= \\ \bigsqcup_{A \in S} \langle f \rangle^* A &= \\ \bigsqcup \langle \langle f \rangle^* \rangle^* S. & \end{aligned}$$

$2^\circ \Rightarrow 4^\circ$. Using theorem 583,

$$\begin{aligned} J \not\leq \langle f \rangle \bigsqcup S &\Leftrightarrow \\ \bigsqcup S [f] J &\Leftrightarrow \\ \exists \mathcal{I} \in S : \mathcal{I} [f] J &\Leftrightarrow \\ \exists \mathcal{I} \in S : J \not\leq \langle f \rangle \mathcal{I} &\Leftrightarrow \\ J \not\leq \bigsqcup \langle \langle f \rangle \rangle^* S. & \end{aligned}$$

$2^\circ \Rightarrow 3^\circ$, $4^\circ \Rightarrow 5^\circ$, $5^\circ \Rightarrow 3^\circ$, $5^\circ \Rightarrow 6^\circ$. Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pretopological spaces.

PROPOSITION 920. To specify a complete funcoid f it is enough to specify $\langle f \rangle^*$ on one-element sets, values of $\langle f \rangle^*$ on one element sets can be specified arbitrarily.

PROOF. From the above theorem is clear that knowing $\langle f \rangle^*$ on one-element sets $\langle f \rangle^*$ can be found on every set and then the value of $\langle f \rangle$ can be inferred for every filter.

Choosing arbitrarily the values of $\langle f \rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f \rangle^* X = \bigsqcup_{\alpha \in \text{atoms } X} \langle f \rangle^* \alpha$ for every $X \in \mathcal{T}(\text{Src } f)$. Obviously it is really a complete funcoid. □

THEOREM 921. A funcoid is principal iff it is both complete and co-complete.

PROOF.

\Rightarrow . Obvious.

\Leftarrow . Let f be both a complete and co-complete funcoid. Consider the relation g defined by that $\uparrow \langle g \rangle^* \alpha = \langle f \rangle^* \alpha$ for one-element sets α (g is correctly defined because f corresponds to a generalized closure). Because f is a complete funcoid f is the funcoid corresponding to g . □

THEOREM 922. If $R \in \mathcal{P}\text{FCD}(A, B)$ is a set of (co-)complete funcoids then $\sqcup R$ is a (co-)complete funcoid (for every sets A and B).

PROOF. It is enough to prove for co-complete funcoids. Let $R \in \mathcal{P}\text{FCD}(A, B)$ be a set of co-complete funcoids. Then for every $X \in \mathcal{T}(\text{Src } f)$

$$\left\langle \sqcup R \right\rangle^* X = \bigsqcup_{f \in R} \langle f \rangle^* X$$

is a principal filter (used theorem 852). \square

COROLLARY 923. If R is a set of binary relations between sets A and B then $\sqcup \langle \uparrow^{\text{FCD}(A, B)} \rangle^* R = \uparrow^{\text{FCD}(A, B)} \cup R$.

PROOF. From two last theorems. \square

LEMMA 924. Every funcoid is representable as meet (on the lattice of funcoids) of binary relations of the form $X \times Y \sqcup \bar{X} \times \top^{\mathcal{T}(B)}$ (where X, Y are typed sets).

PROOF. Let $f \in \text{FCD}(A, B)$, $X \in \mathcal{T}A$, $Y \in \text{up}\langle f \rangle X$, $g(X, Y) \stackrel{\text{def}}{=} X \times Y \sqcup \bar{X} \times \top^{\mathcal{T}(B)}$. Then $g(X, Y) = X \times^{\text{FCD}} Y \sqcup \bar{X} \times^{\text{FCD}} \top^{\mathcal{T}(B)}$. For every $K \in \mathcal{T}A$

$$\begin{aligned} \langle g(X, Y) \rangle^* K &= \langle X \times^{\text{FCD}} Y \rangle^* K \sqcup \langle \bar{X} \times^{\text{FCD}} \top^{\mathcal{T}(B)} \rangle^* K = \\ &= \left(\begin{array}{ll} \perp^{\mathcal{T}(B)} & \text{if } K = \perp^{\mathcal{T}A} \\ Y & \text{if } \perp^{\mathcal{T}A} \neq K \sqsubseteq X \\ \top^{\mathcal{T}(B)} & \text{if } K \not\sqsubseteq X \end{array} \right) \sqsupseteq \langle f \rangle^* K; \end{aligned}$$

so $g(X, Y) \sqsupseteq f$. For every $X \in \mathcal{T}A$

$$\bigsqcap_{Y \in \text{up}\langle f \rangle^* X} \langle g(X, Y) \rangle^* X = \bigsqcap_{Y \in \text{up}\langle f \rangle^* X} Y = \langle f \rangle^* X;$$

consequently

$$\left\langle \bigsqcap \left\{ \frac{g(X, Y)}{X \in \mathcal{T}A, Y \in \text{up}\langle f \rangle^* X} \right\} \right\rangle^* X \sqsubseteq \langle f \rangle^* X$$

that is

$$\bigsqcap \left\{ \frac{g(X, Y)}{X \in \mathcal{T}A, Y \in \text{up}\langle f \rangle^* X} \right\} \sqsubseteq f$$

and finally

$$f = \bigsqcap \left\{ \frac{g(X, Y)}{X \in \mathcal{T}A, Y \in \text{up}\langle f \rangle^* X} \right\}.$$

\square

COROLLARY 925. Filtrators of funcoids are filtered.

THEOREM 926.

- 1°. g is metacomplete if g is a complete funcoid.
- 2°. g is co-metacomplete if g is a co-complete funcoid.

PROOF.

1°. Let R be a set of funcoids from a set A to a set B and g be a funcoid from B to some C . Then

$$\begin{aligned}
\langle g \circ \bigsqcup R \rangle^* X &= \\
\langle g \rangle \langle \bigsqcup R \rangle^* X &= \\
\langle g \rangle \bigsqcup_{f \in R} \langle f \rangle^* X &= \\
\bigsqcup_{f \in R} \langle g \rangle \langle f \rangle^* X &= \\
\bigsqcup_{f \in R} \langle g \circ f \rangle^* X &= \\
\left\langle \bigsqcup_{f \in R} (g \circ f) \right\rangle^* X &= \\
\left\langle \bigsqcup (g \circ)^* R \right\rangle^* X &
\end{aligned}$$

for every typed set $X \in \mathcal{F}A$. So $g \circ \bigsqcup R = \bigsqcup (g \circ)^* R$.

2°. By duality. □

CONJECTURE 927. g is complete if g is a metacomplete funcoid.

I will denote $\mathbf{ComplFCD}$ and $\mathbf{CoComplFCD}$ the sets of small complete and co-complete funcoids correspondingly. $\mathbf{ComplFCD}(A, B)$ are complete funcoids from A to B and likewise with $\mathbf{CoComplFCD}(A, B)$.

OBVIOUS 928. $\mathbf{ComplFCD}$ and $\mathbf{CoComplFCD}$ are closed regarding composition of funcoids.

PROPOSITION 929. $\mathbf{ComplFCD}$ and $\mathbf{CoComplFCD}$ (with induced order) are complete lattices.

PROOF. It follows from theorem 922. □

THEOREM 930. Atoms of the lattice $\mathbf{ComplFCD}(A, B)$ are exactly funcoidal products of the form $\uparrow^A \{\alpha\} \times^{\mathbf{FCD}} b$ where $\alpha \in A$ and b is an ultrafilter on B .

PROOF. First, it's easy to see that $\uparrow^A \{\alpha\} \times^{\mathbf{FCD}} b$ are elements of $\mathbf{ComplFCD}(A, B)$. Also $\perp^{\mathbf{FCD}(A, B)}$ is an element of $\mathbf{ComplFCD}(A, B)$.

$\uparrow^A \{\alpha\} \times^{\mathbf{FCD}} b$ are atoms of $\mathbf{ComplFCD}(A, B)$ because they are atoms of $\mathbf{FCD}(A, B)$.

It remains to prove that if f is an atom of $\mathbf{ComplFCD}(A, B)$ then $f = \uparrow^A \{\alpha\} \times^{\mathbf{FCD}} b$ for some $\alpha \in A$ and an ultrafilter b on B .

Suppose $f \in \mathbf{FCD}(A, B)$ is a non-empty complete funcoid. Then there exists $\alpha \in A$ such that $\langle f \rangle^* @ \{\alpha\} \neq \perp^{\mathcal{F}(B)}$. Thus $\uparrow^A \{\alpha\} \times^{\mathbf{FCD}} b \sqsubseteq f$ for some ultrafilter b on B . If f is an atom then $f = \uparrow^A \{\alpha\} \times^{\mathbf{FCD}} b$. □

THEOREM 931. $G \mapsto \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\mathbf{FCD}} G(\alpha))$ is an order isomorphism from the set of functions $G \in \mathcal{F}(B)^A$ to the set $\mathbf{ComplFCD}(A, B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \langle f \rangle^* @ \{\alpha\}$ where f is a complete funcoid.

PROOF. $\bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha))$ is complete because $G(\alpha) = \bigsqcup \text{atoms } G(\alpha)$ and thus

$$\bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha)) = \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{FCD}} b}{\alpha \in A, b \in \text{atoms } G(\alpha)} \right\}$$

is complete. So $G \mapsto \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha))$ is a function from $G \in \mathcal{F}(B)^A$ to $\text{ComplFCD}(A, B)$.

Let f be complete. Then take

$$G(\alpha) = \bigsqcup \left\{ \frac{b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}}{\uparrow^A \{\alpha\} \times^{\text{FCD}} b \sqsubseteq f} \right\}$$

and we have $f = \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha))$ obviously. So $G \mapsto \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha))$ is surjection onto $\text{ComplFCD}(A, B)$.

Let now prove that it is an injection:

Let

$$f = \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} F(\alpha)) = \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha))$$

for some $F, G \in \mathcal{F}(\text{Dst } f)^{\text{Src } f}$. We need to prove $F = G$. Let $\beta \in \text{Src } f$.

$$\langle f \rangle^* @ \{\beta\} = \bigsqcup_{\alpha \in A} \langle \uparrow^A \{\alpha\} \times^{\text{FCD}} F(\alpha) \rangle^* @ \{\beta\} = F(\beta).$$

Similarly $\langle f \rangle^* @ \{\beta\} = G(\beta)$. So $F(\beta) = G(\beta)$.

We have proved that it is a bijection. To show that it is monotone is trivial.

Denote $f = \bigsqcup_{\alpha \in A} (\uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha))$. Then

$\langle f \rangle^* @ \{\alpha'\} = (\text{because } \uparrow^A \{\alpha'\} \text{ is principal}) =$

$$\bigsqcup_{\alpha \in A} \langle \uparrow^A \{\alpha\} \times^{\text{FCD}} G(\alpha) \rangle @ \{\alpha'\} = \langle \uparrow^A \{\alpha'\} \times^{\text{FCD}} G(\alpha') \rangle @ \{\alpha'\} = G(\alpha').$$

□

COROLLARY 932. $G \mapsto \bigsqcup_{\alpha \in A} (G(\alpha) \times^{\text{FCD}} \uparrow^A \{\alpha\})$ is an order isomorphism from the set of functions $G \in \mathcal{F}(B)^A$ to the set $\text{CoComplFCD}(A, B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \langle f^{-1} \rangle^* @ \{\alpha\}$ where f is a co-complete funcoid.

COROLLARY 933. $\text{ComplFCD}(A, B)$ and $\text{CoComplFCD}(A, B)$ are co-frames.

7.13. Funcoids corresponding to pretopologies

Let Δ be a pretopology on a set U and cl the preclosure corresponding to it (see theorem 777).

Both induce a funcoid, I will show that these two funcoids are reverse of each other:

THEOREM 934. Let f be a complete funcoid defined by the formula $\langle f \rangle^* @ \{x\} = \Delta(x)$ for every $x \in U$, let g be a co-complete funcoid defined by the formula $\langle g \rangle^* X = \uparrow^U \text{cl}(\text{GR } X)$ for every $X \in \mathcal{F}U$. Then $g = f^{-1}$.

REMARK 935. It is obvious that funcoids f and g exist.

PROOF. For $X, Y \in \mathcal{TU}$ we have

$$\begin{aligned}
X [g]^* Y &\Leftrightarrow \\
\uparrow Y \not\prec \langle g \rangle \uparrow X &\Leftrightarrow \\
Y \not\prec \text{cl}(\text{GR } X) &\Leftrightarrow \\
\exists y \in Y : \Delta(y) \not\prec \uparrow X &\Leftrightarrow \\
\exists y \in Y : \langle f \rangle^* \uparrow^U \{y\} \not\prec \uparrow X &\Leftrightarrow \\
(\text{proposition 610 and properties of complete functors}) & \\
\langle f \rangle^* Y \not\prec \uparrow X &\Leftrightarrow \\
Y [f]^* X. &
\end{aligned}$$

So $g = f^{-1}$. □

7.14. Completion of functors

THEOREM 936. $\text{Cor } f = \text{Cor}' f$ for an element f of a filtrator of functors.

PROOF. By theorem 545 and corollary 925. □

DEFINITION 937. *Completion* of a functor $f \in \text{FCD}(A, B)$ is the complete functor $\text{Compl } f \in \text{FCD}(A, B)$ defined by the formula $\langle \text{Compl } f \rangle^* @ \{ \alpha \} = \langle f \rangle^* @ \{ \alpha \}$ for $\alpha \in \text{Src } f$.

DEFINITION 938. *Co-completion* of a functor f is defined by the formula

$$\text{CoCompl } f = (\text{Compl } f^{-1})^{-1}.$$

OBVIOUS 939. $\text{Compl } f \sqsubseteq f$ and $\text{CoCompl } f \sqsubseteq f$.

PROPOSITION 940. The filtrator $(\text{FCD}(A, B), \text{ComplFCD}(A, B))$ is filtered.

PROOF. Because the filtrator of functors is filtered. □

THEOREM 941. $\text{Compl } f = \text{Cor}^{\text{ComplFCD}(A, B)} f = \text{Cor}'^{\text{ComplFCD}(A, B)} f$ for every functor $f \in \text{FCD}(A, B)$.

PROOF. $\text{Cor}^{\text{ComplFCD}(A, B)} f = \text{Cor}'^{\text{ComplFCD}(A, B)} f$ using theorem 545 since the filtrator $(\text{FCD}(A, B), \text{ComplFCD}(A, B))$ is filtered.

Let $g \in \text{up}^{\text{ComplFCD}(A, B)} f$. Then $g \in \text{ComplFCD}(A, B)$ and $g \supseteq f$. Thus $g = \text{Compl } g \supseteq \text{Compl } f$.

Thus $\forall g \in \text{up}^{\text{ComplFCD}(A, B)} f : g \supseteq \text{Compl } f$.

Let $\forall g \in \text{up}^{\text{ComplFCD}(A, B)} f : h \sqsubseteq g$ for some $h \in \text{ComplFCD}(A, B)$.

Then $h \sqsubseteq \prod \text{up}^{\text{ComplFCD}(A, B)} f = f$ and consequently $h = \text{Compl } h \sqsubseteq \text{Compl } f$.

Thus

$$\text{Compl } f = \prod^{\text{ComplFCD}(A, B)} \text{up}^{\text{ComplFCD}(A, B)} f = \text{Cor}^{\text{ComplFCD}(A, B)} f.$$

□

THEOREM 942. $\langle \text{CoCompl } f \rangle^* X = \text{Cor} \langle f \rangle^* X$ for every functor f and typed set $X \in \mathcal{T}(\text{Src } f)$.

PROOF. $\text{CoCompl } f \sqsubseteq f$ thus $\langle \text{CoCompl } f \rangle^* X \sqsubseteq \langle f \rangle^* X$ but $\langle \text{CoCompl } f \rangle^* X$ is a principal filter thus $\langle \text{CoCompl } f \rangle^* X \sqsubseteq \text{Cor} \langle f \rangle^* X$.

Let $\alpha X = \text{Cor} \langle f \rangle^* X$. Then $\alpha \perp_{\mathcal{T}(\text{Src } f)} = \perp_{\mathcal{T}(\text{Dst } f)}$ and

$$\begin{aligned}
\alpha(X \sqcup Y) &= \text{Cor} \langle f \rangle^* (X \sqcup Y) = \text{Cor}(\langle f \rangle^* X \sqcup \langle f \rangle^* Y) = \\
&= \text{Cor} \langle f \rangle^* X \sqcup \text{Cor} \langle f \rangle^* Y = \alpha X \sqcup \alpha Y
\end{aligned}$$

(used theorem 603). Thus α can be continued till $\langle g \rangle$ for some functor g . This functor is co-complete.

Evidently g is the greatest co-complete element of $\text{FCD}(\text{Src } f, \text{Dst } f)$ which is lower than f .

Thus $g = \text{CoCompl } f$ and $\text{Cor}\langle f \rangle^* X = \alpha X = \langle g \rangle^* X = \langle \text{CoCompl } f \rangle^* X$. \square

THEOREM 943. $\text{ComplFCD}(A, B)$ is an atomistic lattice.

PROOF. Let $f \in \text{ComplFCD}(A, B)$, $X \in \mathcal{T}(\text{Src } f)$.

$$\langle f \rangle^* X = \bigsqcup_{x \in \text{atoms } X} \langle f \rangle^* x = \bigsqcup_{x \in \text{atoms } X} \langle f|_x \rangle^* x = \bigsqcup_{x \in \text{atoms } X} \langle f|_x \rangle^* X,$$

thus $f = \bigsqcup_{x \in \text{atoms } X} (f|_x)$. It is trivial that every $f|_x$ is a join of atoms of $\text{ComplFCD}(A, B)$. \square

THEOREM 944. A functor is complete iff it is a join (on the lattice $\text{FCD}(A, B)$) of atomic complete functors.

PROOF. It follows from the theorem 922 and the previous theorem. \square

COROLLARY 945. $\text{ComplFCD}(A, B)$ is join-closed.

THEOREM 946. $\text{Compl} \bigsqcup R = \bigsqcup \langle \text{Compl} \rangle^* R$ for every $R \in \mathcal{P}\text{FCD}(A, B)$ (for every sets A, B).

PROOF. For every typed set X

$$\begin{aligned} \langle \text{Compl} \bigsqcup R \rangle^* X &= \\ \bigsqcup_{x \in \text{atoms } X} \langle \bigsqcup R \rangle^* x &= \\ \bigsqcup_{x \in \text{atoms } X} \bigsqcup_{f \in R} \langle f \rangle^* x &= \\ \bigsqcup_{f \in R} \bigsqcup_{x \in \text{atoms } X} \langle f \rangle^* x &= \\ \bigsqcup_{f \in R} \langle \text{Compl } f \rangle^* X &= \\ \langle \bigsqcup \langle \text{Compl} \rangle^* R \rangle^* X. & \end{aligned}$$

\square

COROLLARY 947. Compl is a lower adjoint.

CONJECTURE 948. Compl is not an upper adjoint (in general).

PROPOSITION 949. $\text{Compl } f = \bigsqcup_{\alpha \in \text{Src } f} (f|_{\uparrow\{\alpha\}})$ for every functor f .

PROOF. Let denote R the right part of the equality to prove.

$\langle R \rangle^* @\{\beta\} = \bigsqcup_{\alpha \in \text{Src } f} \langle f|_{\uparrow\{\alpha\}} \rangle^* @\{\beta\} = \langle f \rangle^* @\{\beta\}$ for every $\beta \in \text{Src } f$ and R is complete as a join of complete functors.

Thus R is the completion of f . \square

CONJECTURE 950. $\text{Compl } f = f \setminus^* (\Omega \times^{\text{FCD}} \mathcal{U})$.

This conjecture may be proved by considerations similar to these in the section “Fréchet filter”.

LEMMA 951. Co-completion of a complete functor is complete.

PROOF. Let f be a complete funcoid.

$$\begin{aligned} \langle \text{CoCompl } f \rangle^* X &= \text{Cor} \langle f \rangle^* X = \text{Cor} \bigsqcup_{x \in \text{atoms } X} \langle f \rangle^* x = \\ &= \bigsqcup_{x \in \text{atoms } X} \text{Cor} \langle f \rangle^* x = \bigsqcup_{x \in \text{atoms } X} \langle \text{CoCompl } f \rangle^* x \end{aligned}$$

for every set typed $X \in \mathcal{T}(\text{Src } f)$. Thus $\text{CoCompl } f$ is complete. \square

THEOREM 952. $\text{Compl } \text{CoCompl } f = \text{CoCompl } \text{Compl } f = \text{Cor } f$ for every funcoid f .

PROOF. $\text{Compl } \text{CoCompl } f$ is co-complete since (used the lemma) $\text{CoCompl } f$ is co-complete. Thus $\text{Compl } \text{CoCompl } f$ is a principal funcoid. $\text{CoCompl } f$ is the greatest co-complete funcoid under f and $\text{Compl } \text{CoCompl } f$ is the greatest complete funcoid under $\text{CoCompl } f$. So $\text{Compl } \text{CoCompl } f$ is greater than any principal funcoid under $\text{CoCompl } f$ which is greater than any principal funcoid under f . Thus $\text{Compl } \text{CoCompl } f$ is the greatest principal funcoid under f . Thus $\text{Compl } \text{CoCompl } f = \text{Cor } f$. Similarly $\text{CoCompl } \text{Compl } f = \text{Cor } f$. \square

7.14.1. More on completion of funcoids.

PROPOSITION 953. For every composable funcoids f and g

- 1°. $\text{Compl}(g \circ f) \supseteq \text{Compl } g \circ \text{Compl } f$;
- 2°. $\text{CoCompl}(g \circ f) \supseteq \text{CoCompl } g \circ \text{CoCompl } f$.

PROOF.

- 1°. $\text{Compl } g \circ \text{Compl } f = \text{Compl}(\text{Compl } g \circ \text{Compl } f) \sqsubseteq \text{Compl}(g \circ f)$.
- 2°. $\text{CoCompl } g \circ \text{CoCompl } f = \text{CoCompl}(\text{CoCompl } g \circ \text{CoCompl } f) \sqsubseteq \text{CoCompl}(g \circ f)$.

\square

PROPOSITION 954. For every composable funcoids f and g

- 1°. $\text{CoCompl}(g \circ f) = (\text{CoCompl } g) \circ f$ if f is a co-complete funcoid.
- 2°. $\text{Compl}(f \circ g) = f \circ \text{Compl } g$ if f is a complete funcoid.

PROOF.

- 1°. For every $X \in \mathcal{T}(\text{Src } f)$

$$\begin{aligned} \langle \text{CoCompl}(g \circ f) \rangle^* X &= \\ \text{Cor} \langle g \circ f \rangle^* X &= \\ \text{Cor} \langle g \rangle \langle f \rangle^* X &= \\ \langle \text{CoCompl } g \rangle \langle f \rangle^* X &= \\ \langle (\text{CoCompl } g) \circ f \rangle^* X. & \end{aligned}$$

2°. $(\text{CoCompl}(g \circ f))^{-1} = f^{-1} \circ (\text{CoCompl } g)^{-1}$; $\text{Compl}(g \circ f)^{-1} = f^{-1} \circ \text{Compl } g^{-1}$; $\text{Compl}(f^{-1} \circ g^{-1}) = f^{-1} \circ \text{Compl } g^{-1}$. After variable replacement we get $\text{Compl}(f \circ g) = f \circ \text{Compl } g$ (after the replacement f is a complete funcoid). \square

COROLLARY 955. For every composable funcoids f and g

- 1°. $\text{Compl } f \circ \text{Compl } g = \text{Compl}(\text{Compl } f \circ g)$.
- 2°. $\text{CoCompl } g \circ \text{CoCompl } f = \text{CoCompl}(g \circ \text{CoCompl } f)$.

PROPOSITION 956. For every composable funcoids f and g

- 1°. $\text{Compl}(g \circ f) = \text{Compl}(g \circ (\text{Compl } f))$;

2°. $\text{CoCompl}(g \circ f) = \text{CoCompl}((\text{CoCompl } g) \circ f)$.

PROOF.

1°.

$$\begin{aligned} \langle g \circ (\text{Compl } f) \rangle^* @ \{x\} &= \langle g \rangle \langle \text{Compl } f \rangle^* @ \{x\} = \\ &= \langle g \rangle \langle f \rangle^* @ \{x\} = \langle g \circ f \rangle^* @ \{x\}. \end{aligned}$$

Thus $\text{Compl}(g \circ (\text{Compl } f)) = \text{Compl}(g \circ f)$.

2°. $(\text{Compl}(g \circ (\text{Compl } f)))^{-1} = (\text{Compl}(g \circ f))^{-1}$; $\text{CoCompl}(g \circ (\text{Compl } f))^{-1} = \text{CoCompl}(g \circ f)^{-1}$; $\text{CoCompl}((\text{Compl } f)^{-1} \circ g^{-1}) = \text{CoCompl}(f^{-1} \circ g^{-1})$; $\text{CoCompl}((\text{CoCompl } f^{-1}) \circ g^{-1}) = \text{CoCompl}(f^{-1} \circ g^{-1})$. After variable replacement $\text{CoCompl}((\text{CoCompl } g) \circ f) = \text{CoCompl}(g \circ f)$. \square

THEOREM 957. The filtrator of functors (from a given set A to a given set B) is with co-separable core.

PROOF. Let $f, g \in \text{FCD}(A, B)$ and $f \sqcup g = \top$. Then for every $X \in \mathcal{T}A$ we have

$$\begin{aligned} \langle f \rangle^* X \sqcup \langle g \rangle^* X = \top &\Leftrightarrow \text{Cor} \langle f \rangle^* X \sqcup \text{Cor} \langle g \rangle^* X = \top \Leftrightarrow \\ &\langle \text{CoCompl } f \rangle^* X \sqcup \langle \text{CoCompl } g \rangle^* X = \top. \end{aligned}$$

Thus $\langle \text{CoCompl } f \sqcup \text{CoCompl } g \rangle^* X = \top$;

$$f \sqcup g = \top \Rightarrow \text{CoCompl } f \sqcup \text{CoCompl } g = \top. \quad (14)$$

Applying the dual of the formulas (14) to the formula (14) we get:

$$f \sqcup g = \top \Rightarrow \text{Compl } \text{CoCompl } f \sqcup \text{Compl } \text{CoCompl } g = \top$$

that is $f \sqcup g = \top \Rightarrow \text{Cor } f \sqcup \text{Cor } g = \top$. So $\text{FCD}(A, B)$ is with co-separable core. \square

COROLLARY 958. The filtrator of complete functors is also with co-separable core.

7.15. Monovalued and injective functors

Following the idea of definition of monovalued morphism let's call *monovalued* such a functor f that $f \circ f^{-1} \sqsubseteq \text{id}_{\text{im } f}^{\text{FCD}}$.

Similarly, I will call a functor injective when $f^{-1} \circ f \sqsubseteq \text{id}_{\text{dom } f}^{\text{FCD}}$.

OBVIOUS 959. A functor f is:

- 1°. monovalued iff $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{FCD}}$;
- 2°. injective iff $f^{-1} \circ f \sqsubseteq 1_{\text{Src } f}^{\text{FCD}}$.

In other words, a functor is monovalued (injective) when it is a monovalued (injective) morphism of the category of functors. Monovaluedness is dual of injectivity.

OBVIOUS 960.

- 1°. A morphism $(\mathcal{A}, \mathcal{B}, f)$ of the category of functor triples is monovalued iff the functor f is monovalued.
- 2°. A morphism $(\mathcal{A}, \mathcal{B}, f)$ of the category of functor triples is injective iff the functor f is injective.

THEOREM 961. The following statements are equivalent for a functor f :

- 1°. f is monovalued.
- 2°. It is metamonovalued.

- 3°. It is weakly metamonovalued.
 4°. $\forall a \in \text{atoms}^{\mathcal{F}(\text{Src } f)} : \langle f \rangle a \in \text{atoms}^{\mathcal{F}(\text{Dst } f)} \cup \{\perp^{\mathcal{F}(\text{Dst } f)}\}$.
 5°. $\forall \mathcal{I}, \mathcal{J} \in \mathcal{F}(\text{Dst } f) : \langle f^{-1} \rangle (\mathcal{I} \sqcap \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \sqcap \langle f^{-1} \rangle \mathcal{J}$.
 6°. $\forall I, J \in \mathcal{I}(\text{Dst } f) : \langle f^{-1} \rangle^* (I \sqcap J) = \langle f^{-1} \rangle^* I \sqcap \langle f^{-1} \rangle^* J$.

PROOF.

4° \Rightarrow 5°. Let $a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}$, $\langle f \rangle a = b$. Then because $b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)} \cup \{\perp^{\mathcal{F}(\text{Dst } f)}\}$

$$\begin{aligned} (\mathcal{I} \sqcap \mathcal{J}) \sqcap b \neq \perp &\Leftrightarrow \mathcal{I} \sqcap b \neq \perp \wedge \mathcal{J} \sqcap b \neq \perp; \\ a [f] \mathcal{I} \sqcap \mathcal{J} &\Leftrightarrow a [f] \mathcal{I} \wedge a [f] \mathcal{J}; \\ \mathcal{I} \sqcap \mathcal{J} [f^{-1}] a &\Leftrightarrow \mathcal{I} [f^{-1}] a \wedge \mathcal{J} [f^{-1}] a; \\ a \sqcap \langle f^{-1} \rangle (\mathcal{I} \sqcap \mathcal{J}) \neq \perp &\Leftrightarrow a \sqcap \langle f^{-1} \rangle \mathcal{I} \neq \perp \wedge a \sqcap \langle f^{-1} \rangle \mathcal{J} \neq \perp; \\ \langle f^{-1} \rangle (\mathcal{I} \sqcap \mathcal{J}) &= \langle f^{-1} \rangle \mathcal{I} \sqcap \langle f^{-1} \rangle \mathcal{J}. \end{aligned}$$

5° \Rightarrow 1°. $\langle f^{-1} \rangle a \sqcap \langle f^{-1} \rangle b = \langle f^{-1} \rangle (a \sqcap b) = \langle f^{-1} \rangle \perp = \perp$ for every two distinct atomic filter objects a and b on $\text{Dst } f$. This is equivalent to $\neg(\langle f^{-1} \rangle a [f] b)$; $b \asymp \langle f \rangle \langle f^{-1} \rangle a$; $b \asymp \langle f \circ f^{-1} \rangle a$; $\neg(a [f \circ f^{-1}] b)$. So $a [f \circ f^{-1}] b \Rightarrow a = b$ for every ultrafilters a and b . This is possible only when $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{FCD}}$.

6° \Rightarrow 5°.

$$\begin{aligned} \langle f^{-1} \rangle (\mathcal{I} \sqcap \mathcal{J}) &= \\ \sqcap \langle \langle f^{-1} \rangle^* \rangle^* \text{up}(\mathcal{I} \sqcap \mathcal{J}) &= \\ \sqcap \langle \langle f^{-1} \rangle^* \rangle^* \left\{ \frac{I \sqcap J}{I \in \text{up } \mathcal{I}, J \in \text{up } \mathcal{J}} \right\} &= \\ \sqcap \left\{ \frac{\langle f^{-1} \rangle^* (I \sqcap J)}{I \in \text{up } \mathcal{I}, J \in \text{up } \mathcal{J}} \right\} &= \\ \sqcap \left\{ \frac{\langle f^{-1} \rangle^* I \sqcap \langle f \rangle^* J}{I \in \text{up } \mathcal{I}, J \in \text{up } \mathcal{J}} \right\} &= \\ \sqcap \left\{ \frac{\langle f \rangle^* I}{I \in \text{up } \mathcal{I}} \right\} \sqcap \sqcap \left\{ \frac{\langle f^{-1} \rangle^* J}{J \in \text{up } \mathcal{J}} \right\} &= \\ \langle f^{-1} \rangle \mathcal{I} \sqcap \langle f^{-1} \rangle \mathcal{J}. & \end{aligned}$$

5° \Rightarrow 6°. Obvious.

$\neg 4^\circ \Rightarrow \neg 1^\circ$. Suppose $\langle f \rangle a \notin \text{atoms}^{\mathcal{F}(\text{Dst } f)} \cup \{\perp^{\mathcal{F}(\text{Dst } f)}\}$ for some $a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}$. Then there exist two atomic filters p and q on $\text{Dst } f$ such that $p \neq q$ and $\langle f \rangle a \sqsupseteq p \wedge \langle f \rangle a \sqsupseteq q$. Consequently $p \neq \langle f \rangle a$; $a \neq \langle f^{-1} \rangle p$; $a \sqsubseteq \langle f^{-1} \rangle p$; $\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \sqsupseteq \langle f \rangle a \sqsupseteq q$; $\langle f \circ f^{-1} \rangle p \not\sqsubseteq p$ and $\langle f \circ f^{-1} \rangle p \neq \perp^{\mathcal{F}(\text{Dst } f)}$. So it cannot be $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{FCD}}$.

2° \Rightarrow 3°. Obvious.

1° \Rightarrow 2°.

$$\begin{aligned} \langle (\sqcap G) \circ f \rangle x &= \langle \sqcap G \rangle \langle f \rangle x = \sqcap_{g \in G} \langle g \rangle \langle f \rangle x = \\ &= \sqcap_{g \in G} \langle g \circ f \rangle x = \left\langle \sqcap_{g \in G} (g \circ f) \right\rangle x \end{aligned}$$

for every atomic filter object $x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}$. Thus $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$.

$3^\circ \Rightarrow 1^\circ$. Take $g = a \times^{\text{FCD}} y$ and $h = b \times^{\text{FCD}} y$ for arbitrary atomic filter objects $a \neq b$ and y . We have $g \sqcap h = \perp$; thus $(g \circ f) \sqcap (h \circ f) = (g \sqcap h) \circ f = \perp$ and thus impossible $x [f] a \wedge x [f] b$ as otherwise $x [g \circ f] y$ and $x [h \circ f] y$ so $x [(g \circ f) \sqcap (h \circ f)] y$. Thus f is monovalued. \square

COROLLARY 962. A binary relation corresponds to a monovalued functor iff it is a function.

PROOF. Because $\forall I, J \in \mathcal{P}(\text{im } f) : \langle f^{-1} \rangle^* (I \sqcap J) = \langle f^{-1} \rangle^* I \sqcap \langle f^{-1} \rangle^* J$ is true for a functor f corresponding to a binary relation if and only if it is a function (see proposition 388). \square

REMARK 963. This corollary can be reformulated as follows: For binary relations (principal functors) the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a functor are the same.

THEOREM 964. If f, g are functors, $f \sqsubseteq g$ and g is monovalued then $g|_{\text{dom } f} = f$.

PROOF. Obviously $g|_{\text{dom } f} \supseteq f$. Suppose for contrary that $g|_{\text{dom } f} \subset f$. Then there exists an atom $a \in \text{atoms dom } f$ such that $\langle g|_{\text{dom } f} \rangle a \neq \langle f \rangle a$ that is $\langle g \rangle a \sqsubset \langle f \rangle a$ what is impossible. \square

7.16. T_0 -, T_1 -, T_2 -, T_3 -, and T_4 -separable functors

For functors it can be generalized T_0 -, T_1 -, T_2 -, and T_3 -separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

DEFINITION 965. Let call T_1 -separable such endofunctor f that for every $\alpha, \beta \in \text{Ob } f$ is true

$$\alpha \neq \beta \Rightarrow \neg(\text{@}\{\alpha\} [f]^* \text{@}\{\beta\}).$$

PROPOSITION 966. An endofunctor f is T_1 -separable iff $\text{Cor } f \sqsubseteq 1_{\text{Ob } f}^{\text{FCD}}$.

PROOF.

$$\forall x, y \in \text{Ob } f : (\text{@}\{x\} [f]^* \text{@}\{y\} \Rightarrow x = y) \Leftrightarrow$$

$$\forall x, y \in \text{Ob } f : (\text{@}\{x\} [\text{Cor } f]^* \text{@}\{y\} \Rightarrow x = y) \Leftrightarrow \text{Cor } f \sqsubseteq 1_{\text{Ob } f}^{\text{FCD}}.$$

\square

PROPOSITION 967. An endofunctor f is T_1 -separable iff $\text{Cor}\langle f \rangle^* \{x\} \sqsubseteq \{x\}$ for every $x \in \text{Ob } f$.

PROOF.

$$\text{Cor}\langle f \rangle^* \{x\} \sqsubseteq \{x\} \Leftrightarrow \langle \text{CoCompl } f \rangle^* \{x\} \sqsubseteq \{x\} \Leftrightarrow$$

$$\text{Compl } \text{CoCompl } f \sqsubseteq 1_{\text{Ob } f}^{\text{FCD}} \Leftrightarrow \text{Cor } f \sqsubseteq 1_{\text{Ob } f}^{\text{FCD}}.$$

\square

DEFINITION 968. Let call T_0 -separable such functor $f \in \text{FCD}(A, A)$ that $f \sqcap f^{-1}$ is T_1 -separable.

DEFINITION 969. Let call T_2 -separable such functor f that $f^{-1} \circ f$ is T_1 -separable.

For symmetric transitive functors T_0 -, T_1 - and T_2 -separability are the same (see theorem 255).

OBVIOUS 970. A functor f is T_2 -separable iff $\alpha \neq \beta \Rightarrow \langle f \rangle^* @ \{ \alpha \} \neq \langle f \rangle^* @ \{ \beta \}$ for every $\alpha, \beta \in \text{Src } f$.

DEFINITION 971. Functor f is *regular* iff for every $C \in \mathcal{T} \text{ Dst } f$ and $p \in \text{Src } f$

$$\langle f \rangle \langle f^{-1} \rangle C \simeq \langle f \rangle @ \{ p \} \Leftarrow \uparrow^{\text{Src } f} \{ p \} \simeq \langle f^{-1} \rangle C.$$

PROPOSITION 972. The following are pairwise equivalent:

- 1°. A functor f is regular.
- 2°. $\text{Compl}(f \circ f^{-1} \circ f) \sqsubseteq \text{Compl } f$.
- 3°. $\text{Compl}(f \circ f^{-1} \circ f) \sqsubseteq f$.

PROOF. Equivalently transform the defining formula for regular functors:

$$\langle f \rangle \langle f^{-1} \rangle C \simeq \langle f \rangle @ \{ p \} \Leftarrow \uparrow^{\text{Src } f} \{ p \} \simeq \langle f^{-1} \rangle C;$$

$$\langle f \rangle \langle f^{-1} \rangle C \neq \langle f \rangle @ \{ p \} \Rightarrow \uparrow^{\text{Src } f} \{ p \} \neq \langle f^{-1} \rangle C;$$

(by definition of functors)

$$C \neq \langle f \rangle \langle f^{-1} \rangle \langle f \rangle @ \{ p \} \Rightarrow C \neq \langle f \rangle @ \{ p \};$$

$$\langle f \rangle \langle f^{-1} \rangle \langle f \rangle @ \{ p \} \sqsubseteq \langle f \rangle @ \{ p \};$$

$$\langle f \circ f^{-1} \circ f \rangle @ \{ p \} \sqsubseteq \langle f \rangle @ \{ p \};$$

$$\text{Compl}(f \circ f^{-1} \circ f) \sqsubseteq \text{Compl } f;$$

$$\text{Compl}(f \circ f^{-1} \circ f) \sqsubseteq f. \quad \square$$

PROPOSITION 973. If f is complete, regularity of functor f is equivalent to $f \circ \text{Compl}(f^{-1} \circ f) \sqsubseteq f$.

PROOF. By proposition 954. □

REMARK 974. After seeing how it collapses into algebraic formulas about functors, the definition for a functor being regular seems quite arbitrary and sucked out of the finger (not an example of algebraic elegance). So I present these formulas only because they coincide with the traditional definition of regular topological spaces. However this is only my personal opinion and it may be wrong.

DEFINITION 975. An endofunctor is T_3 - iff it is both T_2 - and regular.

A topological space S is called T_4 -separable when for any two disjoint closed sets $A, B \subseteq S$ there exist disjoint open sets U, V containing A and B respectively.

Let f be the complete functor corresponding to the topological space.

Since the closed sets are exactly sets of the form $\langle f^{-1} \rangle^* X$ and sets X and Y having non-intersecting open neighborhood is equivalent to $\langle f \rangle^* X \simeq \langle f \rangle^* Y$, the above is equivalent to:

$$\langle f^{-1} \rangle^* A \simeq \langle f^{-1} \rangle^* B \Rightarrow \langle f \rangle^* \langle f^{-1} \rangle^* A \simeq \langle f \rangle^* \langle f^{-1} \rangle^* B;$$

$$\langle f \rangle^* \langle f^{-1} \rangle^* A \neq \langle f \rangle^* \langle f^{-1} \rangle^* B \Rightarrow \langle f^{-1} \rangle^* A \neq \langle f^{-1} \rangle^* B;$$

$$\langle f \rangle^* \langle f^{-1} \rangle^* \langle f \rangle^* \langle f^{-1} \rangle^* A \neq B \Rightarrow \langle f \rangle^* \langle f^{-1} \rangle^* A \neq B;$$

$$\langle f \rangle^* \langle f^{-1} \rangle^* \langle f \rangle^* \langle f^{-1} \rangle^* A \sqsubseteq \langle f \rangle^* \langle f^{-1} \rangle^* A;$$

$$f \circ f^{-1} \circ f \circ f^{-1} \sqsubseteq f \circ f^{-1}.$$

Take the last formula as the definition of T_4 -functor f .

7.17. Filters closed regarding a functor

DEFINITION 976. Let's call *closed* regarding a functor $f \in \text{FCD}(A, A)$ such filter $\mathcal{A} \in \mathcal{F}(\text{Src } f)$ that $\langle f \rangle \mathcal{A} \sqsubseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

PROPOSITION 977. If I and J are closed (regarding some funcoïd f), S is a set of closed filters on $\text{Src } f$, then

- 1°. $\mathcal{I} \sqcup \mathcal{J}$ is a closed filter;
- 2°. $\prod S$ is a closed filter.

PROOF. Let denote the given funcoïd as f . $\langle f \rangle(\mathcal{I} \sqcup \mathcal{J}) = \langle f \rangle \mathcal{I} \sqcup \langle f \rangle \mathcal{J} \sqsubseteq \mathcal{I} \sqcup \mathcal{J}$, $\langle f \rangle \prod S \sqsubseteq \prod \langle \langle f \rangle \rangle^* S \sqsubseteq \prod S$. Consequently the filters $\mathcal{I} \sqcup \mathcal{J}$ and $\prod S$ are closed. \square

PROPOSITION 978. If S is a set of filters closed regarding a complete funcoïd, then the filter $\bigsqcup S$ is also closed regarding our funcoïd.

PROOF. $\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S \sqsubseteq \bigsqcup S$ where f is the given funcoïd. \square

7.18. Proximity spaces

Fix a set U . Let equate typed subsets of U with subsets of U .

We will prove that proximity spaces are essentially the same as reflexive, symmetric, transitive funcoïds.

Our primary interest here is the last axiom (6°) in the definition 797 of proximity spaces.

PROPOSITION 979. If f is a transitive, symmetric funcoïd, then the last axiom of proximity holds.

PROOF.

$$\neg(A [f]^* B) \Leftrightarrow \neg(A [f^{-1} \circ f]^* B) \Leftrightarrow \langle f \rangle^* B \succ \langle f \rangle^* A \Leftrightarrow \\ \exists M \in U : M \succ \langle f \rangle^* A \wedge \overline{M} \succ \langle f \rangle^* B.$$

\square

PROPOSITION 980. For a reflexive funcoïd, the last axiom of proximity implies that it is transitive and symmetric.

PROOF. Let $\neg(A [f]^* B)$ implies $\exists M : M \succ \langle f \rangle^* A \wedge \overline{M} \succ \langle f \rangle^* B$. Then $\neg(A [f]^* B)$ implies $M \succ \langle f \rangle^* A \wedge \langle f \rangle^* B \sqsubseteq M$, thus $\langle f \rangle^* A \succ \langle f \rangle^* B$; $\neg(A [f^{-1} \circ f]^* B)$ that is $f \sqsupseteq f^{-1} \circ f$ and thus $f = f^{-1} \circ f$. By theorem 255 f is transitive and symmetric. \square

THEOREM 981. Reflexive, symmetric, transitive funcoïds endofuncoïds on a set U are essentially the same as proximity spaces on U .

PROOF. Above and theorem 831. \square

Reloids

8.1. Basic definitions

DEFINITION 982. Let A, B be sets. $\mathbf{RLD}\sharp(A, B)$ is the base of an arbitrary but fixed primary filtrator over $\mathbf{Rel}(A, B)$.

OBVIOUS 983. $(\mathbf{RLD}\sharp(A, B), \mathbf{Rel}(A, B))$ is a powerset filtrator.

DEFINITION 984. I call a reloid from a set A to a set B a triple (A, B, F) where $F \in \mathbf{RLD}\sharp(A, B)$.

DEFINITION 985. *Source* and *destination* of every reloid (A, B, F) are defined as

$$\text{Src}(A, B, F) = A \quad \text{and} \quad \text{Dst}(A, B, F) = B.$$

I will denote $\mathbf{RLD}(A, B)$ the set of reloids from A to B .

I will denote \mathbf{RLD} the set of all reloids (for small sets).

DEFINITION 986. I will call *endoreloids* reloids with the same source and destination.

DEFINITION 987.

- $\uparrow^{\mathbf{RLD}\sharp} f$ is the principal filter object corresponding to a \mathbf{Rel} -morphism f .
- $\uparrow^{\mathbf{RLD}\sharp(A, B)} f = \uparrow^{\mathbf{RLD}\sharp}(A, B, f)$ for every binary relation $f \in \mathcal{P}(A \times B)$.
- $\uparrow^{\mathbf{RLD}} f = (\text{Src } f, \text{Dst } f, \uparrow^{\mathbf{RLD}\sharp} f)$ for every \mathbf{Rel} -morphism f .
- $\uparrow^{\mathbf{RLD}(A, B)} f = \uparrow^{\mathbf{RLD}}(A, B, f)$ for every binary relation $f \in \mathcal{P}(A \times B)$.

DEFINITION 988. I call members of a set $\langle \uparrow^{\mathbf{RLD}} \rangle^* \mathbf{Rel}(A, B)$ as *principal reloids*.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations.

DEFINITION 989. $\text{up } f^{-1} = \left\{ \frac{F^{-1}}{F \in \text{up } f} \right\}$ for every $f \in \mathbf{RLD}\sharp(A, B)$.

PROPOSITION 990. f^{-1} exists and $f^{-1} \in \mathbf{RLD}\sharp(B, A)$.

PROOF. We need to prove that $\left\{ \frac{F^{-1}}{F \in \text{up } f} \right\}$ is a filter, but that's obvious. \square

DEFINITION 991. The *reverse* reloid of a reloid is defined by the formula

$$(A, B, F)^{-1} = (B, A, F^{-1}).$$

NOTE 992. The reverse reloid is *not* an inverse in the sense of group theory or category theory.

Reverse reloid is a generalization of conjugate quasi-uniformity.

DEFINITION 993. Every set $\mathbf{RLD}(A, B)$ is a poset by the formula $f \sqsubseteq g \Leftrightarrow \text{GR } f \sqsubseteq \text{GR } g$. We will apply lattice operations to subsets of $\mathbf{RLD}(A, B)$ without explicitly mentioning $\mathbf{RLD}(A, B)$.

Filtrators of reloids are $(\mathbf{RLD}(A, B), \mathbf{Rel}(A, B))$ (for all sets A, B). Here I equate principal reloids with corresponding \mathbf{Rel} -morphisms.

OBVIOUS 994. $(\mathbf{RLD}(A, B), \mathbf{Rel}(A, B))$ is a powerset filtrator isomorphic to the filtrator $(\mathbf{RLD}\sharp(A, B), \mathbf{Rel}(A, B))$. Thus $\mathbf{RLD}(A, B)$ is a special case of $\mathbf{RLD}\sharp(A, B)$.

8.2. Composition of reloids

DEFINITION 995. Reloids f and g are *composable* when $\text{Dst } f = \text{Src } g$.

DEFINITION 996. *Composition* of (composable) reloids is defined by the formula

$$g \circ f = \prod^{\mathbf{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}.$$

OBVIOUS 997. Composition of reloids is a reloid.

OBVIOUS 998. $\uparrow^{\mathbf{RLD}} g \circ \uparrow^{\mathbf{RLD}} f = \uparrow^{\mathbf{RLD}} (g \circ f)$ for composable morphisms f, g of category **Rel**.

THEOREM 999. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable reloids f, g, h .

PROOF. For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow \forall K \in A \exists L \in B : L \subseteq K \wedge \forall K \in B \exists L \in A : L \subseteq K.$$

It is easy to see that \sim is a transitive relation.

I will denote $B \circ A = \left\{ \frac{L \circ K}{K \in A, L \in B} \right\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B : K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ there exists $P' \in B \circ C$ such that $P' \subseteq P$; the vice versa is analogous. So $A \circ C \sim B \circ C$.

$\text{up}((h \circ g) \circ f) \sim \text{up}(h \circ g) \circ \text{up } f$, $\text{up}(h \circ g) \sim (\text{up } h) \circ (\text{up } g)$. By proven above $\text{up}((h \circ g) \circ f) \sim (\text{up } h) \circ (\text{up } g) \circ (\text{up } f)$.

Analogously $\text{up}(h \circ (g \circ f)) \sim (\text{up } h) \circ (\text{up } g) \circ (\text{up } f)$.

So $\text{up}(h \circ (g \circ f)) \sim \text{up}((h \circ g) \circ f)$ what is possible only if $\text{up}(h \circ (g \circ f)) = \text{up}((h \circ g) \circ f)$. Thus $(h \circ g) \circ f = h \circ (g \circ f)$. \square

EXERCISE 1000. Prove $f_n \circ \dots \circ f_0 = \prod^{\mathbf{RLD}} \left\{ \frac{F_n \circ \dots \circ F_0}{F_i \in \text{up } f_i} \right\}$ for every composable reloids f_0, \dots, f_n where n is an integer, independently of the inserted parentheses. (Hint: Use generalized filter bases.)

THEOREM 1001. For every reloid f :

- 1°. $f \circ f = \prod^{\mathbf{RLD}} \left\{ \frac{F \circ F}{F \in \text{up } f} \right\}$ if $\text{Src } f = \text{Dst } f$;
- 2°. $f^{-1} \circ f = \prod^{\mathbf{RLD}} \left\{ \frac{F^{-1} \circ F}{F \in \text{up } f} \right\}$;
- 3°. $f \circ f^{-1} = \prod^{\mathbf{RLD}} \left\{ \frac{F \circ F^{-1}}{F \in \text{up } f} \right\}$.

PROOF. I will prove only 1° and 2° because 3° is analogous to 2°.

1°. It's enough to show that $\forall F, G \in \text{up } f \exists H \in \text{up } f : H \circ H \subseteq G \circ F$. To prove it take $H = F \sqcap G$.

2°. It's enough to show that $\forall F, G \in \text{up } f \exists H \in \text{up } f : H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \sqcap G$. Then $H^{-1} \circ H = (F \sqcap G)^{-1} \circ (F \sqcap G) \subseteq G^{-1} \circ F$. \square

EXERCISE 1002. Prove $f^n = \prod^{\mathbf{RLD}} \left\{ \frac{F^n}{F \in \text{up } f} \right\}$ for every endofunctor f and positive integer n .

THEOREM 1003. For every sets A, B, C if $g, h \in \text{RLD}(A, B)$ then

- 1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for every $f \in \text{RLD}(B, C)$;
- 2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for every $f \in \text{RLD}(C, A)$.

PROOF. We'll prove only the first as the second is dual.
By the infinite distributivity law for filters we have

$$\begin{aligned} f \circ g \sqcup f \circ h &= \\ \prod^{\text{RLD}} \left\{ \frac{F \circ G}{F \in \text{up } f, G \in \text{up } g} \right\} \sqcup \prod^{\text{RLD}} \left\{ \frac{F \circ H}{F \in \text{up } f, H \in \text{up } h} \right\} &= \\ \prod^{\text{RLD}} \left\{ \frac{(F_1 \circ G) \sqcup^{\text{RLD}} (F_2 \circ H)}{F_1, F_2 \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} &= \\ \prod^{\text{RLD}} \left\{ \frac{(F_1 \circ G) \sqcup (F_2 \circ H)}{F_1, F_2 \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\}. \end{aligned}$$

Obviously

$$\begin{aligned} \prod^{\text{RLD}} \left\{ \frac{(F_1 \circ G) \sqcup (F_2 \circ H)}{F_1, F_2 \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} &\sqsupseteq \\ \prod^{\text{RLD}} \left\{ \frac{(((F_1 \sqcap F_2) \circ G) \sqcup ((F_1 \sqcap F_2) \circ H))}{F_1, F_2 \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} &= \\ \prod^{\text{RLD}} \left\{ \frac{(F \circ G) \sqcup (F \circ H)}{F \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} &= \\ \prod^{\text{RLD}} \left\{ \frac{F \circ (G \sqcup H)}{F \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\}. \end{aligned}$$

Because $G \in \text{up } g \wedge H \in \text{up } h \Rightarrow G \sqcup H \in \text{up}(g \sqcup h)$ we have

$$\begin{aligned} \prod^{\text{RLD}} \left\{ \frac{F \circ (G \sqcup H)}{F \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} &\sqsupseteq \\ \prod^{\text{RLD}} \left\{ \frac{F \circ K}{F \in \text{up } f, K \in \text{up}(g \sqcup h)} \right\} &= \\ f \circ (g \sqcup h). \end{aligned}$$

Thus we have proved $f \circ g \sqcup f \circ h \sqsupseteq f \circ (g \sqcup h)$. But obviously $f \circ (g \sqcup h) \sqsupseteq f \circ g$ and $f \circ (g \sqcup h) \sqsupseteq f \circ h$ and so $f \circ (g \sqcup h) \sqsupseteq f \circ g \sqcup f \circ h$. Thus $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$. \square

THEOREM 1004. Let A, B, C be sets, $f \in \text{RLD}(A, B)$, $g \in \text{RLD}(B, C)$, $h \in \text{RLD}(A, C)$. Then

$$g \circ f \neq h \Leftrightarrow g \neq h \circ f^{-1}.$$

PROOF.

$$\begin{aligned}
& g \circ f \not\leq h \Leftrightarrow \\
& \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\} \sqcap \prod^{\text{RLD}} \text{up } h \neq \perp \Leftrightarrow \\
& \prod^{\text{RLD}} \left\{ \frac{(G \circ F) \sqcap^{\text{RLD}} H}{F \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} \neq \perp \Leftrightarrow \\
& \prod^{\text{RLD}} \left\{ \frac{(G \circ F) \sqcap H}{F \in \text{up } f, G \in \text{up } g, H \in \text{up } h} \right\} \neq \perp \Leftrightarrow \\
& \forall F \in \text{up } f, G \in \text{up } g, H \in \text{up } h : \uparrow^{\text{RLD}} ((G \circ F) \sqcap H) \neq \perp \Leftrightarrow \\
& \forall F \in \text{up } f, G \in \text{up } g, H \in \text{up } h : G \circ F \not\leq H
\end{aligned}$$

(used properties of generalized filter bases).

Similarly $g \not\leq h \circ f^{-1} \Leftrightarrow \forall F \in \text{up } f, G \in \text{up } g, H \in \text{up } h : G \not\leq H \circ F^{-1}$.

Thus $g \circ f \not\leq h \Leftrightarrow g \not\leq h \circ f^{-1}$ because $G \circ F \not\leq H \Leftrightarrow G \not\leq H \circ F^{-1}$ by proposition 283. \square

THEOREM 1005. For every composable reloids f and g

$$\begin{aligned}
1^\circ. \quad g \circ f &= \bigsqcup \left\{ \frac{g \circ F}{F \in \text{atoms } f} \right\}. \\
2^\circ. \quad g \circ f &= \bigsqcup \left\{ \frac{G \circ f}{G \in \text{atoms } g} \right\}.
\end{aligned}$$

PROOF. We will prove only the first as the second is dual. \square

Obviously $\bigsqcup \left\{ \frac{g \circ F}{F \in \text{atoms } f} \right\} \sqsubseteq g \circ f$. We need to prove $\bigsqcup \left\{ \frac{g \circ F}{F \in \text{atoms } f} \right\} \sqsupseteq g \circ f$. Really,

$$\begin{aligned}
& \bigsqcup \left\{ \frac{g \circ F}{F \in \text{atoms } f} \right\} \sqsupseteq g \circ f \Leftrightarrow \\
& \forall x \in \text{RLD}(\text{Src } f, \text{Dst } g) : \left(x \not\leq g \circ f \Rightarrow x \not\leq \bigsqcup \left\{ \frac{g \circ F}{F \in \text{atoms } f} \right\} \right) \Leftarrow \\
& \forall x \in \text{RLD}(\text{Src } f, \text{Dst } g) : (x \not\leq g \circ f \Rightarrow \exists F \in \text{atoms } f : x \not\leq g \circ F) \Leftrightarrow \\
& \forall x \in \text{RLD}(\text{Src } f, \text{Dst } g) : (g^{-1} \circ x \not\leq f \Rightarrow \exists F \in \text{atoms } f : g^{-1} \circ x \not\leq F)
\end{aligned}$$

what is obviously true.

COROLLARY 1006. If f and g are composable reloids, then

$$g \circ f = \bigsqcup \left\{ \frac{G \circ F}{F \in \text{atoms } f, G \in \text{atoms } g} \right\}.$$

PROOF. $g \circ f = \bigsqcup_{F \in \text{atoms } f} (g \circ F) = \bigsqcup_{F \in \text{atoms } f} \bigsqcup_{G \in \text{atoms } g} (G \circ F) = \bigsqcup \left\{ \frac{G \circ F}{F \in \text{atoms } f, G \in \text{atoms } g} \right\}$. \square

8.3. Reloidal product of filters

DEFINITION 1007. Reloidal product of filters \mathcal{A} and \mathcal{B} is defined by the formula

$$\mathcal{A} \times^{\text{RLD}} \mathcal{B} \stackrel{\text{def}}{=} \prod^{\text{RLD}} \left\{ \frac{A \times B}{A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\}.$$

OBVIOUS 1008.

- $\uparrow^U A \times^{\text{RLD}} \uparrow^V B = \uparrow^{\text{RLD}(U,V)} (A \times B)$ for every sets $A \subseteq U, B \subseteq V$.
- $\uparrow A \times^{\text{RLD}} \uparrow B = \uparrow^{\text{RLD}} (A \times B)$ for every typed sets A, B .

THEOREM 1009. $\mathcal{A} \times^{\text{RLD}} \mathcal{B} = \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms } \mathcal{A}, b \in \text{atoms } \mathcal{B}} \right\}$ for every filters \mathcal{A} and \mathcal{B} .

PROOF. Obviously $\mathcal{A} \times^{\text{RLD}} \mathcal{B} \supseteq \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms } \mathcal{A}, b \in \text{atoms } \mathcal{B}} \right\}$.

Reversely, let $K \in \text{up} \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms } \mathcal{A}, b \in \text{atoms } \mathcal{B}} \right\}$. Then $K \in \text{up}(a \times^{\text{RLD}} b)$ for every $a \in \text{atoms } \mathcal{A}$, $b \in \text{atoms } \mathcal{B}$. $K \supseteq X_a \times Y_b$ for some $X_a \in \text{up } a$, $Y_b \in \text{up } b$;

$$\begin{aligned} K \supseteq \bigsqcup \left\{ \frac{X_a \times Y_b}{a \in \text{atoms } \mathcal{A}, b \in \text{atoms } \mathcal{B}} \right\} &= \\ \bigsqcup \left\{ \frac{X_a}{a \in \text{atoms } \mathcal{A}} \right\} \times \bigsqcup \left\{ \frac{Y_b}{b \in \text{atoms } \mathcal{B}} \right\} &\supseteq A \times B \end{aligned}$$

where $A \in \text{up } \mathcal{A}$, $B \in \text{up } \mathcal{B}$; $K \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$. \square

THEOREM 1010. If $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathcal{F}(B)$ for some sets A, B then

$$(\mathcal{A}_0 \times^{\text{RLD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\text{RLD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\text{RLD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

PROOF.

$$\begin{aligned} (\mathcal{A}_0 \times^{\text{RLD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\text{RLD}} \mathcal{B}_1) &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{P \sqcap Q}{P \in \text{up}(\mathcal{A}_0 \times^{\text{RLD}} \mathcal{B}_0), Q \in \text{up}(\mathcal{A}_1 \times^{\text{RLD}} \mathcal{B}_1)} \right\} &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{(\mathcal{A}_0 \times \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times \mathcal{B}_1)}{\mathcal{A}_0 \in \text{up } \mathcal{A}_0, \mathcal{B}_0 \in \text{up } \mathcal{B}_0, \mathcal{A}_1 \in \text{up } \mathcal{A}_1, \mathcal{B}_1 \in \text{up } \mathcal{B}_1} \right\} &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times (\mathcal{B}_0 \sqcap \mathcal{B}_1)}{\mathcal{A}_0 \in \text{up } \mathcal{A}_0, \mathcal{B}_0 \in \text{up } \mathcal{B}_0, \mathcal{A}_1 \in \text{up } \mathcal{A}_1, \mathcal{B}_1 \in \text{up } \mathcal{B}_1} \right\} &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{K \times L}{K \in \text{up}(\mathcal{A}_0 \sqcap \mathcal{A}_1), L \in \text{up}(\mathcal{B}_0 \sqcap \mathcal{B}_1)} \right\} &= \\ (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\text{RLD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1). & \end{aligned}$$

\square

THEOREM 1011. If $S \in \mathcal{P}(\mathcal{F}(A) \times \mathcal{F}(B))$ for some sets A, B then

$$\bigsqcap \left\{ \frac{\mathcal{A} \times^{\text{RLD}} \mathcal{B}}{(\mathcal{A}, \mathcal{B}) \in S} \right\} = \bigsqcap \text{dom } S \times^{\text{RLD}} \bigsqcap \text{im } S.$$

PROOF. Let $\mathcal{P} = \bigsqcap \text{dom } S$, $\mathcal{Q} = \bigsqcap \text{im } S$; $l = \bigsqcap \left\{ \frac{\mathcal{A} \times^{\text{RLD}} \mathcal{B}}{(\mathcal{A}, \mathcal{B}) \in S} \right\}$.

$\mathcal{P} \times^{\text{RLD}} \mathcal{Q} \sqsubseteq l$ is obvious.

Let $F \in \text{up}(\mathcal{P} \times^{\text{RLD}} \mathcal{Q})$. Then there exist $P \in \text{up } \mathcal{P}$ and $Q \in \text{up } \mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \sqcap \dots \sqcap P_n$ where $P_i \in \text{dom } S$ and $Q = Q_1 \sqcap \dots \sqcap Q_m$ where $Q_j \in \text{im } S$.

$P \times Q = \bigsqcap_{i,j} (P_i \times Q_j)$.

$P_i \times Q_j \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$ for some $(\mathcal{A}, \mathcal{B}) \in S$. $P \times Q = \bigsqcap_{i,j} (P_i \times Q_j) \in \text{up } l$.

So $F \in \text{up } l$. \square

COROLLARY 1012. $\bigsqcap \langle \mathcal{A} \times^{\text{RLD}} \rangle^* T = \mathcal{A} \times^{\text{RLD}} \bigsqcap T$ if \mathcal{A} is a filter and T is a set of filters with common base.

PROOF. Take $S = \{\mathcal{A}\} \times T$ where T is a set of filters.

Then $\bigsqcap \left\{ \frac{\mathcal{A} \times^{\text{RLD}} \mathcal{B}}{\mathcal{B} \in T} \right\} = \mathcal{A} \times^{\text{RLD}} \bigsqcap T$ that is $\bigsqcap \langle \mathcal{A} \times^{\text{RLD}} \rangle^* T = \mathcal{A} \times^{\text{RLD}} \bigsqcap T$. \square

DEFINITION 1013. I will call a reloid *convex* iff it is a join of direct products.

8.4. Restricting reloid to a filter. Domain and image

DEFINITION 1014. *Identity reloid* for a set A is defined by the formula $1_A^{\text{RLD}} = \uparrow^{\text{RLD}(A,A)} \text{id}_A$.

OBVIOUS 1015. $(1_A^{\text{RLD}})^{-1} = 1_A^{\text{RLD}}$.

DEFINITION 1016. I define *restricting* a reloid f to a filter \mathcal{A} as $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\text{RLD}} \top_{\mathcal{F}(\text{Dst } f)})$.

DEFINITION 1017. *Domain* and *image* of a reloid f are defined as follows:

$$\text{dom } f = \bigsqcap_{\mathcal{F}} \langle \text{dom} \rangle^* \text{up } f; \quad \text{im } f = \bigsqcap_{\mathcal{F}} \langle \text{im} \rangle^* \text{up } f.$$

PROPOSITION 1018. $f \sqsubseteq \mathcal{A} \times^{\text{RLD}} \mathcal{B} \Leftrightarrow \text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$ for every reloid f and filters $\mathcal{A} \in \mathcal{F}(\text{Src } f)$, $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

PROOF.

\Rightarrow . It follows from $\text{dom}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \sqsubseteq \mathcal{A} \wedge \text{im}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \sqsubseteq \mathcal{B}$.

\Leftarrow . $\text{dom } f \sqsubseteq \mathcal{A} \Leftrightarrow \forall A \in \text{up } \mathcal{A} \exists F \in \text{up } f : \text{dom } F \sqsubseteq A$. Analogously $\text{im } f \sqsubseteq \mathcal{B} \Leftrightarrow \forall B \in \text{up } \mathcal{B} \exists G \in \text{up } f : \text{im } G \sqsubseteq B$.

Let $\text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$, $A \in \text{up } \mathcal{A}$, $B \in \text{up } \mathcal{B}$. Then there exist $F, G \in \text{up } f$ such that $\text{dom } F \sqsubseteq A \wedge \text{im } G \sqsubseteq B$. Consequently $F \sqcap G \in \text{up } f$, $\text{dom}(F \sqcap G) \sqsubseteq A$, $\text{im}(F \sqcap G) \sqsubseteq B$ that is $F \sqcap G \sqsubseteq A \times B$. So there exists $H \in \text{up } f$ such that $H \sqsubseteq A \times B$ for every $A \in \text{up } \mathcal{A}$, $B \in \text{up } \mathcal{B}$. So $f \sqsubseteq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$.

□

DEFINITION 1019. I call *restricted identity reloid* for a filter \mathcal{A} the reloid

$$\text{id}_{\mathcal{A}}^{\text{RLD}} = (1_{\text{Base}(\mathcal{A})}^{\text{RLD}})|_{\mathcal{A}}.$$

THEOREM 1020. $\text{id}_{\mathcal{A}}^{\text{RLD}} = \bigsqcap_{A \in \text{up } \mathcal{A}} \text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A})) \text{id}_A$ for every filter \mathcal{A} .

PROOF. Let $K \in \text{up } \bigsqcap_{A \in \text{up } \mathcal{A}} \text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A})) \text{id}_A$, then there exists $A \in \text{up } \mathcal{A}$ such that $\text{GR } K \supseteq \text{id}_A$. Then

$$\begin{aligned} \text{id}_{\mathcal{A}}^{\text{RLD}} &\sqsubseteq \\ \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_{\text{Base}(\mathcal{A})} \sqcap (\mathcal{A} \times^{\text{RLD}} \top) &\sqsubseteq \\ \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_{\text{Base}(\mathcal{A})} \sqcap (\mathcal{A} \times^{\text{RLD}} \top) &= \\ \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_{\text{Base}(\mathcal{A})} \sqcap \uparrow^{\text{RLD}} (A \times \top) &= \\ \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} (\text{id}_{\text{Base}(\mathcal{A})} \sqcap \text{GR}(A \times \top)) &= \\ \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_A &\sqsubseteq K. \end{aligned}$$

Thus $K \in \text{up } \text{id}_{\mathcal{A}}^{\text{RLD}}$.

Reversely let $K \in \text{up } \text{id}_{\mathcal{A}}^{\text{RLD}} = \text{up}(1_{\text{Base}(\mathcal{A})}^{\text{RLD}} \sqcap (\mathcal{A} \times^{\text{RLD}} \top))$, then there exists $A \in \text{up } \mathcal{A}$ such that

$$\begin{aligned} K \in \text{up } \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} (\text{id}_{\text{Base}(\mathcal{A})} \sqcap \text{GR}(A \times \top)) &= \\ \text{up } \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_A &\supseteq \\ \text{up } \prod_{A \in \text{up } \mathcal{A}} \text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A})) \text{id}_A &. \end{aligned}$$

□

COROLLARY 1021. $(\text{id}_{\mathcal{A}}^{\text{RLD}})^{-1} = \text{id}_{\mathcal{A}}^{\text{RLD}}$.

THEOREM 1022. $f|_{\mathcal{A}} = f \circ \text{id}_{\mathcal{A}}^{\text{RLD}}$ for every reloid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$.

PROOF. We need to prove that

$$f \sqcap (\mathcal{A} \times^{\text{RLD}} \top) = f \circ \bigsqcap^{\text{RLD}(\text{Src } f, \text{Src } f)} \left\{ \frac{\text{id}_{\mathcal{A}}}{A \in \text{up } \mathcal{A}} \right\}.$$

We have

$$\begin{aligned} & f \circ \bigsqcap^{\text{RLD}(\text{Src } f, \text{Src } f)} \left\{ \frac{\text{id}_{\mathcal{A}}}{A \in \text{up } \mathcal{A}} \right\} = \\ & \bigsqcap^{\text{RLD}(\text{Src } f, \text{Src } f)} \left\{ \frac{\text{GR}(F) \circ \text{id}_{\mathcal{A}}}{F \in \text{up } f, A \in \text{up } \mathcal{A}} \right\} = \\ & \bigsqcap^{\text{RLD}} \left\{ \frac{F|_{\mathcal{A}}}{F \in \text{up } f, A \in \text{up } \mathcal{A}} \right\} = \\ & \bigsqcap^{\text{RLD}} \left\{ \frac{F \sqcap (A \times \top^{\mathcal{F}(\text{Dst } f)})}{F \in \text{up } f, A \in \text{up } \mathcal{A}} \right\} = \\ & \bigsqcap^{\text{RLD}} \left\{ \frac{F}{F \in \text{up } f} \right\} \sqcap \bigsqcap^{\text{RLD}} \left\{ \frac{A \times \top^{\mathcal{F}(\text{Dst } f)}}{A \in \text{up } \mathcal{A}} \right\} = \\ & f \sqcap (\mathcal{A} \times^{\text{RLD}} \top). \end{aligned}$$

□

THEOREM 1023. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for every composable reloids f and g and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$.

PROOF. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ \text{id}_{\mathcal{A}}^{\text{RLD}} = g \circ (f \circ \text{id}_{\mathcal{A}}^{\text{RLD}}) = g \circ (f|_{\mathcal{A}})$. □

THEOREM 1024. $f \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \text{id}_{\mathcal{B}}^{\text{RLD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{RLD}}$ for every reloid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$, $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

PROOF.

$$\begin{aligned} & f \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \\ & f \sqcap (\mathcal{A} \times^{\text{RLD}} \top^{\mathcal{F}(\text{Dst } f)}) \sqcap (\top^{\mathcal{F}(\text{Src } f)} \times^{\text{RLD}} \mathcal{B}) = \\ & f|_{\mathcal{A}} \sqcap (\top^{\mathcal{F}(\text{Src } f)} \times^{\text{RLD}} \mathcal{B}) = \\ & (f \circ \text{id}_{\mathcal{A}}^{\text{RLD}}) \sqcap (\top^{\mathcal{F}(\text{Src } f)} \times^{\text{RLD}} \mathcal{B}) = \\ & ((f \circ \text{id}_{\mathcal{A}}^{\text{RLD}})^{-1} \sqcap (\top^{\mathcal{F}(\text{Src } f)} \times^{\text{RLD}} \mathcal{B})^{-1})^{-1} = \\ & ((\text{id}_{\mathcal{A}}^{\text{RLD}} \circ f^{-1}) \sqcap (\mathcal{B} \times^{\text{RLD}} \top^{\mathcal{F}(\text{Src } f)}))^{-1} = \\ & (\text{id}_{\mathcal{A}}^{\text{RLD}} \circ f^{-1} \circ \text{id}_{\mathcal{B}}^{\text{RLD}})^{-1} = \\ & \text{id}_{\mathcal{B}}^{\text{RLD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{RLD}}. \end{aligned}$$

□

PROPOSITION 1025. $\text{id}_{\mathcal{B}} \circ \text{id}_{\mathcal{A}} = \text{id}_{\mathcal{A} \cap \mathcal{B}}$ for all filters \mathcal{A}, \mathcal{B} (on some set U).

PROOF. $\text{id}_{\mathcal{B}} \circ \text{id}_{\mathcal{A}} = (\text{id}_{\mathcal{B}})|_{\mathcal{A}} = (1_U^{\text{RLD}}|_{\mathcal{B}})|_{\mathcal{A}} = 1_U^{\text{RLD}}|_{\mathcal{A} \cap \mathcal{B}} = \text{id}_{\mathcal{A} \cap \mathcal{B}}$. □

THEOREM 1026. $f|_{\uparrow\{\alpha\}} = \uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \text{im}(f|_{\uparrow\{\alpha\}})$ for every reloid f and $\alpha \in \text{Src } f$.

PROOF. First,

$$\begin{aligned}
& \text{im}(f|_{\uparrow\{\alpha\}}) = \\
& \prod^{\text{RLD}} \langle \text{im} \rangle^* \text{up}(f|_{\uparrow\{\alpha\}}) = \\
& \prod^{\text{RLD}} \langle \text{im} \rangle^* \text{up}(f \sqcap (\uparrow^{\text{Src } f} \{\alpha\} \times \top^{\mathcal{D}(\text{Dst } f)})) = \\
& \prod^{\text{RLD}} \left\{ \frac{\text{im}(F \cap (\{\alpha\} \times \top^{\mathcal{D}(\text{Dst } f)}))}{F \in \text{up } f} \right\} = \\
& \prod^{\text{RLD}} \left\{ \frac{\text{im}(F|_{\uparrow\{\alpha\}})}{F \in \text{up } f} \right\}.
\end{aligned}$$

Taking this into account we have:

$$\begin{aligned}
& \uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \text{im}(f|_{\uparrow\{\alpha\}}) = \\
& \prod^{\text{RLD}} \left\{ \frac{\uparrow^{\text{Src } f} \{\alpha\} \times K}{K \in \text{im}(f|_{\uparrow\{\alpha\}})} \right\} = \\
& \prod^{\text{RLD}} \left\{ \frac{\uparrow^{\text{Src } f} \{\alpha\} \times \text{im}(F|_{\uparrow\{\alpha\}})}{F \in \text{up } f} \right\} = \\
& \prod^{\text{RLD}} \left\{ \frac{F|_{\uparrow\{\alpha\}}}{F \in \text{up } f} \right\} = \\
& \prod^{\text{RLD}} \left\{ \frac{F \sqcap (\uparrow^{\text{Src } f} \{\alpha\} \times \top^{\mathcal{D}(\text{Dst } f)})}{F \in \text{up } f} \right\} = \\
& \prod^{\text{RLD}} \left\{ \frac{F}{F \in \text{up } f} \right\} \sqcap \uparrow^{\text{RLD}} (\uparrow^{\text{Src } f} \{\alpha\} \times \top^{\mathcal{D}(\text{Dst } f)}) = \\
& f \sqcap \uparrow^{\text{RLD}} (\uparrow^{\text{Src } f} \{\alpha\} \times \top^{\mathcal{D}(\text{Dst } f)}) = \\
& f|_{\uparrow\{\alpha\}}.
\end{aligned}$$

□

LEMMA 1027. $\lambda \mathcal{B} \in \mathcal{F}(B) : \top^{\mathcal{F}} \times^{\text{RLD}} \mathcal{B}$ is an upper adjoint of $\lambda f \in \text{RLD}(A, B) : \text{im } f$ (for every sets A, B).

PROOF. We need to prove $\text{im } f \sqsubseteq \mathcal{B} \Leftrightarrow f \sqsubseteq \top^{\mathcal{F}} \times^{\text{RLD}} \mathcal{B}$ what is obvious. □

COROLLARY 1028. Image and domain of reloids preserve joins.

PROOF. By properties of Galois connections and duality. □

8.5. Categories of reloids

I will define two categories, the *category of reloids* and the *category of reloid triples*.

The *category of reloids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\text{RLD}(A, B)$.
- The composition is the composition of reloids.
- Identity morphism for a set is the identity reloid for that set.

To show it is really a category is trivial.

The *category of reloid triples* is defined as follows:

- Objects are small sets.

- The morphisms from a filter \mathcal{A} to a filter \mathcal{B} are triples $(\mathcal{A}, \mathcal{B}, f)$ where $f \in \text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B}))$ and $\text{dom } f \sqsubseteq \mathcal{A}$, $\text{im } f \sqsubseteq \mathcal{B}$.
- The composition is defined by the formula $(\mathcal{B}, \mathcal{C}, g) \circ (\mathcal{A}, \mathcal{B}, f) = (\mathcal{A}, \mathcal{C}, g \circ f)$.
- Identity morphism for a filter \mathcal{A} is $\text{id}_{\mathcal{A}}^{\text{RLD}}$.

To prove that it is really a category is trivial.

PROPOSITION 1029. \uparrow^{RLD} is a functor from \mathbf{Rel} to RLD .

PROOF. $\uparrow^{\text{RLD}}(g \circ f) = \uparrow^{\text{RLD}} g \circ \uparrow^{\text{RLD}} f$ was proved above. $\uparrow^{\text{RLD}} 1_{\mathcal{A}}^{\mathbf{Rel}} = 1_{\mathcal{A}}^{\text{RLD}}$ is by definition. \square

8.6. Monovalued and injective reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f \circ f^{-1} \sqsubseteq \text{id}_{\text{im } f}^{\text{RLD}}$.

Similarly, I will call a reloid *injective* when $f^{-1} \circ f \sqsubseteq \text{id}_{\text{dom } f}^{\text{RLD}}$.

OBVIOUS 1030. A reloid f is

- monovalued iff $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{RLD}}$;
- injective iff $f^{-1} \circ f \sqsubseteq 1_{\text{Src } f}^{\text{RLD}}$.

In other words, a reloid is monovalued (injective) when it is a monovalued (injective) morphism of the category of reloids.

Monovaluedness is dual of injectivity.

OBVIOUS 1031.

- 1°. A morphism $(\mathcal{A}, \mathcal{B}, f)$ of the category of reloid triples is monovalued iff the reloid f is monovalued.
- 2°. A morphism $(\mathcal{A}, \mathcal{B}, f)$ of the category of reloid triples is injective iff the reloid f is injective.

THEOREM 1032.

- 1°. A reloid f is a monovalued iff there exists a **Set**-morphism (monovalued **Rel**-morphism) $F \in \text{up } f$.
- 2°. A reloid f is a injective iff there exists an injective **Rel**-morphism $F \in \text{up } f$.
- 3°. A reloid f is a both monovalued and injective iff there exists an injection (a monovalued and injective **Rel**-morphism = injective **Set**-morphism) $F \in \text{up } f$.

PROOF. The reverse implications are obvious. Let's prove the direct implications:

- 1°. Let f be a monovalued reloid. Then $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{RLD}}$, that is

$$\prod^{\text{RLD}} \left\{ \frac{F \circ F^{-1}}{F \in \text{up } f} \right\} \sqsubseteq 1_{\text{Dst } f}^{\text{RLD}}.$$

It's simple to show that $\left\{ \frac{F \circ F^{-1}}{F \in \text{up } f} \right\}$ is a filter base. Consequently there exists $F \in \text{up } f$ such that $F \circ F^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{RLD}}$ that is F is monovalued.

- 2°. Similar.

- 3°. Let f be a both monovalued and injective reloid. Then by proved above there exist $F, G \in \text{up } f$ such that F is monovalued and G is injective. Thus $F \sqcap G \in \text{up } f$ is both monovalued and injective.

\square

CONJECTURE 1033. A reloid f is monovalued iff

$$\forall g \in \text{RLD}(\text{Src } f, \text{Dst } f) : (g \sqsubseteq f \Rightarrow \exists \mathcal{A} \in \mathcal{F}(\text{Src } f) : g = f|_{\mathcal{A}}).$$

8.7. Complete reloids and completion of reloids

DEFINITION 1034. A *complete* reloid is a reloid representable as a join of reloidal products $\uparrow^A \{\alpha\} \times^{\text{RLD}} b$ where $\alpha \in A$ and b is an ultrafilter on B for some sets A and B .

DEFINITION 1035. A *co-complete* reloid is a reloid representable as a join of reloidal products $a \times^{\text{RLD}} \uparrow^A \{\beta\}$ where $\beta \in B$ and a is an ultrafilter on A for some sets A and B .

I will denote the sets of complete and co-complete reloids from a set A to a set B as $\text{ComplRLD}(A, B)$ and $\text{CoComplRLD}(A, B)$ correspondingly and set of all (co-)complete reloids (for small sets) as ComplRLD and CoComplRLD .

OBVIOUS 1036. Complete and co-complete are dual.

THEOREM 1037. $G \mapsto \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$ is an order isomorphism from the set of functions $G \in \mathcal{F}(B)^A$ to the set $\text{ComplRLD}(A, B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \text{im}(f|_{\uparrow\{\alpha\}})$ where f is a complete reloid.

PROOF. $\bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$ is complete because $G(\alpha) = \bigsqcup \text{atoms } G(\alpha)$ and thus

$$\bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\} = \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} b}{\alpha \in A, b \in \text{atoms } G(\alpha)} \right\}$$

is complete. So $G \mapsto \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$ is a function from $G \in \mathcal{F}(B)^A$ to $\text{ComplRLD}(A, B)$.

Let f be complete. Then take

$$G(\alpha) = \bigsqcup \left\{ \frac{b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}}{\uparrow^A \{\alpha\} \times^{\text{RLD}} b \sqsubseteq f} \right\}$$

and we have $f = \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$ obviously. So $G \mapsto \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$ is surjection onto $\text{ComplRLD}(A, B)$.

Let now prove that it is an injection:

Let

$$f = \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} F(\alpha)}{\alpha \in A} \right\} = \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$$

for some $F, G \in \mathcal{F}(B)^A$. We need to prove $F = G$. Let $\beta \in \text{Src } f$.

$$\begin{aligned} f \sqcap (\uparrow^A \{\beta\} \times^{\text{RLD}} \top^{\mathcal{F}(B)}) &= \text{(theorem 610)} \\ \bigsqcup \left\{ \frac{(\uparrow^A \{\alpha\} \times^{\text{RLD}} F(\alpha)) \sqcap (\uparrow^A \{\beta\} \times^{\text{RLD}} \top^{\mathcal{F}(B)})}{\alpha \in A} \right\} &= \\ \uparrow^A \{\beta\} \times^{\text{RLD}} F(\beta). & \end{aligned}$$

Similarly $f \sqcap (\uparrow^A \{\beta\} \times^{\text{RLD}} \top^{\mathcal{F}(B)}) = \uparrow^A \{\beta\} \times^{\text{RLD}} G(\beta)$. Thus $\uparrow^A \{\beta\} \times^{\text{RLD}} F(\beta) = \uparrow^A \{\beta\} \times^{\text{RLD}} G(\beta)$ and so $F(\beta) = G(\beta)$.

We have proved that it is a bijection. To show that it is monotone is trivial.

Denote $f = \bigsqcup \left\{ \frac{\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\}$. Then

$$\begin{aligned} \text{im}(f|_{\uparrow\{\alpha'\}}) &= \text{im}(f \sqcap (\uparrow^A \{\alpha'\} \times \top^{\mathcal{F}(B)})) = \\ &\quad (\text{because } \uparrow^A \{\alpha'\} \times \top^{\mathcal{F}(B)} \text{ is principal}) = \\ &= \text{im} \bigsqcup \left\{ \frac{(\uparrow^A \{\alpha\} \times^{\text{RLD}} G(\alpha)) \sqcap (\uparrow^A \{\alpha'\} \times \top^{\mathcal{F}(B)})}{\alpha \in \text{Src } f} \right\} = \\ &= \text{im}(\uparrow^A \{\alpha'\} \times^{\text{RLD}} G(\alpha')) = G(\alpha'). \end{aligned}$$

□

COROLLARY 1038. $G \mapsto \bigsqcup \left\{ \frac{G(\alpha) \times^{\text{RLD}} \uparrow^A \{\alpha\}}{\alpha \in A} \right\}$ is an order isomorphism from the set of functions $G \in \mathcal{F}(B)^A$ to the set $\text{CoComplRLD}(A, B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \text{im}(f^{-1}|_{\uparrow\{\alpha\}})$ where f is a co-complete reloid.

COROLLARY 1039. $\text{ComplRLD}(A, B)$ and $\text{ComplFCD}(A, B)$ are a co-frames.

OBVIOUS 1040. Complete and co-complete reloids are convex.

OBVIOUS 1041. Principal reloids are complete and co-complete.

OBVIOUS 1042. Join (on the lattice of reloids) of complete reloids is complete.

THEOREM 1043. A reloid which is both complete and co-complete is principal.

PROOF. Let f be a complete and co-complete reloid. We have

$$f = \bigsqcup \left\{ \frac{\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in \text{Src } f} \right\} \quad \text{and} \quad f = \bigsqcup \left\{ \frac{H(\beta) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\beta\}}{\beta \in \text{Dst } f} \right\}$$

for some functions $G : \text{Src } f \rightarrow \mathcal{F}(\text{Dst } f)$ and $H : \text{Dst } f \rightarrow \mathcal{F}(\text{Src } f)$. For every $\alpha \in \text{Src } f$ we have

$$\begin{aligned} G(\alpha) &= \\ \text{im } f|_{\uparrow\{\alpha\}} &= \\ \text{im}(f \sqcap (\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \top^{\mathcal{F}(\text{Dst } f)})) &= (*) \\ \text{im} \bigsqcup \left\{ \frac{(H(\beta) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\beta\}) \sqcap (\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \top^{\mathcal{F}(\text{Dst } f)})}{\beta \in \text{Dst } f} \right\} &= \\ \text{im} \bigsqcup \left\{ \frac{(H(\beta) \sqcap \uparrow^{\text{Src } f} \{\alpha\}) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\beta\}}{\beta \in \text{Dst } f} \right\} &= \\ \text{im} \bigsqcup \left\{ \frac{\left(\begin{array}{ll} \uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\beta\} & \text{if } H(\beta) \not\prec \uparrow^{\text{Src } f} \{\alpha\} \\ \perp^{\text{RLD}(\text{Src } f, \text{Dst } f)} & \text{if } H(\beta) \prec \uparrow^{\text{Src } f} \{\alpha\} \end{array} \right)}{\beta \in \text{Dst } f} \right\} &= \\ \text{im} \bigsqcup \left\{ \frac{\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\beta\}}{\beta \in \text{Dst } f, H(\beta) \not\prec \uparrow^{\text{Src } f} \{\alpha\}} \right\} &= \\ \text{im} \bigsqcup \left\{ \frac{\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} \{(\alpha, \beta)\}}{\beta \in \text{Dst } f, H(\beta) \not\prec \uparrow^{\text{Src } f} \{\alpha\}} \right\} &= \\ \bigsqcup \left\{ \frac{\uparrow^{\text{Dst } f} \{\beta\}}{\beta \in \text{Dst } f, H(\beta) \not\prec \uparrow^{\text{Src } f} \{\alpha\}} \right\} & \end{aligned}$$

* theorem 610 was used.

Thus $G(\alpha)$ is a principal filter that is $G(\alpha) = \uparrow^{\text{Dst } f} g(\alpha)$ for some $g : \text{Src } f \rightarrow \text{Dst } f$; $\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} G(\alpha) = \uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} (\{\alpha\} \times g(\alpha))$; f is principal as a join of principal reloids. \square

DEFINITION 1044. *Completion* and *co-completion* of a reloid $f \in \text{RLD}(A, B)$ are defined by the formulas:

$$\text{Compl } f = \text{Cor}^{\text{ComplRLD}(A, B)} f; \quad \text{CoCompl } f = \text{Cor}^{\text{CoComplRLD}(A, B)} f.$$

THEOREM 1045. Atoms of the lattice $\text{ComplRLD}(A, B)$ are exactly reloidal products of the form $\uparrow^A \{\alpha\} \times^{\text{RLD}} b$ where $\alpha \in A$ and b is an ultrafilter on B .

PROOF. First, it's easy to see that $\uparrow^A \{\alpha\} \times^{\text{RLD}} b$ are elements of $\text{ComplRLD}(A, B)$. Also $\perp^{\text{RLD}(A, B)}$ is an element of $\text{ComplRLD}(A, B)$.

$\uparrow^A \{\alpha\} \times^{\text{RLD}} b$ are atoms of $\text{ComplRLD}(A, B)$ because they are atoms of $\text{RLD}(A, B)$.

It remains to prove that if f is an atom of $\text{ComplRLD}(A, B)$ then $f = \uparrow^A \{\alpha\} \times^{\text{RLD}} b$ for some $\alpha \in A$ and an ultrafilter b on B .

Suppose f is a non-empty complete reloid. Then $\uparrow^A \{\alpha\} \times^{\text{RLD}} b \sqsubseteq f$ for some $\alpha \in A$ and an ultrafilter b on B . If f is an atom then $f = \uparrow^A \{\alpha\} \times^{\text{RLD}} b$. \square

OBVIOUS 1046. $\text{ComplRLD}(A, B)$ is an atomistic lattice.

PROPOSITION 1047. $\text{Compl } f = \bigsqcup \left\{ \frac{f|_{\uparrow\{\alpha\}}}{\alpha \in \text{Src } f} \right\}$ for every reloid f .

PROOF. Let's denote R the right part of the equality to be proven. That R is a complete reloid follows from the equality

$$f|_{\uparrow\{\alpha\}} = \uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} \text{im}(f|_{\uparrow\{\alpha\}}).$$

Obviously, $R \sqsubseteq f$.

The only thing left to prove is that $g \sqsubseteq R$ for every complete reloid g such that $g \sqsubseteq f$.

Really let g be a complete reloid such that $g \sqsubseteq f$. Then

$$g = \bigsqcup \left\{ \frac{\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} G(\alpha)}{\alpha \in \text{Src } f} \right\}$$

for some function $G : \text{Src } f \rightarrow \mathcal{F}(\text{Dst } f)$.

We have $\uparrow^{\text{Src } f} \{\alpha\} \times^{\text{RLD}} G(\alpha) = g|_{\uparrow^{\text{Src } f} \{\alpha\}} \sqsubseteq f|_{\uparrow\{\alpha\}}$. Thus $g \sqsubseteq R$. \square

CONJECTURE 1048. $\text{Compl } f \sqcap \text{Compl } g = \text{Compl}(f \sqcap g)$ for every $f, g \in \text{RLD}(A, B)$.

PROPOSITION 1049. Conjecture 1048 is equivalent to the statement that meet of every two complete reloids is a complete reloid.

PROOF. Let conjecture 1048 holds. Then for complete funcoids f and g we have $f \sqcap g = \text{Compl}(f \sqcap g)$ and thus $f \sqcap g$ is complete.

Let meet of every two complete reloid is complete. Then $\text{Compl } f \sqcap \text{Compl } g$ is complete and thus it is greatest complete reloid which is less $\text{Compl } f$ and less $\text{Compl } g$ what is the same as greatest complete reloid which is less than f and g that is $\text{Compl}(f \sqcap g)$. \square

THEOREM 1050. $\text{Compl} \bigsqcup R = \bigsqcup (\text{Compl})^* R$ for every set $R \in \mathcal{P}\text{RLD}(A, B)$ for every sets A, B .

PROOF.

$$\begin{aligned} \text{Compl} \bigsqcup R &= \\ \bigsqcup \left\{ \frac{(\bigsqcup R) \uparrow^A \{\alpha\}}{\alpha \in A} \right\} &= \text{(theorem 610)} \\ \bigsqcup \left\{ \frac{\bigsqcup \left\{ \frac{f \uparrow^A \{\alpha\}}{\alpha \in A} \right\}}{f \in R} \right\} &= \\ \bigsqcup (\text{Compl})^* R. & \end{aligned}$$

□

LEMMA 1051. Completion of a co-complete reloid is principal.

PROOF. Let f be a co-complete reloid. Then there is a function $F : \text{Dst } f \rightarrow \mathcal{F}(\text{Src } f)$ such that

$$f = \bigsqcup \left\{ \frac{F(\alpha) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\alpha\}}{\alpha \in \text{Dst } f} \right\}.$$

So

$$\begin{aligned} \text{Compl } f &= \\ \bigsqcup \left\{ \frac{(\bigsqcup \left\{ \frac{F(\alpha) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\alpha\}}{\alpha \in \text{Dst } f} \right\}) \uparrow^{\{\beta\}}}{\beta \in \text{Src } f} \right\} &= \\ \bigsqcup \left\{ \frac{(\bigsqcup \left\{ \frac{F(\alpha) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\alpha\}}{\alpha \in \text{Dst } f} \right\}) \sqcap (\uparrow^{\text{Src } f} \{\beta\} \times^{\text{RLD}} \top^{\mathcal{F}}(\text{Dst } f))}{\beta \in \text{Src } f} \right\} &= (*) \\ \bigsqcup \left\{ \frac{\bigsqcup \left\{ \frac{(F(\alpha) \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\alpha\}) \sqcap (\uparrow^{\text{Src } f} \{\beta\} \times^{\text{RLD}} \top^{\mathcal{F}}(\text{Dst } f))}{\alpha \in \text{Dst } f} \right\}}{\beta \in \text{Src } f} \right\} &= \\ \bigsqcup \left\{ \frac{\bigsqcup \left\{ \frac{\uparrow^{\text{Src } f} \{\beta\} \times^{\text{RLD}} \uparrow^{\text{Dst } f} \{\alpha\}}{\alpha \in \text{Dst } f} \right\}}{\beta \in \text{Src } f, \uparrow^{\text{Src } f} \{\beta\} \sqsubseteq F(\alpha)} \right\} & \end{aligned}$$

* theorem 610.

Thus $\text{Compl } f$ is principal. □

THEOREM 1052. $\text{Compl CoCompl } f = \text{CoCompl Compl } f = \text{Cor } f$ for every reloid f .

PROOF. We will prove only $\text{Compl CoCompl } f = \text{Cor } f$. The rest follows from symmetry.

From the lemma $\text{Compl CoCompl } f$ is principal. It is obvious $\text{Compl CoCompl } f \sqsubseteq f$. So to finish the proof we need to show only that for every principal reloid $F \sqsubseteq f$ we have $F \sqsubseteq \text{Compl CoCompl } f$.

Really, obviously $F \sqsubseteq \text{CoCompl } f$ and thus $F = \text{Compl } F \sqsubseteq \text{Compl CoCompl } f$. □

CONJECTURE 1053. If f is a complete reloid, then it is metacomplete.

CONJECTURE 1054. If f is a metacomplete reloid, then it is complete.

CONJECTURE 1055. $\text{Compl } f = f \setminus * (\Omega^{\text{Src } f} \times^{\text{RLD}} \top^{\mathcal{F}}(\text{Dst } f))$ for every reloid f .

By analogy with similar properties of funcoids described above:

PROPOSITION 1056. For composable relocks f and g it holds

- 1°. $\text{Compl}(g \circ f) \supseteq (\text{Compl } g) \circ (\text{Compl } f)$
- 2°. $\text{CoCompl}(g \circ f) \supseteq (\text{CoCompl } g) \circ (\text{CoCompl } f)$.

PROOF.

- 1°. $(\text{Compl } g) \circ (\text{Compl } f) \sqsubseteq \text{Compl}((\text{Compl } g) \circ (\text{Compl } f)) \sqsubseteq \text{Compl}(g \circ f)$.
- 2°. By duality.

□

CONJECTURE 1057. For composable relocks f and g it holds

- 1°. $\text{Compl}(g \circ f) = (\text{Compl } g) \circ f$ if f is a co-complete relock;
- 2°. $\text{CoCompl}(f \circ g) = f \circ \text{CoCompl } g$ if f is a complete relock;
- 3°. $\text{CoCompl}((\text{Compl } g) \circ f) = \text{Compl}(g \circ (\text{CoCompl } f)) = (\text{Compl } g) \circ (\text{CoCompl } f)$;
- 4°. $\text{Compl}(g \circ (\text{Compl } f)) = \text{Compl}(g \circ f)$;
- 5°. $\text{CoCompl}((\text{CoCompl } g) \circ f) = \text{CoCompl}(g \circ f)$.

8.8. What uniform spaces are

PROPOSITION 1058. Uniform spaces are exactly reflexive, symmetric, transitive endorelocks.

PROOF. Easy to prove using theorem 1001.

□

Relationships between funcoids and reloids

9.1. Funcoid induced by a reloid

Every reloid f induces a funcoid $(\text{FCD})f \in \text{FCD}(\text{Src } f, \text{Dst } f)$ by the following formulas (for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$):

$$\mathcal{X} [(\text{FCD})f] \mathcal{Y} \Leftrightarrow \forall F \in \text{up } f : \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y};$$

$$\langle (\text{FCD})f \rangle \mathcal{X} = \prod_{F \in \text{up } f} \langle \uparrow^{\text{FCD}} F \rangle \mathcal{X}.$$

We should prove that $(\text{FCD})f$ is really a funcoid.

PROOF. We need to prove that

$$\mathcal{X} [(\text{FCD})f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \langle (\text{FCD})f \rangle \mathcal{X} \neq \perp \Leftrightarrow \mathcal{X} \sqcap \langle (\text{FCD})f^{-1} \rangle \mathcal{Y} \neq \perp.$$

The above formula is equivalent to:

$$\begin{aligned} \forall F \in \text{up } f : \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y} &\Leftrightarrow \\ \mathcal{Y} \sqcap \prod_{F \in \text{up } f} \langle \uparrow^{\text{FCD}} F \rangle \mathcal{X} \neq \perp &\Leftrightarrow \\ \mathcal{X} \sqcap \prod_{F \in \text{up } f} \langle \uparrow^{\text{FCD}} F^{-1} \rangle \mathcal{Y} \neq \perp. & \end{aligned}$$

We have $\mathcal{Y} \sqcap \prod_{F \in \text{up } f} \langle \uparrow^{\text{FCD}} F \rangle \mathcal{X} = \prod_{F \in \text{up } f} (\mathcal{Y} \sqcap \langle \uparrow^{\text{FCD}} F \rangle \mathcal{X})$.

Let's denote $W = \left\{ \frac{\mathcal{Y} \sqcap \langle \uparrow^{\text{FCD}} F \rangle \mathcal{X}}{F \in \text{up } f} \right\}$.

$$\forall F \in \text{up } f : \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y} \Leftrightarrow \forall F \in \text{up } f : \mathcal{Y} \sqcap \langle \uparrow^{\text{FCD}} F \rangle \mathcal{X} \neq \perp \Leftrightarrow \perp \notin W.$$

We need to prove only that $\perp \notin W \Leftrightarrow \prod W \neq \perp$. (The rest follows from symmetry.) To prove it is enough to show that W is a generalized filter base.

Let's prove that W is a generalized filter base. For this it's enough to prove that $V = \left\{ \frac{\langle \uparrow^{\text{FCD}} F \rangle \mathcal{X}}{F \in \text{up } f} \right\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \langle \uparrow^{\text{FCD}} P \rangle \mathcal{X}$, $\mathcal{B} = \langle \uparrow^{\text{FCD}} Q \rangle \mathcal{X}$ where $P, Q \in \text{up } f$. Then for $\mathcal{C} = \langle \uparrow^{\text{FCD}} (P \sqcap Q) \rangle \mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \sqsubseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base. \square

PROPOSITION 1059. $(\text{FCD}) \uparrow^{\text{RLD}} f = \uparrow^{\text{FCD}} f$ for every **Rel**-morphism f .

PROOF. $\mathcal{X} [(\text{FCD}) \uparrow^{\text{RLD}} f] \mathcal{Y} \Leftrightarrow \forall F \in \text{up } \uparrow^{\text{RLD}} f : \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y} \Leftrightarrow \mathcal{X} [\uparrow^{\text{FCD}} f] \mathcal{Y}$ (for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$). \square

THEOREM 1060. $\mathcal{X} [(\text{FCD})f] \mathcal{Y} \Leftrightarrow \mathcal{X} \times^{\text{RLD}} \mathcal{Y} \neq f$ for every reloid f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$.

PROOF.

$$\begin{aligned}
& \mathcal{X} \times^{\text{RLD}} \mathcal{Y} \not\neq f \Leftrightarrow \\
& \forall F \in \text{up } f, P \in \text{up}(\mathcal{X} \times^{\text{RLD}} \mathcal{Y}) : P \not\neq F \Leftrightarrow \\
& \forall F \in \text{up } f, X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \times Y \not\neq F \Leftrightarrow \\
& \forall F \in \text{up } f, X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \uparrow^{\text{FCD}} F \downarrow Y \Leftrightarrow \\
& \forall F \in \text{up } f : \mathcal{X} \uparrow^{\text{FCD}} F \downarrow \mathcal{Y} \Leftrightarrow \\
& \mathcal{X} \uparrow^{\text{FCD}} f \downarrow \mathcal{Y}.
\end{aligned}$$

□

THEOREM 1061. $(\text{FCD})f = \prod^{\text{FCD}} \text{up } f$ for every reloid f .

PROOF. Let a be an ultrafilter on $\text{Src } f$.

$$\langle (\text{FCD})f \rangle a = \prod \left\{ \frac{\langle \uparrow^{\text{FCD}} F \rangle a}{F \in \text{up } f} \right\} \text{ by the definition of } (\text{FCD}).$$

$$\langle \prod^{\text{FCD}} \text{up } f \rangle a = \prod \left\{ \frac{\langle \uparrow^{\text{FCD}} F \rangle a}{F \in \text{up } f} \right\} \text{ by theorem 878.}$$

$$\text{So } \langle (\text{FCD})f \rangle a = \langle \prod^{\text{FCD}} \text{up } f \rangle a \text{ for every ultrafilter } a. \quad \square$$

LEMMA 1062. For every two filter bases S and T of morphisms $\mathbf{Rel}(U, V)$ and every typed set $A \in \mathcal{T}U$

$$\prod^{\text{RLD}} S = \prod^{\text{RLD}} T \Rightarrow \prod_{F \in S}^{\mathcal{F}} \langle F \rangle^* A = \prod_{G \in T}^{\mathcal{F}} \langle G \rangle^* A.$$

PROOF. Let $\prod^{\text{RLD}} S = \prod^{\text{RLD}} T$.

First let prove that $\left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ is a filter base. Let $X, Y \in \left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$. Then $X = \langle F_X \rangle^* A$ and $Y = \langle F_Y \rangle^* A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \sqsubseteq F_X \sqcap F_Y$. So $\langle F_Z \rangle^* A \sqsubseteq X \sqcap Y$ and $\langle F_Z \rangle^* A \in \left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$. So $\left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ is a filter base.

Suppose $X \in \prod_{F \in S}^{\mathcal{F}} \langle F \rangle^* A$. Then there exists $X' \in \left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ where $X \sqsupseteq X'$ because $\left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ is a filter base. That is $X' = \langle F \rangle^* A$ for some $F \in S$. There exists $G \in T$ such that $G \sqsubseteq F$ because T is a filter base. Let $Y' = \langle G \rangle^* A$. We have $Y' \sqsubseteq X' \sqsubseteq X$; $Y' \in \left\{ \frac{\langle G \rangle^* A}{G \in T} \right\}$; $Y' \in \text{up} \prod_{G \in T}^{\mathcal{F}} \langle G \rangle^* A$; $X \in \text{up} \prod_{G \in T}^{\mathcal{F}} \langle G \rangle^* A$. The reverse is symmetric. □

LEMMA 1063. $\left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$ is a filter base for every reloids f and g .

PROOF. Let denote $D = \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \text{up } f$, $G_A, G_B \in \text{up } g$. So $A \sqcap B \sqsupseteq (G_A \sqcap G_B) \circ (F_A \sqcap F_B) \in D$ because $F_A \sqcap F_B \in \text{up } f$ and $G_A \sqcap G_B \in \text{up } g$. □

THEOREM 1064. $(\text{FCD})(g \circ f) = ((\text{FCD})g) \circ ((\text{FCD})f)$ for every composable reloids f and g .

PROOF.

$$\langle (\text{FCD})(g \circ f) \rangle^* X = \prod_{H \in \text{up}(g \circ f)}^{\mathcal{F}} \langle H \rangle^* X = \prod_{H \in \text{up} \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}}^{\mathcal{F}} \langle H \rangle^* X.$$

Obviously

$$\prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\} = \prod^{\text{RLD}} \text{up} \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\};$$

from this by lemma 1062 (taking into account that

$$\left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$$

and

$$\text{up} \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$$

are filter bases)

$$H \in \text{up} \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\} \langle H \rangle^* X = \prod^{\mathcal{F}} \left\{ \frac{\langle G \circ F \rangle^* X}{F \in \text{up } f, G \in \text{up } g} \right\}.$$

On the other side

$$\begin{aligned} \langle ((\text{FCD})g) \circ ((\text{FCD})f) \rangle^* X &= \langle (\text{FCD})g \rangle \langle (\text{FCD})f \rangle^* X = \\ &= \langle (\text{FCD})g \rangle \prod^{\mathcal{F}} \langle F \rangle^* X = \prod_{G \in \text{up } g} \langle \uparrow^{\text{FCD}} G \rangle \prod^{\text{RLD}}_{F \in \text{up } f} \langle F \rangle^* X. \end{aligned}$$

Let's prove that $\left\{ \frac{\langle F \rangle^* X}{F \in \text{up } f} \right\}$ is a filter base. If $A, B \in \left\{ \frac{\langle F \rangle^* X}{F \in \text{up } f} \right\}$ then $A = \langle F_1 \rangle^* X$, $B = \langle F_2 \rangle^* X$ where $F_1, F_2 \in \text{up } f$. $A \cap B \supseteq \langle F_1 \cap F_2 \rangle^* X \in \left\{ \frac{\langle F \rangle^* X}{F \in \text{up } f} \right\}$. So $\left\{ \frac{\langle F \rangle^* X}{F \in \text{up } f} \right\}$ is really a filter base.

By theorem 839 we have

$$\langle \uparrow^{\text{FCD}} G \rangle \prod^{\mathcal{F}}_{F \in \text{up } f} \langle F \rangle^* X = \prod^{\mathcal{F}}_{F \in \text{up } f} \langle G \rangle^* \langle F \rangle^* X.$$

So continuing the above equalities,

$$\begin{aligned} \langle ((\text{FCD})g) \circ ((\text{FCD})f) \rangle^* X &= \\ &= \prod_{G \in \text{up } g} \prod^{\mathcal{F}}_{F \in \text{up } f} \langle G \rangle^* \langle F \rangle^* X = \\ &= \prod^{\mathcal{F}} \left\{ \frac{\langle G \rangle^* \langle F \rangle^* X}{F \in \text{up } f, G \in \text{up } g} \right\} = \\ &= \prod^{\mathcal{F}} \left\{ \frac{\langle G \circ F \rangle^* X}{F \in \text{up } f, G \in \text{up } g} \right\}. \end{aligned}$$

Combining these equalities we get $\langle (\text{FCD})(g \circ f) \rangle^* X = \langle ((\text{FCD})g) \circ ((\text{FCD})f) \rangle^* X$ for every typed set $X \in \mathcal{F}(\text{Src } f)$. \square

PROPOSITION 1065. $(\text{FCD}) \text{id}_A^{\text{RLD}} = \text{id}_A^{\text{FCD}}$ for every filter \mathcal{A} .

PROOF. Recall that $\text{id}_A^{\text{RLD}} = \prod \left\{ \uparrow_{A \in \text{up } \mathcal{A}}^{\text{Base}(\mathcal{A})} \text{id}_A \right\}$. For every $\mathcal{X}, \mathcal{Y} \in \mathcal{F}(\text{Base}(\mathcal{A}))$ we have

$$\begin{aligned} \mathcal{X} \left[(\text{FCD}) \text{id}_A^{\text{RLD}} \right] \mathcal{Y} &\Leftrightarrow \\ \mathcal{X} \times^{\text{RLD}} \mathcal{Y} \not\neq \text{id}_A^{\text{RLD}} &\Leftrightarrow \\ \forall A \in \text{up } \mathcal{A} : \mathcal{X} \times^{\text{RLD}} \mathcal{Y} \not\neq \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_A &\Leftrightarrow \\ \forall A \in \text{up } \mathcal{A} : \mathcal{X} \left[\uparrow^{\text{FCD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{A}))} \text{id}_A \right] \mathcal{Y} &\Leftrightarrow \\ \forall A \in \text{up } \mathcal{A} : \mathcal{X} \sqcap \mathcal{Y} \not\neq A &\Leftrightarrow \\ \mathcal{X} \sqcap \mathcal{Y} \not\neq \mathcal{A} &\Leftrightarrow \\ \mathcal{X} \left[\text{id}_A^{\text{FCD}} \right] \mathcal{Y} & \end{aligned}$$

(used properties of generalized filter bases). □

COROLLARY 1066. $(\text{FCD})1_A^{\text{RLD}} = 1_A^{\text{FCD}}$ for every set A .

PROPOSITION 1067. (FCD) is a functor from RLD to FCD.

PROOF. Preservation of composition and of identity is proved above. □

PROPOSITION 1068.

1°. $(\text{FCD})f$ is a monovalued funcoid if f is a monovalued reloid.

2°. $(\text{FCD})f$ is an injective funcoid if f is an injective reloid.

PROOF. We will prove only the first as the second is dual. Let f be a monovalued reloid. Then $f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{RLD}}$; $(\text{FCD})(f \circ f^{-1}) \sqsubseteq 1_{\text{Dst } f}^{\text{FCD}}$; $(\text{FCD})f \circ ((\text{FCD})f)^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{FCD}}$ that is $(\text{FCD})f$ is a monovalued funcoid. □

PROPOSITION 1069. $(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ for every filters \mathcal{A}, \mathcal{B} .

PROOF. $\mathcal{X} \left[(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \right] \mathcal{Y} \Leftrightarrow \forall F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) : \mathcal{X} \left[\uparrow^{\text{FCD}} F \right] \mathcal{Y}$ (for every $\mathcal{X} \in \mathcal{F}(\text{Base}(\mathcal{A}))$, $\mathcal{Y} \in \mathcal{F}(\text{Base}(\mathcal{B}))$).

Evidently

$$\forall F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) : \mathcal{X} \left[\uparrow^{\text{FCD}} F \right] \mathcal{Y} \Rightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : \mathcal{X} [A \times B] \mathcal{Y}.$$

Let $\forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : \mathcal{X} [A \times B] \mathcal{Y}$. Then if $F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$, there are $A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}$ such that $F \sqsupseteq A \times B$. So $\mathcal{X} \left[\uparrow^{\text{FCD}} F \right] \mathcal{Y}$. We have proved

$$\forall F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) : \mathcal{X} \left[\uparrow^{\text{FCD}} F \right] \mathcal{Y} \Leftrightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : \mathcal{X} [A \times B] \mathcal{Y}.$$

Further

$$\begin{aligned} \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : \mathcal{X} [A \times B] \mathcal{Y} &\Leftrightarrow \\ \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : (\mathcal{X} \not\neq A \wedge \mathcal{Y} \not\neq B) &\Leftrightarrow \\ \mathcal{X} \not\neq A \wedge \mathcal{Y} \not\neq B &\Leftrightarrow \mathcal{X} [A \times^{\text{FCD}} B] \mathcal{Y}. \end{aligned}$$

Thus $\mathcal{X} \left[(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \right] \mathcal{Y} \Leftrightarrow \mathcal{X} [A \times^{\text{FCD}} B] \mathcal{Y}$. □

PROPOSITION 1070. $\text{dom}(\text{FCD})f = \text{dom } f$ and $\text{im}(\text{FCD})f = \text{im } f$ for every reloid f .

PROOF.

$$\begin{aligned} \text{im}(\text{FCD})f &= \langle (\text{FCD})f \rangle_{\top} = \bigsqcap_{F \in \text{up } f} \langle F \rangle^*_{\top} = \\ &= \bigsqcap_{F \in \text{up } f} \text{im } F = \bigsqcap_{F \in \text{up } f} \langle \text{im} \rangle^* \text{up } f = \text{im } f. \end{aligned}$$

$\text{dom}(\text{FCD})f = \text{dom } f$ is similar. \square

PROPOSITION 1071. $(\text{FCD})(f \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{B})) = (\text{FCD})f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B})$ for every reloid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$ and $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

PROOF.

$$\begin{aligned} (\text{FCD})(f \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{B})) &= \\ (\text{FCD})(\text{id}_{\mathcal{B}}^{\text{RLD}} \circ f \circ \text{id}_{\mathcal{A}}^{\text{RLD}}) &= \\ (\text{FCD})\text{id}_{\mathcal{B}}^{\text{RLD}} \circ (\text{FCD})f \circ (\text{FCD})\text{id}_{\mathcal{A}}^{\text{RLD}} &= \\ \text{id}_{\mathcal{B}}^{\text{FCD}} \circ (\text{FCD})f \circ \text{id}_{\mathcal{A}}^{\text{FCD}} &= \\ (\text{FCD})f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}). & \end{aligned}$$

\square

COROLLARY 1072. $(\text{FCD})(f|_{\mathcal{A}}) = ((\text{FCD})f)|_{\mathcal{A}}$ for every reloid f and a filter $\mathcal{A} \in \mathcal{F}(\text{Src } f)$.

PROPOSITION 1073. $\langle (\text{FCD})f \rangle_{\mathcal{X}} = \text{im}(f|_{\mathcal{X}})$ for every reloid f and a filter $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

PROOF. $\text{im}(f|_{\mathcal{X}}) = \text{im}(\text{FCD})(f|_{\mathcal{X}}) = \text{im}(((\text{FCD})f)|_{\mathcal{X}}) = \langle (\text{FCD})f \rangle_{\mathcal{X}}$. \square

PROPOSITION 1074. $(\text{FCD})f = \bigsqcup \left\{ \frac{x \times^{\text{FCD}} y}{x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, x \times^{\text{RLD}} y \not\sqsubseteq f} \right\}$ for every reloid f .

PROOF. $(\text{FCD})f = \bigsqcup \left\{ \frac{x \times^{\text{FCD}} y}{x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, x \times^{\text{FCD}} y \not\sqsubseteq (\text{FCD})f} \right\}$, but
 $x \times^{\text{FCD}} y \not\sqsubseteq (\text{FCD})f \Leftrightarrow x [(\text{FCD})f] y \Leftrightarrow x \times^{\text{RLD}} y \not\sqsubseteq f$,

thus follows the theorem. \square

9.2. Reloids induced by a functor

Every functor $f \in \text{FCD}(A, B)$ induces a reloid from A to B in two ways, intersection of *outward* relations and union of *inward* reloidal products of filters:

$$\begin{aligned} (\text{RLD})_{\text{out}} f &= \bigsqcap^{\text{RLD}} \text{up } f; \\ (\text{RLD})_{\text{in}} f &= \bigsqcup \left\{ \frac{\mathcal{A} \times^{\text{RLD}} \mathcal{B}}{\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f} \right\}. \end{aligned}$$

THEOREM 1075. $(\text{RLD})_{\text{in}} f = \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\}$.

PROOF. It follows from theorem 1009. \square

PROPOSITION 1076. $\text{up } \uparrow^{\text{RLD}} f = \text{up } \uparrow^{\text{FCD}} f$ for every **Rel**-morphism f .

PROOF. $X \in \text{up } \uparrow^{\text{RLD}} f \Leftrightarrow X \sqsupseteq f \Leftrightarrow X \in \text{up } \uparrow^{\text{FCD}} f$. \square

PROPOSITION 1077. $(\text{RLD})_{\text{out}} \uparrow^{\text{FCD}} f = \uparrow^{\text{RLD}} f$ for every **Rel**-morphism f .

PROOF. $(\text{RLD})_{\text{out}} \uparrow^{\text{FCD}} f = \prod^{\text{RLD}} \text{up } f = \uparrow^{\text{RLD}} \text{min up } f = \uparrow^{\text{RLD}} f$ taking into account the previous proposition. \square

Surprisingly, a functor is greater inward than outward:

THEOREM 1078. $(\text{RLD})_{\text{out}} f \sqsubseteq (\text{RLD})_{\text{in}} f$ for every functor f .

PROOF. We need to prove

$$(\text{RLD})_{\text{out}} f \sqsubseteq \bigsqcup \left\{ \frac{\mathcal{A} \times^{\text{RLD}} \mathcal{B}}{\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f} \right\}.$$

Let

$$K \in \text{up} \bigsqcup \left\{ \frac{\mathcal{A} \times^{\text{RLD}} \mathcal{B}}{\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f} \right\}.$$

Then

$$\begin{aligned} K &\in \text{up} \uparrow^{\text{RLD}} \bigsqcup \left\{ \frac{X_{\mathcal{A}} \times Y_{\mathcal{B}}}{\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f} \right\} \\ &= (\text{RLD})_{\text{out}} \uparrow^{\text{FCD}} \bigsqcup \left\{ \frac{X_{\mathcal{A}} \times Y_{\mathcal{B}}}{\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f} \right\} \\ &= (\text{RLD})_{\text{out}} \bigsqcup^{\text{FCD}} \left\{ \frac{\uparrow^{\text{FCD}} (X_{\mathcal{A}} \times Y_{\mathcal{B}})}{\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f} \right\} \\ &\sqsubseteq (\text{RLD})_{\text{out}} \bigsqcup \text{atoms } f \\ &= (\text{RLD})_{\text{out}} f \end{aligned}$$

where $X_{\mathcal{A}} \in \text{up } \mathcal{A}$, $Y_{\mathcal{B}} \in \text{up } \mathcal{B}$. $K \in \text{up}(\text{RLD})_{\text{out}} f$. \square

PROPOSITION 1079. $(\text{RLD})_{\text{out}} f \sqcup (\text{RLD})_{\text{out}} g = (\text{RLD})_{\text{out}} (f \sqcup g)$ for functors f, g .

PROOF.

$$\begin{aligned} (\text{RLD})_{\text{out}} f \sqcup (\text{RLD})_{\text{out}} g &= \prod_{F \in \text{up } f}^{\text{RLD}} F \sqcup \prod_{G \in \text{up } g}^{\text{RLD}} G = \\ &= \prod_{F \in \text{up } f, G \in \text{up } g}^{\text{RLD}} (F \sqcup G) = \prod_{H \in \text{up}(f \sqcup g)}^{\text{RLD}} H = (\text{RLD})_{\text{out}} (f \sqcup g). \end{aligned}$$

\square

THEOREM 1080. $(\text{FCD})(\text{RLD})_{\text{in}} f = f$ for every functor f .

PROOF. For every typed sets $X \in \mathcal{T}(\text{Src } f)$, $Y \in \mathcal{T}(\text{Dst } f)$

$$\begin{aligned} X [(\text{FCD})(\text{RLD})_{\text{in}} f]^* Y &\Leftrightarrow \\ X \times^{\text{RLD}} Y \not\leq (\text{RLD})_{\text{in}} f &\Leftrightarrow \\ \uparrow^{\text{RLD}} (X \times Y) \not\leq \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\} &\Leftrightarrow (*) \\ \exists a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)} : (a \times^{\text{FCD}} b \sqsubseteq f \wedge a \sqsubseteq X \wedge b \sqsubseteq Y) &\Leftrightarrow \\ X [f]^* Y. & \end{aligned}$$

* theorem 583.

Thus $(\text{FCD})(\text{RLD})_{\text{in}} f = f$. \square

REMARK 1081. The above theorem allows to represent functors as relocks ($(\text{RLD})_{\text{in}} f$ is the relock representing functor f). Refer to the section “**Functorial relocks**” below for more details.

OBVIOUS 1082. $(\text{RLD})_{\text{in}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for every filters \mathcal{A}, \mathcal{B} .

CONJECTURE 1083. $(\text{RLD})_{\text{out}} \text{id}_{\mathcal{A}}^{\text{FCD}} = \text{id}_{\mathcal{A}}^{\text{RLD}}$ for every filter \mathcal{A} .

EXERCISE 1084. Prove that generally $(\text{RLD})_{\text{in}} \text{id}_{\mathcal{A}}^{\text{FCD}} \neq \text{id}_{\mathcal{A}}^{\text{RLD}}$. I call $(\text{RLD})_{\text{in}} \text{id}_{\mathcal{A}}^{\text{FCD}}$ *thick identity* or *thick diagonal*, because it is greater (“thicker”) than identity $\text{id}_{\mathcal{A}}^{\text{RLD}}$.

PROPOSITION 1085. $\text{dom}(\text{RLD})_{\text{in}} f = \text{dom } f$ and $\text{im}(\text{RLD})_{\text{in}} f = \text{im } f$ for every funcoid f .

PROOF. We will prove only $\text{dom}(\text{RLD})_{\text{in}} f = \text{dom } f$ as the other formula follows from symmetry. Really:

$$\text{dom}(\text{RLD})_{\text{in}} f = \text{dom} \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\}.$$

By corollary 1028 we have

$$\begin{aligned} \text{dom}(\text{RLD})_{\text{in}} f &= \\ \bigsqcup \left\{ \frac{\text{dom}(a \times^{\text{RLD}} b)}{a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\} &= \\ \bigsqcup \left\{ \frac{\text{dom}(a \times^{\text{FCD}} b)}{a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\}. \end{aligned}$$

By corollary 896 we have

$$\begin{aligned} \text{dom}(\text{RLD})_{\text{in}} f &= \\ \text{dom} \bigsqcup \left\{ \frac{a \times^{\text{FCD}} b}{a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\} &= \\ \text{dom } f. \end{aligned}$$

□

PROPOSITION 1086. $\text{dom}(f|_{\mathcal{A}}) = \mathcal{A} \sqcap \text{dom } f$ for every reloid f and filter $\mathcal{A} \in \mathcal{F}(\text{Src } f)$.

PROOF. $\text{dom}(f|_{\mathcal{A}}) = \text{dom}(\text{FCD})(f|_{\mathcal{A}}) = \text{dom}((\text{FCD})f)|_{\mathcal{A}} = \mathcal{A} \sqcap \text{dom}(\text{FCD})f = \mathcal{A} \sqcap \text{dom } f$. □

THEOREM 1087. For every composable reloids f, g :

- 1°. If $\text{im } f \sqsupseteq \text{dom } g$ then $\text{im}(g \circ f) = \text{im } g$;
- 2°. If $\text{im } f \sqsubseteq \text{dom } g$ then $\text{dom}(g \circ f) = \text{dom } f$.

PROOF.

1°. $\text{im}(g \circ f) = \text{im}(\text{FCD})(g \circ f) = \text{im}((\text{FCD})g \circ (\text{FCD})f) = \text{im}(\text{FCD})g = \text{im } g$.

2°. Similar. □

LEMMA 1088. If a, b, c are filters on powersets and $b \neq \perp$, then

$$\bigsqcup^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{atoms}(a \times^{\text{RLD}} b), G \in \text{atoms}(b \times^{\text{RLD}} c)} \right\} = a \times^{\text{RLD}} c.$$

PROOF.

$$a \times^{\text{RLD}} c = (b \times^{\text{RLD}} c) \circ (a \times^{\text{RLD}} b) = (\text{corollary 1006}) =$$

$$\bigsqcup^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{atoms}(a \times^{\text{RLD}} b), G \in \text{atoms}(b \times^{\text{RLD}} c)} \right\}.$$

□

THEOREM 1089. $a \times^{\text{RLD}} b \sqsubseteq (\text{RLD})_{\text{in}} f \Leftrightarrow a \times^{\text{FCD}} b \sqsubseteq f$ for every funcoid f and $a \in \text{atoms}^{\mathcal{F}(\text{Src } f)}$, $b \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}$.

PROOF. $a \times^{\text{FCD}} b \sqsubseteq f \Rightarrow a \times^{\text{RLD}} b \sqsubseteq (\text{RLD})_{\text{in}} f$ is obvious.

$$\begin{aligned} a \times^{\text{RLD}} b \sqsubseteq (\text{RLD})_{\text{in}} f \Rightarrow a \times^{\text{RLD}} b \not\sqsubseteq (\text{RLD})_{\text{in}} f \Rightarrow \\ a [(\text{FCD})(\text{RLD})_{\text{in}} f] b \Rightarrow a [f] b \Rightarrow a \times^{\text{FCD}} b \sqsubseteq f. \end{aligned}$$

□

CONJECTURE 1090. If $\mathcal{A} \times^{\text{RLD}} \mathcal{B} \sqsubseteq (\text{RLD})_{\text{in}} f$ then $\mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f$ for every funcoid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$, $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

THEOREM 1091. $\text{up}(\text{FCD})g \supseteq \text{up } g$ for every reloid g .

PROOF. Let $K \in \text{up } g$. Then for every typed sets $X \in \mathcal{T} \text{Src } g$, $Y \in \mathcal{T} \text{Dst } g$

$$X [K]^* Y \Leftrightarrow X [\uparrow^{\text{FCD}} K]^* Y \Leftrightarrow X [(\text{FCD}) \uparrow^{\text{RLD}} K]^* Y \Leftrightarrow X [(\text{FCD})g]^* Y.$$

Thus $\uparrow^{\text{FCD}} K \supseteq (\text{FCD})g$ that is $K \in \text{up}(\text{FCD})g$. □

THEOREM 1092. $g \circ (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \circ f = \langle (\text{FCD})f^{-1} \rangle \mathcal{A} \times^{\text{RLD}} \langle (\text{FCD})g \rangle \mathcal{B}$ for every reloids f, g and filters $\mathcal{A} \in \mathcal{F}(\text{Dst } f)$, $\mathcal{B} \in \mathcal{F}(\text{Src } g)$.

PROOF.

$$\begin{aligned} g \circ (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \circ f &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{G \circ (A \times B) \circ F}{F \in \text{up } f, G \in \text{up } g, A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\} &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{\langle F^{-1} \rangle^* A \times \langle G \rangle^* B}{F \in \text{up } f, G \in \text{up } g, A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\} &= \\ \bigsqcap^{\text{RLD}} \left\{ \frac{\langle F^{-1} \rangle^* A \times^{\text{RLD}} \langle G \rangle^* B}{F \in \text{up } f, G \in \text{up } g, A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\} &= \\ &\quad (\text{theorem 1011}) \\ \bigsqcap^{\mathcal{F}} \left\{ \frac{\langle F^{-1} \rangle^* A}{F \in \text{up } f, A \in \text{up } \mathcal{A}} \right\} \times^{\text{RLD}} \bigsqcap^{\mathcal{F}} \left\{ \frac{\langle G \rangle^* B}{G \in \text{up } g, B \in \text{up } \mathcal{B}} \right\} &= \\ \bigsqcap^{\mathcal{F}} \left\{ \frac{\langle \uparrow^{\text{FCD}} F^{-1} \rangle^* A}{F \in \text{up } f, A \in \text{up } \mathcal{A}} \right\} \times^{\text{RLD}} \bigsqcap^{\mathcal{F}} \left\{ \frac{\langle \uparrow^{\text{FCD}} G \rangle^* B}{G \in \text{up } g, B \in \text{up } \mathcal{B}} \right\} &= \\ \bigsqcap^{\mathcal{F}} \left\{ \frac{\langle \uparrow^{\text{FCD}} F^{-1} \rangle \mathcal{A}}{F \in \text{up } f} \right\} \times^{\text{RLD}} \bigsqcap^{\mathcal{F}} \left\{ \frac{\langle \uparrow^{\text{FCD}} G \rangle \mathcal{B}}{G \in \text{up } g} \right\} &= \\ &\quad (\text{by definition of (FCD)}) \\ \langle (\text{FCD})f^{-1} \rangle \mathcal{A} \times^{\text{RLD}} \langle (\text{FCD})g \rangle \mathcal{B}. & \end{aligned}$$

□

COROLLARY 1093.

- 1°. $(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \circ f = \langle (\text{FCD})f^{-1} \rangle \mathcal{A} \times^{\text{RLD}} \mathcal{B}$;
- 2°. $g \circ (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \mathcal{A} \times^{\text{RLD}} \langle (\text{FCD})g \rangle \mathcal{B}$.

9.3. Galois connections between funcoids and reloids

THEOREM 1094. $(\text{FCD}) : \text{RLD}(A, B) \rightarrow \text{FCD}(A, B)$ is the lower adjoint of $(\text{RLD})_{\text{in}} : \text{FCD}(A, B) \rightarrow \text{RLD}(A, B)$ for every sets A, B .

PROOF. Because (FCD) and $(\text{RLD})_{\text{in}}$ are trivially monotone, it's enough to prove (for every $f \in \text{RLD}(A, B)$, $g \in \text{FCD}(A, B)$)

$$f \sqsubseteq (\text{RLD})_{\text{in}}(\text{FCD})f \quad \text{and} \quad (\text{FCD})(\text{RLD})_{\text{in}}g \sqsubseteq g.$$

The second formula follows from the fact that $(\text{FCD})(\text{RLD})_{\text{in}}g = g$.

$$\begin{aligned} & (\text{RLD})_{\text{in}}(\text{FCD})f = \\ & \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq (\text{FCD})f} \right\} = \\ & \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a [(\text{FCD})f] b} \right\} = \\ & \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{RLD}} b \not\sqsubseteq f} \right\} \sqsupseteq \\ & \bigsqcup \left\{ \frac{p \in \text{atoms}(a \times^{\text{RLD}} b)}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, p \not\sqsubseteq f} \right\} = \\ & \bigsqcup \left\{ \frac{p \in \text{atoms}^{\text{RLD}(A, B)}}{p \not\sqsubseteq f} \right\} = \\ & \bigsqcup \left\{ \frac{p}{p \in \text{atoms } f} \right\} = f. \end{aligned}$$

□

COROLLARY 1095.

1°. $(\text{FCD}) \bigsqcup S = \bigsqcup \langle (\text{FCD}) \rangle^* S$ if $S \in \mathcal{P}\text{RLD}(A, B)$.

2°. $(\text{RLD})_{\text{in}} \bigsqcap S = \bigsqcap \langle (\text{RLD})_{\text{in}} \rangle^* S$ if $S \in \mathcal{P}\text{FCD}(A, B)$.

THEOREM 1096. $(\text{RLD})_{\text{in}}(f \sqcup g) = (\text{RLD})_{\text{in}}f \sqcup (\text{RLD})_{\text{in}}g$ for every funcoids $f, g \in \text{FCD}(A, B)$.

PROOF.

$$\begin{aligned} (\text{RLD})_{\text{in}}(f \sqcup g) &= \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq f \sqcup g} \right\} = \\ & \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq f \vee a \times^{\text{FCD}} b \sqsubseteq g} \right\} = \\ & \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq f} \right\} \sqcup \\ & \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(A)}, b \in \text{atoms}^{\mathcal{F}(B)}, a \times^{\text{FCD}} b \sqsubseteq g} \right\} = \\ & (\text{RLD})_{\text{in}}f \sqcup (\text{RLD})_{\text{in}}g. \end{aligned}$$

□

PROPOSITION 1097. $(\text{RLD})_{\text{in}}(f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B})) = ((\text{RLD})_{\text{in}}f) \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{B})$ for every funcoid f and $\mathcal{A} \in \mathcal{F}(\text{Src } f)$, $\mathcal{B} \in \mathcal{F}(\text{Dst } f)$.

PROOF.

$$(\text{RLD})_{\text{in}}(f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B})) = ((\text{RLD})_{\text{in}}f) \sqcap (\text{RLD})_{\text{in}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = ((\text{RLD})_{\text{in}}f) \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{B}).$$

□

COROLLARY 1098. $(\text{RLD})_{\text{in}}(f|_{\mathcal{A}}) = ((\text{RLD})_{\text{in}}f)|_{\mathcal{A}}$.

CONJECTURE 1099. $(\text{RLD})_{\text{in}}$ is not a lower adjoint (in general).

CONJECTURE 1100. $(\text{RLD})_{\text{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

EXERCISE 1101. Prove that $\text{card FCD}(A, B) = 2^{2^{\max\{A, B\}}}$ if A or B is an infinite set (provided that A and B are nonempty).

LEMMA 1102. $\uparrow^{\text{FCD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq (\text{FCD})g \Leftrightarrow \uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq g$ for every reloid g .

PROOF.

$$\begin{aligned} \uparrow^{\text{FCD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq (\text{FCD})g &\Leftrightarrow \\ \uparrow^{\text{FCD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \not\sqsubseteq (\text{FCD})g &\Leftrightarrow @\{x\} [(\text{FCD})g]^* @\{y\} \Leftrightarrow \\ \uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \not\sqsubseteq g &\Leftrightarrow \uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq g. \end{aligned}$$

□

THEOREM 1103. $\text{Cor}(\text{FCD})g = (\text{FCD})\text{Cor } g$ for every reloid g .

PROOF.

$$\begin{aligned} \text{Cor}(\text{FCD})g &= \\ \bigsqcup \left\{ \frac{\uparrow^{\text{FCD}(\text{Src } g, \text{Dst } g)} \{(x, y)\}}{\uparrow^{\text{FCD}} \{(x, y)\} \sqsubseteq (\text{FCD})g} \right\} &= \\ \bigsqcup \left\{ \frac{\uparrow^{\text{FCD}(\text{Src } g, \text{Dst } g)} \{(x, y)\}}{\uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq g} \right\} &= \\ \bigsqcup \left\{ \frac{(\text{FCD}) \uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\}}{\uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq g} \right\} &= \\ (\text{FCD}) \bigsqcup \left\{ \frac{\uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\}}{\uparrow^{\text{RLD}(\text{Src } g, \text{Dst } g)} \{(x, y)\} \sqsubseteq g} \right\} &= \\ &(\text{FCD})\text{Cor } g. \end{aligned}$$

□

CONJECTURE 1104.

- 1°. $\text{Cor}(\text{RLD})_{\text{in}}g = (\text{RLD})_{\text{in}}\text{Cor } g$;
- 2°. $\text{Cor}(\text{RLD})_{\text{out}}g = (\text{RLD})_{\text{out}}\text{Cor } g$.

THEOREM 1105. For every reloid f :

- 1°. $\text{Compl}(\text{FCD})f = (\text{FCD})\text{Compl } f$;
- 2°. $\text{CoCompl}(\text{FCD})f = (\text{FCD})\text{CoCompl } f$.

PROOF. We will prove only the first, because the second is dual.

$$\begin{aligned} \text{Compl}(\text{FCD})f &= \bigsqcup_{\alpha \in \text{Src } f} ((\text{FCD})f)|_{\uparrow\{\alpha\}} = (\text{proposition 1071}) = \\ &\bigsqcup_{\alpha \in \text{Src } f} (\text{FCD})(f|_{\uparrow\{\alpha\}}) = (\text{FCD}) \bigsqcup_{\alpha \in \text{Src } f} f|_{\uparrow\{\alpha\}} = (\text{FCD})\text{Compl } f. \end{aligned}$$

□

CONJECTURE 1106.

- 1°. $\text{Compl}(\text{RLD})_{\text{in}}g = (\text{RLD})_{\text{in}}\text{Compl } g$;

$$2^\circ. \text{Compl}(\text{RLD})_{\text{out}}g = (\text{RLD})_{\text{out}} \text{Compl}g.$$

Note that the above Galois connection between funcoids and reloids is a Galois surjection.

$$\text{PROPOSITION 1107. } (\text{RLD})_{\text{in}}g = \max \left\{ \frac{f \in \text{RLD}}{(\text{FCD})f \sqsubseteq g} \right\} = \max \left\{ \frac{f \in \text{RLD}}{(\text{FCD})f = g} \right\}.$$

PROOF. By theorem 131 and proposition 323. \square

9.4. Funcoidal reloids

DEFINITION 1108. I call *funcoidal* such a reloid ν that

$$\begin{aligned} \mathcal{X} \times^{\text{RLD}} \mathcal{Y} \not\sqsubseteq \nu \Rightarrow \\ \exists \mathcal{X}' \in \mathcal{F}(\text{Base}(\mathcal{X})) \setminus \{\perp\}, \mathcal{Y}' \in \mathcal{F}(\text{Base}(\mathcal{Y})) \setminus \{\perp\} : \\ (\mathcal{X}' \sqsubseteq \mathcal{X} \wedge \mathcal{Y}' \sqsubseteq \mathcal{Y} \wedge \mathcal{X}' \times^{\text{RLD}} \mathcal{Y}' \sqsubseteq \nu) \end{aligned}$$

for every $\mathcal{X} \in \mathcal{F}(\text{Src } \nu)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } \nu)$.

REMARK 1109. See theorem 1114 below for how they are bijectively related with funcoids (and thus named funcoidal).

PROPOSITION 1110. A reloid ν is funcoidal iff $x \times^{\text{RLD}} y \not\sqsubseteq \nu \Rightarrow x \times^{\text{RLD}} y \sqsubseteq \nu$ for every atomic filter objects x and y on respective sets.

PROOF.

$$\Rightarrow. x \times^{\text{RLD}} y \not\sqsubseteq \nu \Rightarrow \exists \mathcal{X}' \in \text{atoms } \mathcal{X}, \mathcal{Y}' \in \text{atoms } \mathcal{Y} : \mathcal{X}' \times^{\text{RLD}} \mathcal{Y}' \sqsubseteq \nu \Rightarrow x \times^{\text{RLD}} y \sqsubseteq \nu.$$

$\Leftarrow.$

$$\begin{aligned} \mathcal{X} \times^{\text{RLD}} \mathcal{Y} \not\sqsubseteq \nu \Rightarrow \\ \exists x \in \text{atoms } \mathcal{X}, y \in \text{atoms } \mathcal{Y} : x \times^{\text{RLD}} y \not\sqsubseteq \nu \Rightarrow \\ \exists x \in \text{atoms } \mathcal{X}, y \in \text{atoms } \mathcal{Y} : x \times^{\text{RLD}} y \sqsubseteq \nu \Rightarrow \\ \exists \mathcal{X}' \in \mathcal{F}(\text{Base}(\mathcal{X})) \setminus \{\perp\}, \mathcal{Y}' \in \mathcal{F}(\text{Base}(\mathcal{Y})) \setminus \{\perp\} : \\ (\mathcal{X}' \sqsubseteq \mathcal{X} \wedge \mathcal{Y}' \sqsubseteq \mathcal{Y} \wedge \mathcal{X}' \times^{\text{RLD}} \mathcal{Y}' \sqsubseteq \nu). \end{aligned}$$

\square

PROPOSITION 1111.

$$(\text{RLD})_{\text{in}}(\text{FCD})f = \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(\text{Src } \nu)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } \nu)}, a \times^{\text{RLD}} b \not\sqsubseteq f} \right\}.$$

PROOF.

$$\begin{aligned} (\text{RLD})_{\text{in}}(\text{FCD})f = \\ \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(\text{Src } \nu)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } \nu)}, a \times^{\text{FCD}} b \sqsubseteq (\text{FCD})f} \right\} = \\ \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(\text{Src } \nu)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } \nu)}, a [(\text{FCD})f] b} \right\} = \\ \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}(\text{Src } \nu)}, b \in \text{atoms}^{\mathcal{F}(\text{Dst } \nu)}, a \times^{\text{RLD}} b \not\sqsubseteq f} \right\}. \end{aligned}$$

\square

DEFINITION 1112. I call $(\text{RLD})_{\text{in}}(\text{FCD})f$ *funcoidization* of a reloid f .

LEMMA 1113. $(\text{RLD})_{\text{in}}(\text{FCD})f$ is funcoidal for every reloid f .

PROOF. $x \times^{\text{RLD}} y \not\sqsubseteq (\text{RLD})_{\text{in}}(\text{FCD})f \Rightarrow x \times^{\text{RLD}} y \sqsubseteq (\text{RLD})_{\text{in}}(\text{FCD})f$ for atomic filters x and y . \square

THEOREM 1114. $(\text{RLD})_{\text{in}}$ is a bijection from $\text{FCD}(A, B)$ to the set of funcoidal reloids from A to B . The reverse bijection is given by (FCD) .

PROOF. Let $f \in \text{FCD}(A, B)$. Prove that $(\text{RLD})_{\text{in}}f$ is funcoidal.

Really $(\text{RLD})_{\text{in}}f = (\text{RLD})_{\text{in}}(\text{FCD})(\text{RLD})_{\text{in}}f$ and thus we can use the lemma stating that it is funcoidal.

It remains to prove $(\text{RLD})_{\text{in}}(\text{FCD})f = f$ for a funcoidal reloid f . ($(\text{FCD})(\text{RLD})_{\text{in}}g = g$ for every funcoid g is already proved above.)

$$\begin{aligned} & (\text{RLD})_{\text{in}}(\text{FCD})f = \\ & \bigsqcup \left\{ \frac{x \times^{\text{RLD}} y}{x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, x \times^{\text{RLD}} y \not\sqsubseteq f} \right\} = \\ & \bigsqcup \left\{ \frac{p \in \text{atoms}(x \times^{\text{RLD}} y)}{x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, x \times^{\text{RLD}} y \not\sqsubseteq f} \right\} = \\ & \bigsqcup \left\{ \frac{p \in \text{atoms}(x \times^{\text{RLD}} y)}{x \in \text{atoms}^{\mathcal{F}(\text{Src } f)}, y \in \text{atoms}^{\mathcal{F}(\text{Dst } f)}, x \times^{\text{RLD}} y \sqsubseteq f} \right\} = \\ & \bigsqcup \text{atoms } f = f. \end{aligned}$$

\square

COROLLARY 1115. Funcoidal reloids are convex.

PROOF. Every $(\text{RLD})_{\text{in}}f$ is obviously convex. \square

THEOREM 1116. $(\text{RLD})_{\text{in}}(g \circ f) = (\text{RLD})_{\text{in}}g \circ (\text{RLD})_{\text{in}}f$ for every composable funcoids f and g .

PROOF.

$$(\text{RLD})_{\text{in}}g \circ (\text{RLD})_{\text{in}}f = (\text{corollary 1006}) =$$

$$\bigsqcup^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \text{atoms}(\text{RLD})_{\text{in}}f, G \in \text{atoms}(\text{RLD})_{\text{in}}g} \right\}$$

Let F be an atom of the poset $\text{RLD}(\text{Src } f, \text{Dst } f)$.

$$F \in \text{atoms}(\text{RLD})_{\text{in}}f \Rightarrow \text{dom } F \times^{\text{RLD}} \text{im } F \not\sqsubseteq (\text{RLD})_{\text{in}}f \Rightarrow$$

$$(\text{because } (\text{RLD})_{\text{in}}f \text{ is a funcoidal reloid}) \Rightarrow$$

$$\text{dom } F \times^{\text{RLD}} \text{im } F \sqsubseteq (\text{RLD})_{\text{in}}f$$

but $\text{dom } F \times^{\text{RLD}} \text{im } F \sqsubseteq (\text{RLD})_{\text{in}}f \Rightarrow F \sqsubseteq (\text{RLD})_{\text{in}}f$ is obvious.

So

$$F \in \text{atoms}(\text{RLD})_{\text{in}}f \Leftrightarrow \text{dom } F \times^{\text{RLD}} \text{im } F \sqsubseteq (\text{RLD})_{\text{in}}f \Rightarrow$$

$$(\text{FCD})(\text{dom } F \times^{\text{RLD}} \text{im } F) \sqsubseteq (\text{FCD})(\text{RLD})_{\text{in}}f \Leftrightarrow \text{dom } F \times^{\text{FCD}} \text{im } F \sqsubseteq f.$$

But

$$\text{dom } F \times^{\text{FCD}} \text{im } F \sqsubseteq f \Rightarrow (\text{RLD})_{\text{in}}(\text{dom } F \times^{\text{FCD}} \text{im } F) \sqsubseteq (\text{RLD})_{\text{in}}f \Leftrightarrow$$

$$\text{dom } F \times^{\text{RLD}} \text{im } F \sqsubseteq (\text{RLD})_{\text{in}}f.$$

So $F \in \text{atoms}(\text{RLD})_{\text{in}}f \Leftrightarrow \text{dom } F \times^{\text{FCD}} \text{im } F \sqsubseteq f$.

Let $F \in \text{atoms}(\text{RLD})_{\text{in}f}$, $G \in \text{atoms}(\text{RLD})_{\text{in}g}$. Then $\text{dom } F \times^{\text{FCD}} \text{im } F \sqsubseteq f$ and $\text{dom } G \times^{\text{FCD}} \text{im } G \sqsubseteq g$. Provided that $\text{im } F \not\sqsubseteq \text{dom } G$, we have:

$$\begin{aligned} \text{dom } F \times^{\text{RLD}} \text{im } G &= (\text{dom } G \times^{\text{RLD}} \text{im } G) \circ (\text{dom } F \times^{\text{RLD}} \text{im } F) = \\ \bigsqcup^{\text{RLD}} \left\{ \frac{G' \circ F'}{F' \in \text{atoms}(\text{dom } F \times^{\text{RLD}} \text{im } F), G' \in \text{atoms}(\text{dom } G \times^{\text{RLD}} \text{im } G)} \right\} &\sqsubseteq (*) \\ \bigsqcup^{\text{RLD}} \left\{ \frac{G' \circ F'}{F' \in \text{atoms}^{\text{RLD}}(\text{Src } F, \text{Dst } F), G' \in \text{atoms}^{\text{RLD}}(\text{Src } G, \text{Dst } G),} \right. & \\ \left. F' \sqsubseteq (\text{RLD})_{\text{in}f}, G' \sqsubseteq (\text{RLD})_{\text{in}g} \right\} &= \\ \bigsqcup^{\text{RLD}} \left\{ \frac{G' \circ F'}{F' \in \text{atoms}(\text{RLD})_{\text{in}f}, G' \in \text{atoms}(\text{RLD})_{\text{in}g}} \right\} &= (\text{RLD})_{\text{in}g} \circ (\text{RLD})_{\text{in}f}. \end{aligned}$$

(*) $F' \in \text{atoms}(\text{dom } F \times^{\text{RLD}} \text{im } F)$ and $\text{dom } F \times^{\text{FCD}} \text{im } F \sqsubseteq f$ implies $\text{dom } F' \times^{\text{FCD}} \text{im } F' \sqsubseteq f$; thus $\text{dom } F' \times^{\text{RLD}} \text{im } F' \sqsubseteq (\text{RLD})_{\text{in}f}$ and thus $F' \sqsubseteq (\text{RLD})_{\text{in}f}$. Likewise for G and G' .

Thus $(\text{RLD})_{\text{in}g} \circ (\text{RLD})_{\text{in}f} \supseteq \bigsqcup^{\text{RLD}} \left\{ \frac{\text{dom } F \times^{\text{RLD}} \text{im } G}{F \in \text{atoms}(\text{RLD})_{\text{in}f}, G \in \text{atoms}(\text{RLD})_{\text{in}g}, \text{im } F \not\sqsubseteq \text{dom } G} \right\}$.
But

$$\begin{aligned} (\text{RLD})_{\text{in}g} \circ (\text{RLD})_{\text{in}f} &\sqsubseteq \bigsqcup^{\text{RLD}} \left\{ \frac{(\text{dom } G \times^{\text{RLD}} \text{im } G) \circ (\text{dom } F \times^{\text{RLD}} \text{im } F)}{F \in \text{atoms}(\text{RLD})_{\text{in}f}, G \in \text{atoms}(\text{RLD})_{\text{in}g}} \right\} = \\ \bigsqcup^{\text{RLD}} \left\{ \frac{\text{dom } F \times^{\text{RLD}} \text{im } G}{F \in \text{atoms}(\text{RLD})_{\text{in}f}, G \in \text{atoms}(\text{RLD})_{\text{in}g}, \text{im } F \not\sqsubseteq \text{dom } G} \right\}. & \end{aligned}$$

Thus

$$\begin{aligned} (\text{RLD})_{\text{in}g} \circ (\text{RLD})_{\text{in}f} &= \\ \bigsqcup^{\text{RLD}} \left\{ \frac{\text{dom } F \times^{\text{RLD}} \text{im } G}{F \in \text{atoms}(\text{RLD})_{\text{in}f}, G \in \text{atoms}(\text{RLD})_{\text{in}g}, \text{im } F \not\sqsubseteq \text{dom } G} \right\} &= \\ \bigsqcup^{\text{RLD}} \left\{ \frac{\text{dom } F \times^{\text{RLD}} \text{im } G}{F \in \text{atoms}^{\text{RLD}}(\text{Src } f, \text{Dst } f), G \in \text{atoms}^{\text{RLD}}(\text{Dst } f, \text{Dst } g),} \right. & \\ \left. \text{dom } F \times^{\text{FCD}} \text{im } F \sqsubseteq f, \text{dom } G \times^{\text{FCD}} \text{im } G \sqsubseteq g, \text{im } F \not\sqsubseteq \text{dom } G \right\}. & \end{aligned}$$

But

$$\begin{aligned} (\text{RLD})_{\text{in}}(g \circ f) &= \bigsqcup \left\{ \frac{a \times^{\text{RLD}} c}{a \times^{\text{FCD}} c \in \text{atoms}(g \circ f)} \right\} = (\text{proposition 910}) = \\ \bigsqcup \left\{ \frac{a \times^{\text{RLD}} c}{a \in \mathcal{F}(\text{Src } f), c \in \mathcal{F}(\text{Dst } g),} \right. & \\ \left. \exists b \in \mathcal{F}(\text{Dst } f) : (a \times^{\text{FCD}} b \in \text{atoms } f \wedge b \times^{\text{FCD}} c \in \text{atoms } g) \right\} &= \\ \bigsqcup \left\{ \frac{a \times^{\text{RLD}} c}{a \in \mathcal{F}(\text{Src } f), c \in \mathcal{F}(\text{Dst } g),} \right. & \\ \left. \exists b_0, b_1 \in \mathcal{F}(\text{Dst } f) : (a \times^{\text{FCD}} b \in \text{atoms } f \wedge b \times^{\text{FCD}} c \in \text{atoms } g \wedge b_0 \not\sqsubseteq b_1) \right\}. & \end{aligned}$$

Now it becomes obvious that $(\text{RLD})_{\text{in}g} \circ (\text{RLD})_{\text{in}f} = (\text{RLD})_{\text{in}}(g \circ f)$. \square

9.5. Complete funcoids and reloids

For the proof below assume

$$\theta = \left(\bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\}) \mapsto \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{FCD}} \langle f \rangle^* @\{x\}) \right)$$

(where f ranges the set of complete funcoids).

LEMMA 1117. θ is a bijection from complete reloids into complete funcoids.

PROOF. Theorems 931 and 1037. \square

LEMMA 1118. $(\text{FCD})g = \theta g$ for every complete reloid g .

PROOF. Really, $g = \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\})$ for a complete reloid g and thus

$$\begin{aligned} (\text{FCD})g &= \bigsqcup_{x \in \text{Src } f} (\text{FCD})(\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\}) = \\ &= \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{FCD}} \langle f \rangle^* @\{x\}) = \theta g. \end{aligned}$$

\square

LEMMA 1119. $(\text{RLD})_{\text{out}} f = \theta^{-1} f$ for every complete funcoid f .

PROOF. We have $f = \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{FCD}} \langle f \rangle^* @\{x\})$. We need to prove $(\text{RLD})_{\text{out}} f = \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\})$.

Really, $(\text{RLD})_{\text{out}} f \supseteq \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\})$.

It remains to prove that $\bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\}) \supseteq (\text{RLD})_{\text{out}} f$.

Let $L \in \text{up} \bigsqcup_{x \in \text{Src } f} (\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\})$. We will prove $L \in \text{up}(\text{RLD})_{\text{out}} f$.

We have

$$L \in \bigcap_{x \in \text{Src } f} \text{up}(\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\}).$$

$\langle L \rangle^* \{x\} = G(x)$ for some $G(x) \in \text{up} \langle f \rangle^* @\{x\}$ (because $L \in \text{up}(\uparrow^{\text{Src } f} \{x\} \times^{\text{RLD}} \langle f \rangle^* @\{x\})$).

Thus $L = G \in \text{up } f$ (because f is complete). Thus $L \in \text{up } f$ and so $L \in \text{up}(\text{RLD})_{\text{out}} f$. \square

PROPOSITION 1120. (FCD) and $(\text{RLD})_{\text{out}}$ form mutually inverse bijections between complete reloids and complete funcoids.

PROOF. From two last lemmas. \square

THEOREM 1121. The diagram at the figure 8 (with the “unnamed” arrow from $\text{ComplRLD}(A, B)$ to $\mathcal{F}(B)^A$ defined as the inverse isomorphism of its opposite arrow) is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore “parallel” arrows are mutually inverse). The arrows preserve order.

PROOF. It’s proved above, that all morphisms (except the “unnamed” arrow, which is the inverse morphism by definition) depicted on the diagram are bijections and the depicted “opposite” morphisms are mutually inverse.

That arrows preserve order is obvious.

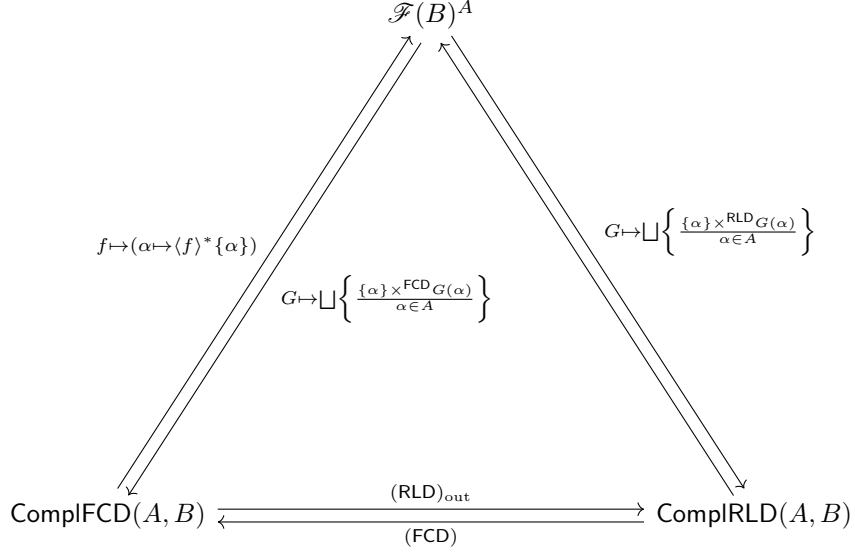


FIGURE 8

It remains to apply lemma 196 (taking into account that θ can be decomposed into $(G \mapsto \bigsqcup \left\{ \frac{\{ \alpha \} \times^{\text{RLD}} G(\alpha)}{\alpha \in A} \right\})^{-1}$ and $G \mapsto \bigsqcup \left\{ \frac{\{ \alpha \} \times^{\text{FCD}} G(\alpha)}{\alpha \in A} \right\}$). \square

THEOREM 1122. Composition of complete reloids is complete.

PROOF. Let f, g be complete reloids. Then $(\text{FCD})(g \circ f) = (\text{FCD})g \circ (\text{FCD})f$. Thus (because $(\text{FCD})(g \circ f)$ is a complete funcoid) we have $g \circ f = (\text{RLD})_{\text{out}}((\text{FCD})g \circ (\text{FCD})f)$, but $(\text{FCD})g \circ (\text{FCD})f$ is a complete funcoid, thus $g \circ f$ is a complete reloid. \square

THEOREM 1123.

- 1°. $(\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f = (\text{RLD})_{\text{out}}(g \circ f)$ for composable complete funcoids f and g .
- 2°. $(\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f = (\text{RLD})_{\text{out}}(g \circ f)$ for composable co-complete funcoids f and g .

PROOF. Let f, g be composable complete funcoids.

$$(\text{FCD})((\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f) = (\text{FCD})(\text{RLD})_{\text{out}}g \circ (\text{FCD})(\text{RLD})_{\text{out}}f = g \circ f.$$

Thus (taking into account that $(\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f$ is complete) we have $(\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f = (\text{RLD})_{\text{out}}(g \circ f)$.

For co-complete funcoids it's dual. \square

PROPOSITION 1124. If f is a (co-)complete funcoid then $\text{up } f$ is a filter.

PROOF. It is enough to consider the case if f is complete.

We need to prove that $\forall F, G \in \text{up } f : F \sqcap G \in \text{up } f$.

For every $F \in \mathbf{Rel}(\text{Src } f, \text{Dst } f)$ we have

$$F \in \text{up } f \Leftrightarrow F \sqsupseteq f \Leftrightarrow \langle F \rangle^* \{x\} \sqsupseteq \langle f \rangle^* \{x\}.$$

Thus

$$\begin{aligned} F, G \in \text{up } f &\Rightarrow \langle F \rangle^* \{x\} \sqsupseteq \langle f \rangle^* \{x\} \wedge \langle G \rangle^* \{x\} \sqsupseteq \langle f \rangle^* \{x\} \Rightarrow \\ &\langle F \sqcap G \rangle^* \{x\} = \langle F \rangle^* \{x\} \cap \langle G \rangle^* \{x\} \sqsupseteq \langle f \rangle^* \{x\} \Rightarrow F \sqcap G \in \text{up } f. \end{aligned}$$

That $\text{up } f$ is nonempty and up-directed is obvious. \square

COROLLARY 1125.

- 1°. If f is a (co-)complete funcoid then $\text{up } f = \text{up}(\text{RLD})_{\text{out}} f$.
 2°. If f is a (co-)complete reloid then $\text{up } f = \text{up}(\text{FCD})f$.

PROOF. By order isomorphism, it is enough to prove the first.

Because $\text{up } f$ is a filter, by properties of generalized filter bases we have $F \in \text{up}(\text{RLD})_{\text{out}} f \Leftrightarrow \exists g \in \text{up } f : F \sqsupseteq g \Leftrightarrow F \in \text{up } f$. \square

9.6. Properties preserved by relationships

PROPOSITION 1126. $(\text{FCD})f$ is reflexive iff f is reflexive (for every endoreloid f).

PROOF.

$$\begin{aligned} f \text{ is reflexive} &\Leftrightarrow 1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq f \Leftrightarrow \forall F \in \text{up } f : 1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq F \Leftrightarrow \\ &1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq \bigsqcap^{\text{FCD}} \text{up } f \Leftrightarrow 1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq (\text{FCD})f \Leftrightarrow (\text{FCD})f \text{ is reflexive.} \end{aligned}$$

\square

PROPOSITION 1127. $(\text{RLD})_{\text{out}} f$ is reflexive iff f is reflexive (for every endofuncoid f).

PROOF.

$$\begin{aligned} f \text{ is reflexive} &\Leftrightarrow 1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq f \Leftrightarrow (\text{corollary 925}) \Leftrightarrow \\ &\forall F \in \text{up } f : 1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq F \Leftrightarrow 1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq \bigsqcap^{\text{RLD}} \text{up } f \Leftrightarrow \\ &1_{\text{Ob } f}^{\text{Rel}} \sqsubseteq (\text{RLD})_{\text{out}} f \Leftrightarrow (\text{RLD})_{\text{out}} f \text{ is reflexive.} \end{aligned}$$

\square

PROPOSITION 1128. $(\text{RLD})_{\text{in}} f$ is reflexive iff f is reflexive (for every endofuncoid f).

PROOF. $(\text{RLD})_{\text{in}} f$ is reflexive iff $(\text{FCD})(\text{RLD})_{\text{in}} f$ is reflexive iff f is reflexive. \square

OBVIOUS 1129. (FCD) , $(\text{RLD})_{\text{in}}$, and $(\text{RLD})_{\text{out}}$ preserve symmetry of the argument funcoid or reloid.

PROPOSITION 1130. $a \times_F^{\text{RLD}} a = \perp$ for every nontrivial ultrafilter a .

PROOF.

$$\begin{aligned} a \times_F^{\text{RLD}} a &= (\text{RLD})_{\text{out}}(a \times^{\text{FCD}} a) = \\ &\bigsqcap^{\text{RLD}} \text{up}(a \times^{\text{FCD}} a) \sqsubseteq 1^{\text{FCD}} \cap (\top^{\text{FCD}} \setminus 1^{\text{FCD}}) = \perp^{\text{FCD}}. \end{aligned}$$

\square

EXAMPLE 1131. There exist filters \mathcal{A} and \mathcal{B} such that $(\text{FCD})(\mathcal{A} \times_F^{\text{RLD}} \mathcal{B}) \sqsubset \mathcal{A} \times^{\text{FCD}} \mathcal{B}$.

PROOF. Take $\mathcal{A} = \mathcal{B} = a$ for a nontrivial ultrafilter a . $a \times_F^{\text{RLD}} a = \perp$. Thus $(\text{FCD})(a \times_F^{\text{RLD}} a) = \perp \sqsubset a \times^{\text{FCD}} a$. \square

CONJECTURE 1132. There exist filters \mathcal{A} and \mathcal{B} such that $(\text{FCD})(\mathcal{A} \times \mathcal{B}) \sqsubset \mathcal{A} \times^{\text{FCD}} \mathcal{B}$.

EXAMPLE 1133. There is such a non-symmetric reloid f that $(\text{FCD})f$ is symmetric.

PROOF. Take $f = ((\text{RLD})_{\text{in}}(=)|_{\mathbb{R}}) \sqcap (\geq)_{\mathbb{R}}$. f is non-symmetric because $f \not\asymp (>)_{\mathbb{R}}$ but $f \asymp (<)_{\mathbb{R}}$. $(\text{FCD})f = (=)|_{\mathbb{R}}$ because $(=)|_{\mathbb{R}} \sqsubseteq f \sqsubseteq (\text{RLD})_{\text{in}}(=)|_{\mathbb{R}}$. \square

PROPOSITION 1134. If $(\text{RLD})_{\text{in}}f$ is symmetric then endofunctor f is symmetric.

PROOF. Suppose $(\text{RLD})_{\text{in}}f$ is symmetric then $f = (\text{FCD})(\text{RLD})_{\text{in}}f$ is symmetric. \square

CONJECTURE 1135. If $(\text{RLD})_{\text{out}}f$ is symmetric then endofunctor f is symmetric.

PROPOSITION 1136. If f is a transitive endoreloid, then $(\text{FCD})f$ is a transitive functor.

PROOF. $f = f \circ f$; $(\text{FCD})f = (\text{FCD})(f \circ f)$; $(\text{FCD})f = (\text{FCD})f \circ (\text{FCD})f$. \square

CONJECTURE 1137. There exists a non-transitive endoreloid f such that $(\text{FCD})f$ is a transitive functor.

PROPOSITION 1138. $(\text{RLD})_{\text{in}}f$ is transitive iff f is transitive (for every endofunctor f).

PROOF.

$$\begin{aligned} f = f \circ f &\Rightarrow (\text{RLD})_{\text{in}}f = (\text{RLD})_{\text{in}}(f \circ f) \Leftrightarrow (\text{theorem 1116}) \Leftrightarrow \\ &(\text{RLD})_{\text{in}}f = (\text{RLD})_{\text{in}}f \circ (\text{RLD})_{\text{in}}f \Rightarrow \\ &(\text{FCD})(\text{RLD})_{\text{in}}f = (\text{FCD})(\text{RLD})_{\text{in}}f \circ (\text{FCD})(\text{RLD})_{\text{in}}f \Leftrightarrow f = f \circ f. \end{aligned}$$

Thus $f = f \circ f \Leftrightarrow (\text{RLD})_{\text{in}}f \circ (\text{RLD})_{\text{in}}f$. \square

CONJECTURE 1139.

- 1°. There exists such a transitive endofunctor f , that $(\text{RLD})_{\text{out}}f$ is not a transitive reloid.
- 2°. There exists such a non-transitive endofunctor f , that $(\text{RLD})_{\text{out}}f$ is transitive reloid.

9.7. Some sub-posets of functors and reloids

PROPOSITION 1140. The following are complete sub-meet-semilattices (that is subsets closed for arbitrary meets) of $\text{RLD}(A, A)$ (for every set A):

- 1°. symmetric reloids on A ;
- 2°. reflexive reloids on A ;
- 3°. symmetric reflexive reloids on A ;
- 4°. transitive reloids on A ;
- 5°. symmetric reflexive transitive reloids (= reloids of equivalence = uniform spaces) on A .

PROOF. The first three items are obvious.

Fourth: Let S be a set of transitive reloids on A . That is $f \circ f \sqsubseteq f$ for every $f \in S$. Then $(\sqcap S) \circ (\sqcap S) \sqsubseteq f \circ f \sqsubseteq f$. Consequently $(\sqcap S) \circ (\sqcap S) \sqsubseteq \sqcap S$.

The last item follows from the previous. \square

PROPOSITION 1141. The following are complete sub-meet-semilattices (that is subsets closed for arbitrary meets) of $\text{FCD}(A, A)$ (for every set A):

- 1°. symmetric functors on A ;
- 2°. reflexive functors on A ;
- 3°. symmetric reflexive functors on A ;
- 4°. transitive functors on A ;

5°. symmetric reflexive transitive funcoids (= funcoids of equivalence = proximity spaces) on A .

PROOF. Analogous. \square

Obvious corollaries:

COROLLARY 1142. The following are complete lattices (for every set A):

- 1°. symmetric reloids on A ;
- 2°. reflexive reloids on A ;
- 3°. symmetric reflexive reloids on A ;
- 4°. transitive reloids on A ;
- 5°. symmetric reflexive transitive reloids (= reloids of equivalence = uniform spaces) on A .

COROLLARY 1143. The following are complete lattices (for every set A):

- 1°. symmetric funcoids on A ;
- 2°. reflexive funcoids on A ;
- 3°. symmetric reflexive funcoids on A ;
- 4°. transitive funcoids on A ;
- 5°. symmetric reflexive transitive funcoids (= funcoids of equivalence = proximity spaces) on A .

The following conjecture was inspired by theorem 2.2 in [41]:

CONJECTURE 1144. Join of a set S on the lattice of transitive reloids is the join (on the lattice of reloids) of all compositions of finite sequences of elements of S .

The similar question can be asked about uniform spaces.
Does the same hold for funcoids?

9.8. Double filtrators

Below I show that it's possible to describe (\mathbf{FCD}) , $(\mathbf{RLD})_{\text{out}}$, and $(\mathbf{RLD})_{\text{in}}$ entirely in terms of filtrators (order). This seems not to lead to really interesting results but it's curious.

DEFINITION 1145. *Double filtrator* is a triple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{Z})$ of posets such that \mathfrak{Z} is a sub-poset of both \mathfrak{A} and \mathfrak{B} .

In other words, a double filtrator $(\mathfrak{A}, \mathfrak{B}, \mathfrak{Z})$ is a triple such that both $(\mathfrak{A}, \mathfrak{Z})$ and $(\mathfrak{B}, \mathfrak{Z})$ are filtrators.

DEFINITION 1146. *Double filtrator of funcoids and reloids* is $(\mathbf{FCD}, \mathbf{RLD}, \mathbf{Rel})$.

DEFINITION 1147. $(\mathbf{FCD})f = \prod^{\mathfrak{A}} \text{up}^{\mathfrak{Z}} f$ for $f \in \mathfrak{B}$.

DEFINITION 1148. $(\mathbf{RLD})_{\text{out}}f = \prod^{\mathfrak{B}} \text{up}^{\mathfrak{Z}} f$ for $f \in \mathfrak{A}$.

DEFINITION 1149. If (\mathbf{FCD}) is a lower adjoint, define $(\mathbf{RLD})_{\text{in}}$ as the upper adjoint of (\mathbf{FCD}) .

9.8.1. Embedding of \mathfrak{A} into \mathfrak{B} . In this section we will suppose that (\mathbf{FCD}) and $(\mathbf{RLD})_{\text{in}}$ form a Galois surjection, that is $(\mathbf{FCD})(\mathbf{RLD})_{\text{in}}f = f$ for every $f \in \mathfrak{A}$. Then $(\mathbf{RLD})_{\text{in}}$ is an order embedding from \mathfrak{A} to \mathfrak{B} .

9.8.2. One more core part. In this section we will assume that (FCD) and $(\text{RLD})_{\text{in}}$ form a Galois surjection and equate \mathfrak{A} with its image by $(\text{RLD})_{\text{in}}$ in \mathfrak{B} . We will also assume $(\mathfrak{A}, \mathfrak{B})$ being a filtered filtrator.

PROPOSITION 1150. $(\text{FCD})f = \text{Cor}^{\mathfrak{A}} f$ for every $f \in \mathfrak{B}$.

PROOF. $\text{Cor}^{\mathfrak{A}} f = \prod^{\mathfrak{A}} \text{up}^{\mathfrak{A}} f \sqsubseteq \prod^{\mathfrak{A}} \text{up}^{\mathfrak{B}} f = (\text{FCD})f$. But for every $g \in \text{up}^{\mathfrak{A}} f$ we have $g = \prod^{\mathfrak{A}} \text{up}^{\mathfrak{B}} g \supseteq \prod^{\mathfrak{A}} \text{up}^{\mathfrak{B}} f$, thus $\prod^{\mathfrak{A}} \text{up}^{\mathfrak{A}} f \supseteq \prod^{\mathfrak{A}} \text{up}^{\mathfrak{B}} f$. \square

EXAMPLE 1151. $(\text{FCD})f \neq \text{Cor}'^{\mathfrak{A}} f$ for the double filtrator of funcoids and reloids.

PROOF. Consider a nontrivial ultrafilter a and the reloid $f = \text{id}_a^{\text{RLD}}$.

$$\begin{aligned} \text{Cor}'^{\mathfrak{A}} f &= \text{Cor}'^{\text{FCD}} \text{id}_a^{\text{RLD}} = \prod^{\text{FCD}} \text{down}^{\text{FCD}} \text{id}_a^{\text{RLD}} = \\ &= \prod^{\text{FCD}} \emptyset = \perp^{\text{FCD}} \neq a \times^{\text{FCD}} a = (\text{FCD}) \text{id}_a^{\text{RLD}}. \end{aligned}$$

\square

I leave to a reader's exercise to apply the above theory to complete funcoids and reloids.

On distributivity of composition with a principal reloid

10.1. Decomposition of composition of binary relations

REMARK 1152. Sorry for an unfortunate choice of terminology: “composition” and “decomposition” are unrelated.

The idea of the proof below is that composition of binary relations can be decomposed into two operations: \otimes and dom :

$$g \otimes f = \left\{ \frac{((x, z), y)}{x f y \wedge y g z} \right\}.$$

Composition of binary relations can be decomposed: $g \circ f = \text{dom}(g \otimes f)$.

It can be decomposed even further: $g \otimes f = \Theta_0 f \cap \Theta_1 g$ where

$$\Theta_0 f = \left\{ \frac{((x, z), y)}{x f y, z \in \mathcal{U}} \right\} \quad \text{and} \quad \Theta_1 g = \left\{ \frac{((x, z), y)}{y f z, x \in \mathcal{U}} \right\}.$$

(Here \mathcal{U} is the Grothendieck universe.)

Now we will do a similar trick with reloids.

10.2. Decomposition of composition of reloids

A similar thing for reloids:

In this chapter we will equate reloids with filters on cartesian products of sets.

For composable reloids f and g we have

$$\begin{aligned} g \circ f &= \\ & \text{RLD}(\text{Src } f, \text{Dst } g) \left\{ \frac{G \circ F}{F \in \text{GR } f, G \in \text{GR } g} \right\} = \\ & \text{RLD}(\text{Src } f, \text{Dst } g) \left\{ \frac{\text{dom}(G \otimes F)}{F \in \text{GR } f, G \in \text{GR } g} \right\}. \end{aligned}$$

LEMMA 1153. $\left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}$ is a filter base.

PROOF. Let $P, Q \in \left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}$. Then $P = G_0 \otimes F_0$, $Q = G_1 \otimes F_1$ for some $F_0, F_1 \in f$, $G_0, G_1 \in g$. Then $F_0 \cap F_1 \in \text{up } f$, $G_0 \cap G_1 \in \text{up } g$ and thus

$$P \cap Q \supseteq (F_0 \cap F_1) \otimes (G_0 \cap G_1) \in \left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}.$$

□

COROLLARY 1154. $\left\{ \frac{\uparrow^{\mathcal{F}(\text{Src } f \times \text{Dst } g)}(G \otimes F)}{F \in \text{GR } f, G \in \text{GR } g} \right\}$ is a generalized filter base.

PROPOSITION 1155. $g \circ f = \text{dom} \prod^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}$.

PROOF. $\uparrow^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \text{dom}(G \otimes F) \supseteq \text{dom} \prod^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}$.

Thus

$$g \circ f \supseteq \text{dom} \prod^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}.$$

Let $X \in \text{up} \text{dom} \prod^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \left\{ \frac{G \otimes F}{F \in \text{up } f, G \in \text{up } g} \right\}$. Then there exist Y such that

$$X \times Y \in \text{up} \prod^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \left\{ \frac{G \otimes F}{F \in \text{up } f, G \in \text{up } g} \right\}.$$

So because it is a generalized filter base $X \times Y \supseteq G \otimes F$ for some $F \in \text{up } f, G \in \text{up } g$. Thus $X \in \text{up} \text{dom}(G \otimes F)$. $X \in \text{up}(g \circ f)$. \square

We can define $g \otimes f$ for reloids f, g :

$$g \otimes f = \left\{ \frac{G \otimes F}{F \in \text{GR } f, G \in \text{GR } g} \right\}.$$

Then

$$g \circ f = \prod^{\mathcal{F}(\text{Src } f \times \text{Dst } g)} \langle \text{dom} \rangle^*(g \otimes f) = \text{dom} \left\langle \uparrow^{\text{RLD}(\text{Src } f \times \text{Dst } g, \mathcal{U})} \right\rangle^*(g \otimes f).$$

10.3. Lemmas for the main result

LEMMA 1156. $(g \otimes f) \cap (h \otimes f) = (g \cap h) \otimes f$ for binary relations f, g, h .

PROOF.

$$(g \cap h) \otimes f = \Theta_0 f \cap \Theta_1 (g \cap h) = \Theta_0 f \cap (\Theta_1 g \cap \Theta_1 h) = (\Theta_0 f \cap \Theta_1 g) \cap (\Theta_0 f \cap \Theta_1 h) = (g \otimes f) \cap (h \otimes f).$$

\square

LEMMA 1157. Let $F = \uparrow^{\text{RLD}} f$ be a principal reloid (for a **Rel**-morphism f), T be a set of reloids from $\text{Dst } F$ to a set V .

$$\prod_{G \in \text{up} \sqcup T}^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes f) = \bigsqcup_{G \in T} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F).$$

PROOF. $\prod_{G \in \text{up} \sqcup T}^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes f) \supseteq \bigsqcup_{G \in T} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F)$ is obvious.

Let $K \in \text{up} \bigsqcup_{G \in T} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F)$. Then for each $G \in T$

$$K \in \text{up} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F);$$

$K \in \text{up} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} \left\{ \frac{\Gamma \otimes f}{\Gamma \in G} \right\}$. Then $K \in \left\{ \frac{\Gamma \otimes f}{\Gamma \in G} \right\}$ by properties of generalized filter bases.

$$K \in \left\{ \frac{(\Gamma_0 \cap \dots \cap \Gamma_n) \otimes f}{n \in \mathbb{N}, \Gamma_i \in G} \right\} = \left\{ \frac{(\Gamma_0 \otimes f) \cap \dots \cap (\Gamma_n \otimes f)}{n \in \mathbb{N}, \Gamma_i \in G} \right\}.$$

$\forall G \in T : K \supseteq (\Gamma_{G,0} \otimes f) \cap \dots \cap (\Gamma_{G,n} \otimes f)$ for some $n \in \mathbb{N}, \Gamma_{G,i} \in G$.

$$K \supseteq \left\{ \frac{(\Gamma_0 \otimes f) \cap \dots \cap (\Gamma_n \otimes f)}{n \in \mathbb{N}, \Gamma_i \in G} \right\} \text{ where } \Gamma_i = \bigcup_{g \in G} \Gamma_{g,i} \in \text{up} \sqcup T.$$

$$\begin{aligned}
K &\in \left\{ \frac{(\Gamma_0 \otimes f) \cap \cdots \cap (\Gamma_n \otimes f)}{n \in \mathbb{N}} \right\}. \text{ So} \\
K &\in \left\{ \frac{(\Gamma'_0 \otimes f) \cap \cdots \cap (\Gamma'_n \otimes f)}{n \in \mathbb{N}, \Gamma'_i \in \text{up} \bigsqcup T} \right\} = \\
&\quad \left\{ \frac{(\Gamma'_0 \cap \cdots \cap \Gamma'_n) \otimes f}{n \in \mathbb{N}, \Gamma'_i \in \text{up} \bigsqcup T} \right\} = \\
&\quad \text{up} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} \left\{ \frac{G \otimes f}{G \in \text{up} \bigsqcup T} \right\}. \quad \square
\end{aligned}$$

10.4. Proof of the main result

Let's prove a special case of conjecture 1053:

THEOREM 1158. $(\bigsqcup T) \circ F = \bigsqcup \left\{ \frac{G \circ F}{G \in T} \right\}$ for every principal reloid $F = \uparrow^{\text{RLD}} f$ (for a **Rel**-morphism f) and a set T of reloids from $\text{Dst } F$ to some set V . (In other words principal reloids are co-metacomplete and thus also metacomplete by duality.)

PROOF.

$$\begin{aligned}
&(\bigsqcup T) \circ F = \\
&\prod^{\text{RLD}(\text{Src } f, V)} \langle \text{dom} \rangle^* \left((\bigsqcup T) \otimes F \right) = \\
&\text{dom} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} \left((\bigsqcup T) \otimes F \right) = \\
&\quad \text{dom} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})}_{G \in \text{up} \bigsqcup T} (G \otimes f); \\
&\quad \bigsqcup_{G \in T} (G \circ F) = \\
&\quad \bigsqcup_{G \in T} \prod^{\text{RLD}(\text{Src } f, V)} \langle \text{dom} \rangle^* (G \otimes F) = \\
&\quad \bigsqcup_{G \in T} \text{dom} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F) = \\
&\quad \text{dom} \bigsqcup_{G \in T} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F).
\end{aligned}$$

It's enough to prove

$$\prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})}_{G \in \text{up} \bigsqcup T} (G \otimes f) = \bigsqcup_{G \in T} \prod^{\text{RLD}(\text{Src } f \times V, \mathcal{U})} (G \otimes F)$$

but this is the statement of the lemma. □

10.5. Embedding reloids into functors

DEFINITION 1159. Let f be a reloid. The functor

$$\rho f = \text{FCD}(\mathcal{P}(\text{Src } f \times \text{Src } f), \mathcal{P}(\text{Dst } f \times \text{Dst } f))$$

is defined by the formulas:

$$\langle \rho f \rangle x = f \circ x \quad \text{and} \quad \langle \rho f^{-1} \rangle y = f^{-1} \circ y$$

where x are endoreloids on $\text{Src } f$ and y are endoreloids on $\text{Dst } f$.

PROPOSITION 1160. It is really a functor (if we equate reloids x and y with corresponding filters on Cartesian products of sets).

PROOF. $y \neq \langle \rho f \rangle x \Leftrightarrow y \neq f \circ x \Leftrightarrow f^{-1} \circ y \neq x \Leftrightarrow \langle \rho f^{-1} \rangle y \neq x$. \square

COROLLARY 1161. $(\rho f)^{-1} = \rho f^{-1}$.

DEFINITION 1162. It can be continued to arbitrary functors x having destination $\text{Src } f$ by the formula $\langle \rho^* f \rangle x = \langle \rho f \rangle \text{id}_{\text{Src } f} \circ x = f \circ x$.

PROPOSITION 1163. ρ is an injection.

PROOF. Consider $x = \text{id}_{\text{Src } f}$. \square

PROPOSITION 1164. $\rho(g \circ f) = (\rho g) \circ (\rho f)$.

PROOF. $\langle \rho(g \circ f) \rangle x = g \circ f \circ x = \langle \rho g \rangle \langle \rho f \rangle x = (\langle \rho g \rangle \circ \langle \rho f \rangle) x$. Thus $\langle \rho(g \circ f) \rangle = \langle \rho g \rangle \circ \langle \rho f \rangle = \langle (\rho g) \circ (\rho f) \rangle$ and so $\rho(g \circ f) = (\rho g) \circ (\rho f)$. \square

THEOREM 1165. $\rho \sqcup F = \sqcup \langle \rho \rangle^* F$ for a set F of reloids.

PROOF. It's enough to prove $\langle \rho \sqcup F \rangle^* X = \langle \sqcup \langle \rho \rangle^* F \rangle^* X$ for a set X . Really,

$$\begin{aligned} \langle \rho \sqcup F \rangle^* X &= \\ \langle \rho \sqcup F \rangle \uparrow X &= \\ \sqcup F \circ \uparrow X &= \\ \sqcup \left\{ \frac{f \circ \uparrow X}{f \in F} \right\} &= \\ \sqcup \left\{ \frac{\langle \rho f \rangle \uparrow X}{f \in F} \right\} &= \\ \left\langle \sqcup \left\{ \frac{\rho f}{f \in F} \right\} \right\rangle X &= \\ \left\langle \sqcup \langle \rho \rangle^* F \right\rangle^* X. & \end{aligned}$$

\square

CONJECTURE 1166. $\rho \sqcap F = \sqcap \langle \rho \rangle^* F$ for a set F of reloids.

PROPOSITION 1167. $\rho 1_A^{\text{RLD}} = 1_{\mathcal{P}(A \times A)}^{\text{FCD}}$.

PROOF. $\langle \rho 1_A^{\text{RLD}} \rangle x = 1_A^{\text{RLD}} \circ x = x = \left\langle 1_{\mathcal{P}(A \times A)}^{\text{FCD}} \right\rangle x$. \square

We can try to develop further theory by applying embedding of reloids into functors for researching of properties of reloids.

THEOREM 1168. Reloid f is monovalued iff functor ρf is monovalued.

PROOF.

$$\begin{aligned}
& \rho f \text{ is monovalued} \Leftrightarrow \\
& (\rho f) \circ (\rho f)^{-1} \sqsubseteq 1_{\text{Dst } \rho f} \Leftrightarrow \\
& \rho(f \circ f^{-1}) \sqsubseteq 1_{\text{Dst } \rho f} \Leftrightarrow \\
& \rho(f \circ f^{-1}) \sqsubseteq 1_{\mathcal{P}(\text{Dst } f \times \text{Dst } f)}^{\text{FCD}} \Leftrightarrow \\
& \rho(f \circ f^{-1}) \sqsubseteq \rho 1_{\text{Dst } f}^{\text{RLD}} \Leftrightarrow \\
& f \circ f^{-1} \sqsubseteq 1_{\text{Dst } f}^{\text{RLD}} \Leftrightarrow \\
& f \text{ is monovalued.}
\end{aligned}$$

□

Continuous morphisms

This chapter uses the apparatus from the section “Partially ordered dagger categories”.

11.1. Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \sqsubseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

11.1.1. Pretopology. Let (A, cl_A) and (B, cl_B) be preclosure spaces. Then by definition a function $f : A \rightarrow B$ is continuous iff $f \text{cl}_A(X) \subseteq \text{cl}_B(fX)$ for every $X \in \mathcal{P}A$. Let now μ and ν be endofuncoids corresponding correspondingly to cl_A and cl_B . Then the condition for continuity can be rewritten as

$$\uparrow^{\text{FCD}(\text{Ob } \mu, \text{Ob } \nu)} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\text{FCD}(\text{Ob } \mu, \text{Ob } \nu)} f.$$

11.1.2. Proximity spaces. Let μ and ν be proximity spaces (which I consider a special case of endofuncoids). By definition a **Set**-morphism f is a proximity-continuous map from μ to ν iff

$$\forall X, Y \in \mathcal{T}(\text{Ob } \mu) : (X [\mu]^* Y \Rightarrow \langle f \rangle^* X [\nu]^* \langle f \rangle^* Y).$$

Equivalently transforming this formula we get

$$\begin{aligned} \forall X, Y \in \mathcal{T}(\text{Ob } \mu) : (X [\mu]^* Y \Rightarrow \langle f \rangle \uparrow X [\nu] \langle f \rangle \uparrow Y); \\ \forall X, Y \in \mathcal{T}(\text{Ob } \mu) : (X [\mu]^* Y \Rightarrow \uparrow X [f^{-1} \circ \nu \circ f] \uparrow Y); \\ \forall X, Y \in \mathcal{T}(\text{Ob } \mu) : (X [\mu]^* Y \Rightarrow X [f^{-1} \circ \nu \circ f]^* Y); \\ \mu \sqsubseteq f^{-1} \circ \nu \circ f. \end{aligned}$$

So a function f is proximity continuous iff $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

11.1.3. Uniform spaces. Uniform spaces are a special case of endoreloids. Let μ and ν be uniform spaces. By definition a **Set**-morphism f is a uniformly continuous map from μ to ν iff

$$\forall \varepsilon \in \text{up } \nu \exists \delta \in \text{up } \mu \forall (x, y) \in \delta : (fx, fy) \in \varepsilon.$$

Equivalently transforming this formula we get:

$$\begin{aligned} \forall \varepsilon \in \text{up } \nu \exists \delta \in \text{up } \mu \forall (x, y) \in \delta : \{(fx, fy)\} \subseteq \varepsilon; \\ \forall \varepsilon \in \text{up } \nu \exists \delta \in \text{up } \mu \forall (x, y) \in \delta : f \circ \{(x, y)\} \circ f^{-1} \subseteq \varepsilon; \\ \forall \varepsilon \in \text{up } \nu \exists \delta \in \text{up } \mu : f \circ \delta \circ f^{-1} \subseteq \varepsilon; \\ \forall \varepsilon \in \text{up } \nu : \uparrow^{\text{RLD}(\text{Ob } \mu, \text{Ob } \nu)} f \circ \mu \circ (\uparrow^{\text{RLD}(\text{Ob } \mu, \text{Ob } \nu)} f)^{-1} \sqsubseteq \uparrow^{\text{RLD}(\text{Ob } \mu, \text{Ob } \nu)} \varepsilon; \\ \uparrow^{\text{RLD}(\text{Ob } \mu, \text{Ob } \nu)} f \circ \mu \circ (\uparrow^{\text{RLD}(\text{Ob } \mu, \text{Ob } \nu)} f)^{-1} \sqsubseteq \nu. \end{aligned}$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.

11.2. Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν be endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in C(\mu, \nu) \Leftrightarrow f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu) \wedge f \circ \mu \sqsubseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f \in C'(\mu, \nu) \Leftrightarrow f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu) \wedge \mu \sqsubseteq f^\dagger \circ \nu \circ f;$$

$$f \in C''(\mu, \nu) \Leftrightarrow f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu) \wedge f \circ \mu \circ f^\dagger \sqsubseteq \nu.$$

REMARK 1169. In the examples (above) about functors and relicts the “dagger functor” is the reverse of a functor or relict, that is $f^\dagger = f^{-1}$.

PROPOSITION 1170. Every of these three definitions of continuity forms a wide sub-precategory (wide subcategory if the original precategory is a category).

PROOF.

C. Let $f \in C(\mu, \nu)$, $g \in C(\nu, \pi)$. Then $f \circ \mu \sqsubseteq \nu \circ f$, $g \circ \nu \sqsubseteq \pi \circ g$, $g \circ f \circ \mu \sqsubseteq g \circ \nu \circ f \sqsubseteq \pi \circ g \circ f$. So $g \circ f \in C(\mu, \pi)$. $1_{\text{Ob } \mu} \in C(\mu, \mu)$ is obvious.

C'. Let $f \in C'(\mu, \nu)$, $g \in C'(\nu, \pi)$. Then $\mu \sqsubseteq f^\dagger \circ \nu \circ f$, $\nu \sqsubseteq g^\dagger \circ \pi \circ g$;

$$\mu \sqsubseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \sqsubseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in C'(\mu, \pi)$. $1_{\text{Ob } \mu} \in C'(\mu, \mu)$ is obvious.

C''. Let $f \in C''(\mu, \nu)$, $g \in C''(\nu, \pi)$. Then $f \circ \mu \circ f^\dagger \sqsubseteq \nu$, $g \circ \nu \circ g^\dagger \sqsubseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \sqsubseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \sqsubseteq \pi.$$

So $g \circ f \in C''(\mu, \pi)$. $1_{\text{Ob } \mu} \in C''(\mu, \mu)$ is obvious. □

PROPOSITION 1171. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in C'(\mu, \nu) \Rightarrow f \in C(\mu, \nu) \Rightarrow f \in C''(\mu, \nu).$$

PROOF. Let $f \in C'(\mu, \nu)$. Then $\mu \sqsubseteq f^\dagger \circ \nu \circ f$;

$$f \circ \mu \sqsubseteq f \circ f^\dagger \circ \nu \circ f \sqsubseteq 1_{\text{Dst } f} \circ \nu \circ f = \nu \circ f; \quad f \in C(\mu, \nu).$$

Let $f \in C(\mu, \nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$;

$$f \circ \mu \circ f^\dagger \sqsubseteq \nu \circ f \circ f^\dagger \sqsubseteq \nu \circ 1_{\text{Dst } f} = \nu; \quad f \in C''(\mu, \nu). \quad \square$$

PROPOSITION 1172. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in C''(\mu, \nu) \Rightarrow f \in C(\mu, \nu) \Rightarrow f \in C'(\mu, \nu).$$

PROOF. Let $f \in C''(\mu, \nu)$. Then $f \circ \mu \circ f^\dagger \sqsubseteq \nu$; $f \circ \mu \circ f^\dagger \circ f \sqsubseteq \nu \circ f$; $f \circ \mu \circ 1_{\text{Src } f} \sqsubseteq \nu \circ f$; $f \circ \mu \sqsubseteq \nu \circ f$; $f \in C(\mu, \nu)$.

Let $f \in C(\mu, \nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$; $f^\dagger \circ f \circ \mu \sqsubseteq f^\dagger \circ \nu \circ f$; $1_{\text{Src } \mu} \circ \mu \sqsubseteq f^\dagger \circ \nu \circ f$; $\mu \sqsubseteq f^\dagger \circ \nu \circ f$; $f \in C'(\mu, \nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

THEOREM 1173. If f is a monovalued and entirely defined morphism of a partially ordered dagger precategory then

$$f \in C'(\mu, \nu) \Leftrightarrow f \in C(\mu, \nu) \Leftrightarrow f \in C''(\mu, \nu).$$

PROOF. From two previous propositions. \square

The classical general topology theorem that uniformly continuous function from a uniform space to another uniform space is proximity-continuous regarding the proximities generated by the uniformities, generalized for reroids and funcoids takes the following form:

THEOREM 1174. If an entirely defined morphism of the category of reroids $f \in C''(\mu, \nu)$ for some endomorphisms μ and ν of the category of reroids, then $(\text{FCD})f \in C'((\text{FCD})\mu, (\text{FCD})\nu)$.

EXERCISE 1175. I leave a simple exercise for the reader to prove the last theorem.

THEOREM 1176. Let μ and ν be endomorphisms of some partially ordered dagger precategory and $f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu)$ be a monovalued, entirely defined morphism. Then

$$f \in C(\mu, \nu) \Leftrightarrow f \in C(\mu^\dagger, \nu^\dagger).$$

PROOF.

$$\begin{aligned} f \circ \mu \sqsubseteq \nu \circ f &\Leftrightarrow \mu \sqsubseteq f^\dagger \circ \nu \circ f \Rightarrow \\ &\mu \circ f^\dagger \sqsubseteq f^\dagger \circ \nu \circ f \circ f^\dagger \Rightarrow \mu \circ f^\dagger \sqsubseteq f^\dagger \circ \nu \Leftrightarrow \\ f \circ \mu^\dagger \sqsubseteq \nu^\dagger \circ f &\Rightarrow f^\dagger \circ f \circ \mu^\dagger \sqsubseteq f^\dagger \circ \nu^\dagger \circ f \Rightarrow \\ &\mu^\dagger \sqsubseteq f^\dagger \circ \nu^\dagger \circ f \Leftrightarrow \mu \sqsubseteq f^\dagger \circ \nu \circ f. \end{aligned}$$

Thus $f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow \mu \Leftrightarrow f \circ \mu^\dagger \sqsubseteq \nu^\dagger \circ f$. \square

11.3. Continuity for topological spaces

PROPOSITION 1177. The following are pairwise equivalent for funcoids μ, ν and a monovalued, entirely defined morphism $f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu)$:

- 1°. $\forall A \in \mathcal{T} \text{Ob } \mu, B \in \text{up}\langle \nu \rangle \langle f \rangle^* A : \langle f^{-1} \rangle^* B \in \text{up}\langle \mu \rangle^* A$.
- 2°. $f \in C(\mu, \nu)$.
- 3°. $f \in C(\mu^{-1}, \nu^{-1})$.

PROOF.

2° \Leftrightarrow **3°**. By general $f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow f \circ \mu^\dagger \sqsubseteq \nu^\dagger \circ f$ formula above.

1° \Leftrightarrow **2°**. **1°** is equivalent to $\langle \langle f^{-1} \rangle^* \rangle^* \text{up}\langle \nu \rangle \langle f \rangle^* A \subseteq \text{up}\langle \mu \rangle^* A$ equivalent to $\langle \nu \rangle \langle f \rangle^* A \supseteq \langle f \rangle \langle \mu \rangle^* A$ (used ‘‘Orderings of filters’’ chapter). \square

COROLLARY 1178. The following are pairwise equivalent for topological spaces μ, ν and a monovalued, entirely defined morphism $f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu)$:

- 1°. $\forall x \in \text{Ob } \mu, B \in \text{up}\langle \nu \rangle \langle f \rangle^* \{x\} : \langle f^{-1} \rangle^* B \in \text{up}\langle \mu \rangle^* \{x\}$.
- 2°. Preimages (by f) of open sets are open.
- 3°. $f \in C(\mu, \nu)$ that is $\langle f \rangle \langle \mu \rangle^* \{x\} \sqsubseteq \langle \nu \rangle \langle f \rangle^* \{x\}$ for every $x \in \text{Ob } \mu$.
- 4°. $f \in C(\mu^{-1}, \nu^{-1})$ that is $\langle f \rangle \langle \mu^{-1} \rangle^* A \sqsubseteq \langle \nu^{-1} \rangle \langle f \rangle^* A$ for every $A \in \mathcal{T} \text{Ob } \mu$.

PROOF. 2° from the previous proposition is equivalent to $\langle f \rangle \langle \mu \rangle^* \{x\} \sqsubseteq \langle \nu \rangle \langle f \rangle^* \{x\}$ equivalent to $\langle \langle f^{-1} \rangle^* \rangle^* \text{up} \langle \nu \rangle \langle f \rangle^* \{x\} \subseteq \text{up} \langle \mu \rangle^* \{x\}$ for every $x \in \text{Ob } \mu$, equivalent to 1° (used “Orderings of filters” chapter).

It remains to prove $3^\circ \Leftrightarrow 2^\circ$.

$3^\circ \Rightarrow 2^\circ$. Let B be an open set in ν . For every $x \in \langle f^{-1} \rangle^* B$ we have $f(x) \in B$ that is B is a neighborhood of $f(x)$, thus $\langle f^{-1} \rangle^* B$ is a neighborhood of x . We have proved that $\langle f^{-1} \rangle^* B$ is open.

$2^\circ \Rightarrow 3^\circ$. Let B be a neighborhood of $f(x)$. Then there is an open neighborhood $B' \subseteq B$ of $f(x)$. $\langle f^{-1} \rangle^* B'$ is open and thus is a neighborhood of x ($x \in \langle f^{-1} \rangle^* B'$ because $f(x) \in B'$). Consequently $\langle f^{-1} \rangle^* B$ is a neighborhood of x .

Alternative proof of $2^\circ \Leftrightarrow 4^\circ$: <http://math.stackexchange.com/a/1855782/4876> \square

11.4. $C(\mu \circ \mu^{-1}, \nu \circ \nu^{-1})$

PROPOSITION 1179. $f \in C(\mu, \nu) \Rightarrow f \in C''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1})$ for endofunctors μ, ν and monovalued functor $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$.

PROOF. Let $f \in C(\mu, \nu)$.

$$\begin{aligned} X [f \circ \mu \circ \mu^{-1} \circ f^{-1}]^* Z &\Leftrightarrow \\ \exists p \in \text{atoms}^{\mathcal{F}} : (X [\mu^{-1} \circ f^{-1}]^* p \wedge p [f \circ \mu]^* Z) &\Leftrightarrow \\ \exists p \in \text{atoms}^{\mathcal{F}} : (p [f \circ \mu]^* X \wedge p [f \circ \mu]^* Z) &\Rightarrow \\ \exists p \in \text{atoms}^{\mathcal{F}} : (p [\nu \circ f]^* X \wedge p [\nu \circ f]^* Z) &\Leftrightarrow \\ \exists p \in \text{atoms}^{\mathcal{F}} : (\langle f \rangle^* p [\nu]^* X \wedge \langle f \rangle^* p [\nu]^* Z) &\Rightarrow X [\nu \circ \nu^{-1}]^* Z \end{aligned}$$

(taken into account monovaluedness of f and thus that $\langle f \rangle^* p$ is atomic or least). Thus $f \circ \mu \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1}$ that is $f \in C''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1})$. \square

PROPOSITION 1180. $f \in C''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) \Rightarrow f \in C''(\mu, \nu)$ for complete endofunctors μ, ν and principal functor $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$, provided that μ is reflexive, and ν is T_1 -separable.

PROOF.

$$\begin{aligned} f \in C''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) &\Leftrightarrow \\ f \circ \mu \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1} &\Rightarrow (\text{reflexivity of } \mu) \Rightarrow \\ f \circ \mu \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1} &\Leftrightarrow f \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1} \Rightarrow \\ \langle f \circ \mu^{-1} \circ f^{-1} \rangle^* X \sqsubseteq \langle \nu \rangle^* \langle \nu^{-1} \rangle^* X &\Rightarrow \\ \text{Cor} \langle f \circ \mu^{-1} \circ f^{-1} \rangle^* X \sqsubseteq \text{Cor} \langle \nu \rangle^* \langle \nu^{-1} \rangle^* X &\Leftrightarrow \\ \langle f \circ \mu^{-1} \circ f^{-1} \rangle^* X \sqsubseteq \text{Cor} \langle \nu \rangle^* \langle \nu^{-1} \rangle^* X &\Rightarrow \\ (T_1\text{-separability}) &\Rightarrow \\ \langle f \circ \mu^{-1} \circ f^{-1} \rangle^* X \sqsubseteq \langle \nu^{-1} \rangle^* X &\text{ for any typed set } X \text{ on } \text{Ob } \nu. \end{aligned}$$

Thus

$$\begin{aligned} f \in C''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) &\Rightarrow f \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu^{-1} \Leftrightarrow \\ f \circ \mu \circ f^{-1} \sqsubseteq \nu &\Leftrightarrow f \in C''(\mu, \nu). \end{aligned}$$

\square

THEOREM 1181. $f \in C(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) \Leftrightarrow f \in C(\mu, \nu)$ for complete endofuncoids μ, ν and principal monovalued and entirely defined funcoid $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$, provided that μ is reflexive, and ν is T_1 -separable.

PROOF. Two above propositions and theorem 1173. \square

11.5. Continuity of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids on some set regarding the composition.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to so called *restricted identity* element I_A of the semigroup (for example restricted identity funcoid or restricted identity reloid). For identity elements we will require

- 1°. $I_A \circ I_B = I_{A \sqcap B}$;
- 2°. $f \circ I_A \sqsubseteq f$; $I_A \circ f \sqsubseteq f$.

In the case when our semigroup is “dagger” (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define restricting an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element f of our semigroup to objects A and B as $I_B \circ f \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$f \sqcap (A \times B) = I_B \circ f \circ I_A.$$

Square restricting of an element f to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ f \circ I_A$ (or by the formula $f \sqcap (A \times A)$).

THEOREM 1182. For every elements f, μ, ν of our semigroup and an object A

- 1°. $f \in C(\mu, \nu) \Rightarrow f|_A \in C(I_A \circ \mu \circ I_A, \nu)$;
- 2°. $f \in C'(\mu, \nu) \Rightarrow f|_A \in C'(I_A \circ \mu \circ I_A, \nu)$;
- 3°. $f \in C''(\mu, \nu) \Rightarrow f|_A \in C''(I_A \circ \mu \circ I_A, \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

PROOF.

1°.

$$\begin{aligned} f|_A \in C(I_A \circ \mu \circ I_A, \nu) &\Leftrightarrow \\ f|_A \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f|_A &\Leftrightarrow \\ f \circ I_A \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f|_A &\Leftrightarrow \\ f \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f \circ I_A &\Leftrightarrow \\ f \circ I_A \circ \mu \sqsubseteq \nu \circ f &\Leftrightarrow \\ f \circ \mu \sqsubseteq \nu \circ f &\Leftrightarrow \\ f \in C(\mu, \nu). & \end{aligned}$$

2°.

$$\begin{aligned}
f|_A \in C'(I_A \circ \mu \circ I_A, \nu) &\Leftrightarrow \\
I_A \circ \mu \circ I_A \sqsubseteq (f|_A)^\dagger \circ \nu \circ f|_A &\Leftarrow \\
I_A \circ \mu \circ I_A \sqsubseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A &\Leftrightarrow \\
I_A \circ \mu \circ I_A \sqsubseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A &\Leftarrow \\
\mu \sqsubseteq f^\dagger \circ \nu \circ f &\Leftrightarrow \\
f \in C'(\mu, \nu). &
\end{aligned}$$

3°.

$$\begin{aligned}
f|_A \in C''(I_A \circ \mu \circ I_A, \nu) &\Leftrightarrow \\
f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \sqsubseteq \nu &\Leftrightarrow \\
f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \sqsubseteq \nu &\Leftrightarrow \\
f \circ I_A \circ \mu \circ I_A \circ f^\dagger \sqsubseteq \nu &\Leftarrow \\
f \circ \mu \circ f^\dagger \sqsubseteq \nu &\Leftrightarrow \\
f \in C''(\mu, \nu). &
\end{aligned}$$

□

11.6. Anticontinuous morphisms

Let μ and ν be endomorphisms of some partially ordered precategory. *Anticontinuous* functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in C_*(\mu, \nu) \Leftrightarrow f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu) \wedge f \circ \mu \supseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then *anticontinuity* also can be defined in two other ways:

$$\begin{aligned}
f \in C'_*(\mu, \nu) &\Leftrightarrow f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu) \wedge \mu \supseteq f^\dagger \circ \nu \circ f; \\
f \in C''_*(\mu, \nu) &\Leftrightarrow f \in \text{Hom}(\text{Ob } \mu, \text{Ob } \nu) \wedge f \circ \mu \circ f^\dagger \supseteq \nu.
\end{aligned}$$

Anticontinuity is the order dual of continuity.

THEOREM 1183. For partially ordered dagger categories:

- 1°. $f \in C_*(\mu, \nu) \Leftrightarrow f^\dagger \in C(\nu^\dagger, \mu^\dagger)$;
- 2°. $f \in C'_*(\mu, \nu) \Leftrightarrow f^\dagger \in C''(\nu^\dagger, \mu^\dagger)$;
- 3°. $f \in C''_*(\mu, \nu) \Leftrightarrow f^\dagger \in C'(\nu^\dagger, \mu^\dagger)$.

PROOF.

- 1°. $f \in C_*(f, g) \Leftrightarrow f \circ \mu \supseteq \nu \circ f \Leftrightarrow \mu^\dagger \circ f^\dagger \supseteq f^\dagger \circ \nu^\dagger \Leftrightarrow f^\dagger \in C(\nu^\dagger, \mu^\dagger)$.
- 2°. $f \in C'_*(\mu, \nu) \Leftrightarrow \mu \supseteq f^\dagger \circ \nu \circ f \Leftrightarrow f^\dagger \circ \nu^\dagger \circ f \sqsubseteq \mu^\dagger \Leftrightarrow f^\dagger \in C''(\nu^\dagger, \mu^\dagger)$.
- 3°. By duality.

□

DEFINITION 1184. An *open map* from a topological space to a topological space is a function which maps open sets into open sets.

THEOREM 1185. For topological spaces considered as complete funcoids, a principal anticontinuous morphism is the same as open map.

PROOF. Because f, μ, ν are complete functors, we have

$$f \in C_*(\mu, \nu) \Leftrightarrow f \circ \mu \supseteq \nu \circ f \Leftrightarrow \text{Compl}(f \circ \mu) \supseteq \text{Compl}(\nu \circ f).$$

Equivalently transforming further, we get

$$\begin{aligned} \forall x \in \text{Ob } \mu : \langle f \rangle \langle \mu \rangle^* @ \{x\} &\supseteq \langle \nu \rangle \langle f \rangle^* @ \{x\}; \\ \forall x \in \text{Ob } \mu, V \in \langle \mu \rangle^* \{x\} : \langle f \rangle^* V &\supseteq \langle \nu \rangle \langle f \rangle^* @ \{x\}, \end{aligned}$$

what is the criterion of f being an open map. □

Connectedness regarding functors and relicts

12.1. Some lemmas

LEMMA 1186. Let U be a set, $A, B \in \mathcal{T}U$ be typed sets, f be an endo-functor on U . If $\neg(A [f]^* B) \wedge A \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f)$ then f is closed on A .

PROOF. Let $A \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f)$.

$$\begin{aligned} \neg(A [f]^* B) &\Leftrightarrow \\ B \sqcap \langle f \rangle^* A &= \perp \Rightarrow \\ (\text{dom } f \sqcup \text{im } f) \sqcap B \sqcap \langle f \rangle^* A &= \perp \Rightarrow \\ ((\text{dom } f \sqcup \text{im } f) \setminus A) \sqcap \langle f \rangle^* A &= \perp \Leftrightarrow \\ \langle f \rangle^* A &\subseteq A. \end{aligned}$$

□

COROLLARY 1187. If $\neg(A [f]^* B) \wedge A \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f)$ then f is closed on $A \setminus B$ for a functor $f \in \text{FCD}(U, U)$ for every sets U and typed sets $A, B \in \mathcal{T}U$.

PROOF. Let $\neg(A [f]^* B) \wedge A \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f)$. Then

$$\neg((A \setminus B) [f]^* B) \wedge (A \setminus B) \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f).$$

□

LEMMA 1188. If $\neg(A [f]^* B) \wedge A \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f)$ then $\neg(A [f^n]^* B)$ for every whole positive n .

PROOF. Let $\neg(A [f]^* B) \wedge A \sqcup B \in \text{up}(\text{dom } f \sqcup \text{im } f)$. From the above lemma $\langle f \rangle^* A \subseteq A$. $B \sqcap \langle f \rangle^* A = \perp$, consequently $\langle f \rangle^* A \subseteq A \setminus B$. Because (by the above corollary) f is closed on $A \setminus B$, then $\langle f \rangle \langle f \rangle^* A \subseteq A \setminus B$, $\langle f \rangle \langle f \rangle \langle f \rangle^* A \subseteq A \setminus B$, etc. So $\langle f^n \rangle^* A \subseteq A \setminus B$, $B \simeq \langle f^n \rangle^* A$, $\neg(A [f^n]^* B)$. □

12.2. Endomorphism series

DEFINITION 1189. $S_1(\mu) = \mu \sqcup \mu^2 \sqcup \mu^3 \sqcup \dots$ for an endomorphism μ of a pre-category with countable join of morphisms (that is join defined for every countable set of morphisms).

DEFINITION 1190. $S(\mu) = \mu^0 \sqcup S_1(\mu) = \mu^0 \sqcup \mu \sqcup \mu^2 \sqcup \mu^3 \sqcup \dots$ where $\mu^0 = 1_{\text{Ob } \mu}$ (identity morphism for the object $\text{Ob } \mu$) where $\text{Ob } \mu$ is the object of endomorphism μ for an endomorphism μ of a category with countable join of morphisms.

I call S_1 and S *endomorphism series*.

PROPOSITION 1191. The relation $S(\mu)$ is transitive for the category **Rel**.

PROOF.

$$\begin{aligned} S(\mu) \circ S(\mu) &= \mu^0 \sqcup S(\mu) \sqcup \mu \circ S(\mu) \sqcup \mu^2 \circ S(\mu) \sqcup \dots = \\ &(\mu^0 \sqcup \mu^1 \sqcup \mu^2 \sqcup \dots) \sqcup (\mu^1 \sqcup \mu^2 \sqcup \mu^3 \sqcup \dots) \sqcup (\mu^2 \sqcup \mu^3 \sqcup \mu^4 \sqcup \dots) = \\ &\mu^0 \sqcup \mu^1 \sqcup \mu^2 \sqcup \dots = S(\mu). \end{aligned}$$

□

12.3. Connectedness regarding binary relations

Before going to research connectedness for functors and relicts we will excursion into the basic special case of connectedness regarding binary relations on a set \mathcal{U} .

This is commonly studied in “graph theory” courses. *Digraph* as commonly defined is essentially the same as an endomorphism of the category **Rel**.

DEFINITION 1192. A set A is called (*strongly*) *connected* regarding a binary relation μ on U when

$$\forall X, Y \in \mathcal{P}U \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X [\mu]^* Y).$$

DEFINITION 1193. A typed set A of type U is called (*strongly*) *connected* regarding a **Rel**-endomorphism μ on U when

$$\forall X, Y \in \mathcal{T}(\text{Ob } \mu) \setminus \{\perp^{\mathcal{T}(\text{Ob } \mu)}\} : (X \sqcup Y = A \Rightarrow X [\mu]^* Y).$$

OBVIOUS 1194. A typed set A is connected regarding **Rel**-endomorphism μ on its type iff $\text{GR } A$ is connected regarding $\text{GR } \mu$.

Let \mathcal{U} be a set.

DEFINITION 1195. *Path* between two elements $a, b \in \mathcal{U}$ in a set $A \subseteq \mathcal{U}$ through binary relation μ is the finite sequence $x_0 \dots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i (\mu \cap A \times A) x_{i+1}$ for every $i = 0, \dots, n - 1$. n is called *path length*.

PROPOSITION 1196. There exists path between every element $a \in \mathcal{U}$ and that element itself.

PROOF. It is the path consisting of one vertex (of length 0). □

PROPOSITION 1197. There is a path from element a to element b in a set A through a binary relation μ iff $a (S(\mu \cap A \times A)) b$ (that is $(a, b) \in S(\mu \cap A \times A)$).

PROOF.

⇒. If a path from a to b exists, then $\{b\} \subseteq \langle (\mu \cap A \times A)^n \rangle^* \{a\}$ where n is the path length. Consequently $\{b\} \subseteq \langle S(\mu \cap A \times A) \rangle^* \{a\}$; $a (S(\mu \cap A \times A)) b$.

⇐. If $a (S(\mu \cap A \times A)) b$ then there exists $n \in \mathbb{N}$ such that $a (\mu \cap A \times A)^n b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \dots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i (\mu \cap A \times A) x_{i+1}$ for every $i = 0, \dots, n - 1$. That is there is a path from a to b . □

PROPOSITION 1198. There is a path from element a to element b in a set A through a binary relation μ iff $a (S_1(\mu \cap A \times A)) b$ (that is $(a, b) \in S_1(\mu \cap A \times A)$).

PROOF. Similar to the previous proof. □

THEOREM 1199. The following statements are equivalent for a binary relation μ and a set A :

- 1°. For every $a, b \in A$ there is a nonzero-length path between a and b in A through μ .

- 2°. $S_1(\mu \cap (A \times A)) \supseteq A \times A$.
 3°. $S_1(\mu \cap (A \times A)) = A \times A$.
 4°. A is connected regarding μ .

PROOF.

- 1° \Rightarrow 2°. Let for every $a, b \in A$ there is a nonzero-length path between a and b in A through μ . Then $a (S_1(\mu \cap A \times A)) b$ for every $a, b \in A$. It is possible only when $S_1(\mu \cap (A \times A)) \supseteq A \times A$.
 3° \Rightarrow 1°. For every two vertices a and b we have $a (S_1(\mu \cap A \times A)) b$. So (by the previous) for every two vertices a and b there exists a nonzero-length path from a to b .
 3° \Rightarrow 4°. Suppose $\neg(X [\mu \cap (A \times A)]^* Y)$ for some $X, Y \in \mathcal{P}\mathcal{U} \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg(X [(\mu \cap (A \times A))^n]^* Y)$ for every $n \in \mathbb{Z}_+$. Consequently $\neg(X [S_1(\mu \cap (A \times A))]^* Y)$. So $S_1(\mu \cap (A \times A)) \neq A \times A$.
 4° \Rightarrow 3°. If $\langle S_1(\mu \cap (A \times A)) \rangle^* \{v\} = A$ for every vertex v then $S_1(\mu \cap (A \times A)) = A \times A$. Consider the remaining case when $V \stackrel{\text{def}}{=} \langle S_1(\mu \cap (A \times A)) \rangle^* \{v\} \subset A$ for some vertex v . Let $W = A \setminus V$. If $\text{card } A = 1$ then $S_1(\mu \cap (A \times A)) \supseteq \text{id}_A = A \times A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V [\mu]^* W$ what is equivalent to $V [\mu \cap (A \times A)]^* W$ that is $\langle \mu \cap (A \times A) \rangle^* V \cap W \neq \emptyset$. This is impossible because

$$\begin{aligned} \langle \mu \cap (A \times A) \rangle^* V &= \langle \mu \cap (A \times A) \rangle^* \langle S_1(\mu \cap (A \times A)) \rangle^* V = \\ &= \langle (\mu \cap (A \times A))^2 \cup (\mu \cap (A \times A))^3 \cup \dots \cup \rangle^* V \subseteq \langle S_1(\mu \cap (A \times A)) \rangle^* V = V. \end{aligned}$$

- 2° \Rightarrow 3°. Because $S_1(\mu \cap (A \times A)) \subseteq A \times A$. □

COROLLARY 1200. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap (A \times A)$.

DEFINITION 1201. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A .

THEOREM 1202. The set A is partitioned into connected components (regarding every binary relation F).

PROOF. Consider the binary relation $a \sim b \Leftrightarrow a (S(F)) b \wedge b (S(F)) a$. \sim is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q . Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. □

PROPOSITION 1203. A set is connected (regarding a binary relation) iff it has one connected component.

PROOF. Direct implication is obvious. Reverse is proved by contradiction. □

12.4. Connectedness regarding funcoids and reloids

DEFINITION 1204. *Connectivity reloid* $S_1^*(\mu) = \prod_{M \in \text{up } \mu}^{\text{RLD}} S_1(M)$ for an endoreloid μ .

DEFINITION 1205. $S^*(\mu)$ for an endoreloid μ is defined as follows:

$$S^*(\mu) = \prod_{M \in \text{up } \mu}^{\text{RLD}} S(M).$$

Do not mess the word *connectivity* with the word *connectedness* which means being connected.¹

PROPOSITION 1206. $S^*(\mu) = 1_{\text{Ob } \mu}^{\text{RLD}} \sqcup S_1^*(\mu)$ for every endoreloid μ .

PROOF. By the proposition 610. □

PROPOSITION 1207. $S^*(\mu) = S(\mu)$ and $S_1^*(\mu) = S_1(\mu)$ if μ is a principal reloid.

PROOF. $S^*(\mu) = \prod\{S(\mu)\} = S(\mu)$; $S_1^*(\mu) = \prod\{S_1(\mu)\} = S_1(\mu)$. □

DEFINITION 1208. A filter $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$ is called *connected* regarding an endoreloid μ when $S_1^*(\mu \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{A})) \supseteq \mathcal{A} \times^{\text{RLD}} \mathcal{A}$.

OBVIOUS 1209. A filter $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$ is connected regarding an endoreloid μ iff $S_1^*(\mu \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{A})) = \mathcal{A} \times^{\text{RLD}} \mathcal{A}$.

DEFINITION 1210. A filter $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$ is called *connected* regarding an endofunctor μ when

$$\forall \mathcal{X}, \mathcal{Y} \in \mathcal{F}(\text{Ob } \mu) \setminus \{\perp^{\mathcal{F}(\text{Ob } \mu)}\} : (\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A} \Rightarrow \mathcal{X} [\mu] \mathcal{Y}).$$

PROPOSITION 1211. Let A be a typed set of type U . The filter $\uparrow A$ is connected regarding an endofunctor μ on U iff

$$\forall X, Y \in \mathcal{F}(\text{Ob } \mu) \setminus \{\perp^{\mathcal{F}(\text{Ob } \mu)}\} : (X \sqcup Y = A \Rightarrow X [\mu]^* Y).$$

PROOF.

\Rightarrow . Obvious.

\Leftarrow . It follows from co-separability of filters. □

THEOREM 1212. The following are equivalent for every typed set A of type U and **Rel**-endomorphism μ on a set U :

- 1°. A is connected regarding μ .
- 2°. $\uparrow A$ is connected regarding $\uparrow^{\text{RLD}} \mu$.
- 3°. $\uparrow A$ is connected regarding $\uparrow^{\text{FCD}} \mu$.

PROOF.

$1^\circ \Leftrightarrow 2^\circ$.

$$\begin{aligned} S_1^*(\uparrow^{\text{RLD}} \mu \sqcap (A \times^{\text{RLD}} A)) &= \\ S_1^*(\uparrow^{\text{RLD}} (\mu \sqcap (A \times A))) &= \\ \uparrow^{\text{RLD}} S_1(\mu \sqcap (A \times A)). & \end{aligned}$$

So

$$\begin{aligned} S_1^*(\uparrow^{\text{RLD}} \mu \sqcap (A \times^{\text{RLD}} A)) \supseteq A \times^{\text{RLD}} A &\Leftrightarrow \\ \uparrow^{\text{RLD}} S_1(\mu \sqcap (A \times A)) \supseteq \uparrow^{\text{RLD}} (A \times A) &= A \times^{\text{RLD}} A. \end{aligned}$$

$1^\circ \Leftrightarrow 3^\circ$. It follows from the previous proposition. □

Next is conjectured a statement more strong than the above theorem:

CONJECTURE 1213. Let \mathcal{A} be a filter on a set U and F be a **Rel**-endomorphism on U .

\mathcal{A} is connected regarding $\uparrow^{\text{FCD}} F$ iff \mathcal{A} is connected regarding $\uparrow^{\text{RLD}} F$.

¹In some math literature these two words are used interchangeably.

OBVIOUS 1214. A filter \mathcal{A} is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \sqcap (\mathcal{A} \times^{\text{RLD}} \mathcal{A})$.

OBVIOUS 1215. A filter \mathcal{A} is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{A})$.

THEOREM 1216. A filter \mathcal{A} is connected regarding a reloid f iff \mathcal{A} is connected regarding every $F \in \langle \uparrow^{\text{RLD}} \rangle^* \text{up } f$.

PROOF.

\Rightarrow . Obvious.

\Leftarrow . \mathcal{A} is connected regarding $\uparrow^{\text{RLD}} F$ iff $S_1(F) = F^1 \sqcup F^2 \sqcup \dots \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{A})$.

$$S_1^*(f) = \prod_{F \in \text{up } f}^{\text{RLD}} S_1(F) \supseteq \prod_{F \in \text{up } f} (\mathcal{A} \times^{\text{RLD}} \mathcal{A}) = \mathcal{A} \times^{\text{RLD}} \mathcal{A}.$$

□

CONJECTURE 1217. A filter \mathcal{A} is connected regarding a funcoid f iff \mathcal{A} is connected regarding every $F \in \langle \uparrow^{\text{FCD}} \rangle^* \text{up } f$.

The above conjecture is open even for the case when \mathcal{A} is a principal filter.

CONJECTURE 1218. A filter \mathcal{A} is connected regarding a reloid f iff it is connected regarding the funcoid $(\text{FCD})f$.

The above conjecture is true in the special case of principal filters:

PROPOSITION 1219. A filter $\uparrow A$ (for a typed set A) is connected regarding an endoreloid f on the suitable object iff it is connected regarding the endofuncoid $(\text{FCD})f$.

PROOF. $\uparrow A$ is connected regarding a reloid f iff A is connected regarding every $F \in \text{up } f$ that is when (taken into account that connectedness for $\uparrow^{\text{RLD}} F$ is the same as connectedness of $\uparrow^{\text{FCD}} F$)

$$\begin{aligned} \forall F \in \text{up } f \forall \mathcal{X}, \mathcal{Y} \in \mathcal{F}(\text{Ob } f) \setminus \{\perp^{\mathcal{F}(\text{Ob } f)}\} : (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y}) &\Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathcal{F}(\text{Ob } f) \setminus \{\perp^{\mathcal{F}(\text{Ob } f)}\} \forall F \in \text{up } f : (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y}) &\Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathcal{F}(\text{Ob } f) \setminus \{\perp^{\mathcal{F}(\text{Ob } f)}\} (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \forall F \in \text{up } f : \mathcal{X} [\uparrow^{\text{FCD}} F] \mathcal{Y}) &\Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathcal{F}(\text{Ob } f) \setminus \{\perp^{\mathcal{F}(\text{Ob } f)}\} (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \mathcal{X} [(\text{FCD})f] \mathcal{Y}) & \end{aligned}$$

that is when the set $\uparrow A$ is connected regarding the funcoid $(\text{FCD})f$. □

CONJECTURE 1220. A set A is connected regarding an endofuncoid μ iff for every $a, b \in A$ there exists a totally ordered set $P \subseteq A$ such that $\min P = a$, $\max P = b$ and

$$\forall q \in P \setminus \{b\} : \left\{ \frac{x \in P}{x \leq q} \right\} [\mu]^* \left\{ \frac{x \in P}{x > q} \right\}.$$

Weaker condition:

$$\forall q \in P \setminus \{b\} : \left\{ \frac{x \in P}{x \leq q} \right\} [\mu]^* \left\{ \frac{x \in P}{x > q} \right\} \vee \forall q \in P \setminus \{a\} : \left\{ \frac{x \in P}{x < q} \right\} [\mu]^* \left\{ \frac{x \in P}{x \geq q} \right\}.$$

12.5. Algebraic properties of S and S^*

THEOREM 1221. $S^*(S^*(f)) = S^*(f)$ for every endoreloid f .

PROOF.

$$S^*(S^*(f)) = \prod_{R \in \text{up } S^*(f)}^{\text{RLD}} S(R) \sqsubseteq \prod_{R \in \left\{ \frac{S(F)}{F \in \text{up } f} \right\}}^{\text{RLD}} S(R) = \prod_{R \in \text{up } f}^{\text{RLD}} S(S(R)) = \prod_{R \in \text{up } f}^{\text{RLD}} S(R) = S^*(f).$$

So $S^*(S^*(f)) \sqsubseteq S^*(f)$. That $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. \square

COROLLARY 1222. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for every endoreloid f .

PROOF. Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.

But $S^*(S(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$. \square

CONJECTURE 1223. $S(S(f)) = S(f)$ for

1 $^\circ$. every endoreloid f ;

2 $^\circ$. every endofuncoïd f .

CONJECTURE 1224. $S(f) \circ S(f) = S(f)$ for every endoreloid f .

THEOREM 1225. $S^*(f) \circ S^*(f) = S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$ for every endoreloid f .

PROOF. ²

It is enough to prove $S^*(f) \circ S^*(f) = S^*(f)$ because $S^*(f) \sqsubseteq S(f) \circ S^*(f) \sqsubseteq S^*(f) \circ S^*(f)$ and likewise for $S^*(f) \circ S(f)$.

$$S^*(\mu) \circ S^*(\mu) = \prod_{F \in \text{up } S^*(\mu)}^{\text{RLD}} (F \circ F) = (\text{see below}) = \prod_{X \in \text{up } \mu}^{\text{RLD}} (S(X) \circ S(X)) = \prod_{X \in \text{up } \mu}^{\text{RLD}} S(X) = S^*(\mu).$$

$$F \in \text{up } S^*(\mu) \Leftrightarrow F \in \text{up } \prod_{F \in \text{up } \mu}^{\mathcal{F}} S(F) \Rightarrow$$

(by properties of filter bases) $\Rightarrow \exists X \in \text{up } \mu : F \supseteq S(X) \Rightarrow$
 $\exists X \in \text{up } \mu : F \circ F \supseteq S(X) \circ S(X)$

thus

$$\prod_{F \in \text{up } S^*(\mu)}^{\text{RLD}} F \circ F \supseteq \prod_{X \in \text{up } \mu}^{\text{RLD}} (S(X) \circ S(X));$$

$X \in \text{up } \mu \Rightarrow S(X) \in \text{up } S^*(\mu) \Rightarrow \exists F \in \text{up } S^*(\mu) : S(X) \circ S(X) \supseteq F \circ F$ thus

$$\prod_{F \in \text{up } S^*(\mu)}^{\text{RLD}} F \circ F \sqsubseteq \prod_{X \in \text{up } \mu}^{\text{RLD}} (S(X) \circ S(X)).$$

\square

CONJECTURE 1226. $S(f) \circ S(f) = S(f)$ for every endofuncoïd f .

²Can be more succintly proved considering $\mu \mapsto S^*(\mu)$ as a pointfree funcoïd?

12.6. Irreflexive reloids

DEFINITION 1227. Endoreloid f is irreflexive iff $f \simeq 1^{\text{Ob}} f$.

PROPOSITION 1228. Endoreloid f is irreflexive iff $f \sqsubseteq \top \setminus 1$.

PROOF. By theorem 604. □

OBVIOUS 1229. $f \setminus 1$ is an irreflexive endoreloid if f is an endoreloid.

PROPOSITION 1230. $S(f) = S(f \sqcup 1)$ if f is an endoreloid, endofunctor, or endorelation.

PROOF. First prove $(f \sqcup 1)^n = 1 \sqcup f \sqcup \dots \sqcup f^n$ for $n \in \mathbb{N}$. For $n = 0$ it's obvious. By induction we have

$$\begin{aligned} (f \sqcup 1)^{n+1} &= \\ (f \sqcup 1)^n \circ (f \sqcup 1) &= \\ (1 \sqcup f \sqcup \dots \sqcup f^n) \circ (f \sqcup 1) &= \\ (f \sqcup f^2 \sqcup \dots \sqcup f^{n+1}) \sqcup (1 \sqcup f \sqcup \dots \sqcup f^n) &= \\ 1 \sqcup f \sqcup \dots \sqcup f^{n+1}. & \end{aligned}$$

So $S(f \sqcup 1) = 1 \sqcup (1 \sqcup f) \sqcup (1 \sqcup f \sqcup f^2) \sqcup \dots = 1 \sqcup f \sqcup f^2 \sqcup \dots = S(f)$. □

COROLLARY 1231. $S(f) = S(f \sqcup 1) = S(f \setminus 1)$ if f is an endoreloid (or just an endorelation).

PROOF. $S(f \setminus 1) = S((f \setminus 1) \sqcup 1) \supseteq S(f)$. But $S(f \setminus 1) \sqsubseteq S(f)$ is obvious. So $S(f \setminus 1) = S(f)$. □

12.7. Micronization

“Micronization” was a thoroughly wrong idea with several errors in the proofs. This section is removed from the book.

Total boundness of reloids

13.1. Thick binary relations

DEFINITION 1232. I will call α -*thick* and denote $\text{thick}_\alpha(E)$ a **Rel**-endomorphism E when there exists a finite cover S of $\text{Ob } E$ such that $\forall A \in S : A \times A \subseteq \text{GR } E$.

DEFINITION 1233. $\text{CS}(S) = \bigcup \left\{ \frac{A \times A}{A \in S} \right\}$ for a collection S of sets.

REMARK 1234. CS means “Cartesian squares”.

OBVIOUS 1235. A **Rel**-endomorphism is α -thick iff there exists a finite cover S of $\text{Ob } E$ such that $\text{CS}(S) \subseteq \text{GR } E$.

DEFINITION 1236. I will call β -*thick* and denote $\text{thick}_\beta(E)$ a **Rel**-endomorphism E when there exists a finite set B such that $\langle \text{GR } E \rangle^* B = \text{Ob } E$.

PROPOSITION 1237. $\text{thick}_\alpha(E) \Rightarrow \text{thick}_\beta(E)$.

PROOF. Let $\text{thick}_\alpha(E)$. Then there exists a finite cover S of the set $\text{Ob } E$ such that $\forall A \in S : A \times A \subseteq \text{GR } E$. Without loss of generality assume $A \neq \emptyset$ for every $A \in S$. So $A \subseteq \langle \text{GR } E \rangle^* \{x_A\}$ for some x_A for every $A \in S$. So

$$\langle \text{GR } E \rangle^* \left\{ \frac{x_A}{A \in S} \right\} = \bigcup \left\{ \frac{\langle \text{GR } E \rangle^* \{x_A\}}{A \in S} \right\} = \text{Ob } E$$

and thus E is β -thick. □

OBVIOUS 1238. Let X be a set, A and B be **Rel**-endomorphisms on X and $B \sqsupseteq A$. Then:

- $\text{thick}_\alpha(A) \Rightarrow \text{thick}_\alpha(B)$;
- $\text{thick}_\beta(A) \Rightarrow \text{thick}_\beta(B)$.

EXAMPLE 1239. There is a β -thick Rel-morphism which is not α -thick.

PROOF. Consider the **Rel**-morphism on $[0; 1]$ with the graph on figure 9:

$$\Gamma = \left\{ \frac{(x, x)}{x \in [0; 1]} \right\} \cup \left\{ \frac{(x, 0)}{x \in [0; 1]} \right\} \cup \left\{ \frac{(0, x)}{x \in [0; 1]} \right\}.$$

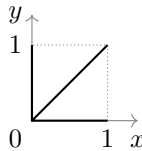


FIGURE 9. Thickness counterexample graph

Γ is β -thick because $\langle \Gamma \rangle^* \{0\} = [0; 1]$.

To prove that Γ is not α -thick it's enough to prove that every set A such that $A \times A \subseteq \Gamma$ is finite.

Suppose for the contrary that A is infinite. Then A contains more than one non-zero points y, z ($y \neq z$). Without loss of generality $y < z$. So we have that (y, z) is not of the form (y, y) nor $(0, y)$ nor $(y, 0)$. Therefore $A \times A$ isn't a subset of Γ . \square

13.2. Totally bounded endoreloids

The below is a straightforward generalization of the customary definition of totally bounded sets on uniform spaces (it's proved below that for uniform spaces the below definitions are equivalent).

DEFINITION 1240. An endoreloid f is α -totally bounded ($\text{totBound}_\alpha(f)$) if every $E \in \text{up } f$ is α -thick.

DEFINITION 1241. An endoreloid f is β -totally bounded ($\text{totBound}_\beta(f)$) if every $E \in \text{up } f$ is β -thick.

REMARK 1242. We could rewrite the above definitions in a more algebraic way like $\text{up } f \subseteq \text{thick}_\alpha$ (with thick_α would be defined as a set rather than as a predicate), but we don't really need this simplification.

PROPOSITION 1243. If an endoreloid is α -totally bounded then it is β -totally bounded.

PROOF. Because $\text{thick}_\alpha(E) \Rightarrow \text{thick}_\beta(E)$. \square

PROPOSITION 1244. If an endoreloid f is reflexive and $\text{Ob } f$ is finite then f is both α -totally bounded and β -totally bounded.

PROOF. It enough to prove that f is α -totally bounded. Really, every $E \in \text{up } f$ is reflexive. Thus $\{x\} \times \{x\} \subseteq \text{GR } E$ for $x \in \text{Ob } f$ and thus $\left\{ \frac{\{x\}}{x \in \text{Ob } f} \right\}$ is a sought for finite cover of $\text{Ob } f$. \square

OBVIOUS 1245.

- A principal endoreloid induced by a **Rel**-morphism E is α -totally bounded iff E is α -thick.
- A principal endoreloid induced by a **Rel**-morphism E is β -totally bounded iff E is β -thick.

EXAMPLE 1246. There is a β -totally bounded endoreloid which is not α -totally bounded.

PROOF. It follows from the example above and properties of principal endoreloids. \square

13.3. Special case of uniform spaces

Remember that *uniform space* is essentially the same as symmetric, reflexive and transitive endoreloid.

THEOREM 1247. Let f be such an endoreloid that $f \circ f^{-1} \subseteq f$. Then f is α -totally bounded iff it is β -totally bounded.

PROOF.

\Rightarrow . Proved above.

\Leftarrow . For every $\epsilon \in \text{up } f$ we have that $\langle \text{GR } \epsilon \rangle^* \{c_0\}, \dots, \langle \text{GR } \epsilon \rangle^* \{c_n\}$ covers the space. $\langle \text{GR } \epsilon \rangle^* \{c_i\} \times \langle \text{GR } \epsilon \rangle^* \{c_i\} \subseteq \text{GR}(\epsilon \circ \epsilon^{-1})$ because for $x \in \langle \text{GR } \epsilon \rangle^* \{c_i\}$ (the same as $c_i \in \langle \text{GR } \epsilon \rangle^* \{x\}$) we have

$$\langle \langle \text{GR } \epsilon \rangle^* \{c_i\} \times \langle \text{GR } \epsilon \rangle^* \{c_i\} \rangle^* \{x\} = \langle \text{GR } \epsilon \rangle^* \{c_i\} \subseteq \langle \text{GR } \epsilon \rangle^* \langle \text{GR } \epsilon^{-1} \rangle^* \{x\} = \langle \text{GR}(\epsilon \circ \epsilon^{-1}) \rangle^* \{x\}.$$

For every $\epsilon' \in \text{up } f$ exists $\epsilon \in \text{up } f$ such that $\epsilon \circ \epsilon^{-1} \sqsubseteq \epsilon'$ because $f \circ f^{-1} \sqsubseteq f$. Thus for every ϵ' we have $\langle \text{GR } \epsilon \rangle^* \{c_i\} \times \langle \text{GR } \epsilon \rangle^* \{c_i\} \subseteq \text{GR } \epsilon'$ and so $\langle \text{GR } \epsilon \rangle^* \{c_0\}, \dots, \langle \text{GR } \epsilon \rangle^* \{c_n\}$ is a sought for finite cover. \square

COROLLARY 1248. A uniform space is α -totally bounded iff it is β -totally bounded.

PROOF. From the theorem and the definition of uniform spaces. \square

Thus we can say about just *totally bounded* uniform spaces (without specifying whether it is α or β).

13.4. Relationships with other properties

THEOREM 1249. Let μ and ν be endoreloids. Let f be a principal $C'(\mu, \nu)$ continuous, monovalued, surjective reloid. Then if μ is β -totally bounded then ν is also β -totally bounded.

PROOF. Let φ be the monovalued, surjective function, which induces the reloid f .

We have $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

Let $F \in \text{up } \nu$. Then there exists $E \in \text{up } \mu$ such that $E \subseteq \varphi^{-1} \circ F \circ \varphi$.

Since μ is β -totally bounded, there exists a finite typed subset A of $\text{Ob } \mu$ such that $\langle \text{GR } E \rangle^* A = \text{Ob } \mu$.

We claim $\langle \text{GR } F \rangle^* \langle \varphi \rangle^* A = \text{Ob } \nu$.

Indeed let $y \in \text{Ob } \nu$ be an arbitrary point. Since φ is surjective, there exists $x \in \text{Ob } \mu$ such that $\varphi x = y$. Since $\langle \text{GR } E \rangle^* A = \text{Ob } \mu$ there exists $a \in A$ such that $a \langle \text{GR } E \rangle x$ and thus $a \langle \varphi^{-1} \circ F \circ \varphi \rangle x$. So $(\varphi a, y) = (\varphi a, \varphi x) \in \text{GR } F$. Therefore $y \in \langle \text{GR } F \rangle^* \langle \varphi \rangle^* A$. \square

THEOREM 1250. Let μ and ν be endoreloids. Let f be a principal $C''(\mu, \nu)$ continuous, surjective reloid. Then if μ is α -totally bounded then ν is also α -totally bounded.

PROOF. Let φ be the surjective binary relation which induces the reloid f .

We have $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.

Let $F \in \text{up } \nu$. Then there exists $E \in \text{up } \mu$ such that $\varphi \circ E \circ \varphi^{-1} \subseteq F$.

There exists a finite cover S of $\text{Ob } \mu$ such that $\bigcup \left\{ \frac{A \times A}{A \in S} \right\} \subseteq \text{GR } E$.

Thus $\varphi \circ \left(\bigcup \left\{ \frac{A \times A}{A \in S} \right\} \right) \circ \varphi^{-1} \subseteq \text{GR } F$ that is $\bigcup \left\{ \frac{\langle \varphi \rangle^* A \times \langle \varphi \rangle^* A}{A \in S} \right\} \subseteq \text{GR } F$.

It remains to prove that $\left\{ \frac{\langle \varphi \rangle^* A}{A \in S} \right\}$ is a cover of $\text{Ob } \nu$. It is true because φ is a surjection and S is a cover of $\text{Ob } \mu$. \square

A stronger statement (principality requirement removed):

CONJECTURE 1251. The image of a uniformly continuous entirely defined monovalued surjective reloid from a $(\alpha$ -, β -)totally bounded endoreloid is also $(\alpha$ -, β -)totally bounded.

Can we remove the requirement to be entirely defined from the above conjecture?

QUESTION 1252. Under which conditions it's true that join of $(\alpha\text{-}, \beta\text{-})$ totally bounded reloids is also totally bounded?

13.5. Additional predicates

We may consider also the following predicates expressing different kinds of what is intuitively is understood as boundness. Their usefulness is unclear, but I present them for completeness.

- $\text{totBound}_\alpha(f)$
- $\text{totBound}_\beta(f)$
- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\alpha(E^n)$
- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\beta(E^n)$
- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\alpha(E^0 \sqcup \dots \sqcup E^n)$
- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\beta(E^0 \sqcup \dots \sqcup E^n)$
- $\exists n \in \mathbb{N} : \text{totBound}_\alpha(f^n)$
- $\exists n \in \mathbb{N} : \text{totBound}_\beta(f^n)$
- $\exists n \in \mathbb{N} : \text{totBound}_\alpha(f^0 \sqcup \dots \sqcup f^n)$
- $\exists n \in \mathbb{N} : \text{totBound}_\beta(f^0 \sqcup \dots \sqcup f^n)$
- $\text{totBound}_\alpha(S(f))$
- $\text{totBound}_\beta(S(f))$

Some of the above defined predicates are equivalent:

PROPOSITION 1253.

- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\alpha(E^n) \Leftrightarrow \exists n \in \mathbb{N} : \text{totBound}_\alpha(f^n)$.
- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\beta(E^n) \Leftrightarrow \exists n \in \mathbb{N} : \text{totBound}_\beta(f^n)$.

PROOF. Because for every $E \in \text{up } f$ some $F \in \text{up } f^n$ is a subset of E^n , we have

$$\forall E \in \text{up } f : \text{thick}_\alpha(E^n) \Leftrightarrow \forall F \in \text{up } f^n : \text{thick}_\alpha(F)$$

and likewise for thick_β . □

PROPOSITION 1254.

- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\alpha(E^0 \sqcup \dots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N} : \text{totBound}_\alpha(f^0 \sqcup \dots \sqcup f^n)$
- $\exists n \in \mathbb{N} \forall E \in \text{up } f : \text{thick}_\beta(E^0 \sqcup \dots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N} : \text{totBound}_\beta(f^0 \sqcup \dots \sqcup f^n)$

PROOF. It's enough to prove

$$\forall E \in \text{up } f \exists F \in \text{up}(f^0 \sqcup \dots \sqcup f^n) : F \sqsubseteq E^0 \sqcup \dots \sqcup E^n \text{ and} \quad (15)$$

$$\forall F \in \text{up}(f^0 \sqcup \dots \sqcup f^n) \exists E \in \text{up } f : E^0 \sqcup \dots \sqcup E^n \sqsubseteq F. \quad (16)$$

For the formula (15) take $F = E^0 \sqcup \dots \sqcup E^n$.

Let's prove (16). Let $F \in \text{up}(f^0 \sqcup \dots \sqcup f^n)$. Using the fact that $F \in \text{up } f^i$ take $E_i \in \text{up } f$ for $i = 0, \dots, n$ such that $E_i^i \sqsubseteq F$ (exercise 1002 and properties of generalized filter bases) and then $E = E_0 \sqcap \dots \sqcap E_n \in \text{up } f$. We have $E^0 \sqcup \dots \sqcup E^n \sqsubseteq F$. □

PROPOSITION 1255. All predicates in the above list are pairwise equivalent in the case if f is a uniform space.

PROOF. Because $f \circ f = f$ and thus $f^n = f^0 \sqcup \dots \sqcup f^n = S(f) = f$. □

Orderings of filters in terms of reloids

Whilst the other chapters of this book use filters to research functors and reloids, here the opposite thing is discussed, the theory of reloids is used to describe properties of filters.

In this chapter the word *filter* is used to denote a filter on a set (not on an arbitrary poset) only.

14.1. Ordering of filters

Below I will define some categories having filters (with possibly different bases) as their objects and some relations having two filters (with possibly different bases) as arguments induced by these categories (defined as existence of a morphism between these two filters).

THEOREM 1256. $\text{card } a = \text{card } U$ for every ultrafilter a on U if U is infinite.

PROOF. Let $f(X) = X$ if $X \in a$ and $f(X) = U \setminus X$ if $X \notin a$. Obviously f is a surjection from U to a .

Every $X \in a$ appears as a value of f exactly twice, as $f(X)$ and $f(U \setminus X)$. So $\text{card } a = (\text{card } U)/2 = \text{card } U$. \square

COROLLARY 1257. Cardinality of every two ultrafilters on a set U is the same.

PROOF. For infinite U it follows from the theorem. For finite case it is obvious. \square

PROPOSITION 1258. $\langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} = \left\{ \frac{C \in \mathcal{P}(\text{Dst } f)}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\}$ for every **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$. (Here a functor is considered as a pair of functions $\mathfrak{F}(\text{Base}(\mathcal{A})) \rightarrow \mathfrak{F}(\text{Base}(\mathcal{B}))$, $\mathfrak{F}(\text{Base}(\mathcal{B})) \rightarrow \mathfrak{F}(\text{Base}(\mathcal{A}))$ rather than as a pair of functions $\mathcal{F}(\text{Base}(\mathcal{A})) \rightarrow \mathcal{F}(\text{Base}(\mathcal{B}))$, $\mathcal{F}(\text{Base}(\mathcal{B})) \rightarrow \mathcal{F}(\text{Base}(\mathcal{A}))$.)

PROOF. For every set $C \in \mathcal{P} \text{Base}(\mathcal{B})$ we have

$$\begin{aligned} \langle f^{-1} \rangle^* C \in \mathcal{A} &\Rightarrow \\ \exists K \in \mathcal{A} : \langle f^{-1} \rangle^* C = K &\Rightarrow \\ \exists K \in \mathcal{A} : \langle f \rangle^* \langle f^{-1} \rangle^* C = \langle f \rangle^* K &\Rightarrow \\ \exists K \in \mathcal{A} : C \supseteq \langle f \rangle^* K &\Leftrightarrow \\ \exists K \in \mathcal{A} : C \in \langle \uparrow^{\text{FCD}} f \rangle^* K &\Rightarrow \\ C \in \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}. & \end{aligned}$$

So $C \in \left\{ \frac{C \in \mathcal{P}(\text{Dst } f)}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\} \Rightarrow C \in \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.

Let now $C \in \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$. Then $\uparrow \langle f^{-1} \rangle^* C \supseteq \langle \uparrow^{\text{FCD}} f^{-1} \rangle \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} \supseteq \mathcal{A}$ and thus $\langle f^{-1} \rangle^* C \in \mathcal{A}$. \square

Below I'll define some directed multigraphs. By an abuse of notation, I will denote these multigraphs the same as (below defined) categories based on some

of these directed multigraphs with added composition of morphisms (of directed multigraphs edges). As such I will call vertices of these multigraphs objects and edges morphisms.

DEFINITION 1259. I will denote $\mathbf{GreFunc}_1$ the multigraph whose objects are filters and whose morphisms between objects \mathcal{A} and \mathcal{B} are **Set**-morphisms from $\text{Base}(\mathcal{A})$ to $\text{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.

DEFINITION 1260. I will denote $\mathbf{GreFunc}_2$ the multigraph whose objects are filters and whose morphisms between objects \mathcal{A} and \mathcal{B} are **Set**-morphisms from $\text{Base}(\mathcal{A})$ to $\text{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.

DEFINITION 1261. Let \mathcal{A} be a filter on a set X and \mathcal{B} be a filter on a set Y . $\mathcal{A} \geq_1 \mathcal{B}$ iff $\text{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B})$ is not empty.

DEFINITION 1262. Let \mathcal{A} be a filter on a set X and \mathcal{B} be a filter on a set Y . $\mathcal{A} \geq_2 \mathcal{B}$ iff $\text{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$ is not empty.

PROPOSITION 1263.

1°. $f \in \text{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B})$ iff f is a **Set**-morphism from $\text{Base}(\mathcal{A})$ to $\text{Base}(\mathcal{B})$ such that

$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}$$

for every $C \in \mathcal{P} \text{Base}(\mathcal{B})$.

2°. $f \in \text{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$ iff f is a **Set**-morphism from $\text{Base}(\mathcal{A})$ to $\text{Base}(\mathcal{B})$ such that

$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}$$

for every $C \in \mathcal{P} \text{Base}(\mathcal{B})$.

PROOF.

1°.

$$\begin{aligned} f \in \text{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B}) &\Leftrightarrow \mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} \Leftrightarrow \\ &\forall C \in \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} : C \in \mathcal{B} \Leftrightarrow \forall C \in \mathcal{P} \text{Base}(\mathcal{B}) : (\langle f^{-1} \rangle^* C \in \mathcal{A} \Rightarrow C \in \mathcal{B}). \end{aligned}$$

2°.

$$\begin{aligned} f \in \text{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B}) &\Leftrightarrow \mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} \Leftrightarrow \forall C : (C \in \mathcal{B} \Leftrightarrow C \in \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}) \Leftrightarrow \\ &\forall C \in \mathcal{P} \text{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow C \in \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}) \Leftrightarrow \\ &\forall C \in \mathcal{P} \text{Base}(\mathcal{B}) : (\langle f^{-1} \rangle^* C \in \mathcal{A} \Leftrightarrow C \in \mathcal{B}). \end{aligned}$$

□

DEFINITION 1264. The directed multigraph $\mathbf{FuncBij}$ is the directed multigraph got from $\mathbf{GreFunc}_2$ by restricting to only bijective morphisms.

DEFINITION 1265. A filter \mathcal{A} is *directly isomorphic* to a filter \mathcal{B} iff there is a morphism $f \in \text{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$.

OBVIOUS 1266. $f \in \text{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ for every **Set**-morphism from $\text{Base}(\mathcal{A})$ to $\text{Base}(\mathcal{B})$.

OBVIOUS 1267. $f \in \text{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ for every **Set**-morphism from $\text{Base}(\mathcal{A})$ to $\text{Base}(\mathcal{B})$.

COROLLARY 1268. $\mathcal{A} \geq_1 \mathcal{B}$ iff it exists a **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.

COROLLARY 1269. $\mathcal{A} \geq_2 \mathcal{B}$ iff it exists a **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.

PROPOSITION 1270. For a bijective **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ the following are equivalent:

- 1°. $\mathcal{B} = \left\{ \frac{C \in \mathcal{P} \text{Base}(\mathcal{B})}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\}$.
- 2°. $\forall C \in \text{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A})$.
- 3°. $\forall C \in \text{Base}(\mathcal{A}) : (C \in \langle f \rangle^* \mathcal{B} \Leftrightarrow C \in \mathcal{A})$.
- 4°. $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}}$ is a bijection from \mathcal{A} to \mathcal{B} .
- 5°. $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}}$ is a function onto \mathcal{B} .
- 6°. $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.
- 7°. $f \in \text{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$.
- 8°. $f \in \text{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$.

PROOF.

1° \Leftrightarrow 2°.

$$\mathcal{B} = \left\{ \frac{C \in \mathcal{P} \text{Base}(\mathcal{B})}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\} \Leftrightarrow \forall C \in \mathcal{P} \text{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}).$$

2° \Leftrightarrow 3°. Because f is a bijection.

2° \Rightarrow 5°. For every $C \in \mathcal{B}$ we have $\langle f^{-1} \rangle^* C \in \mathcal{A}$ and thus $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}} \langle \uparrow^{\text{FCD}} f^{-1} \rangle C = \langle f \rangle^* \langle f^{-1} \rangle^* C = C$. Thus $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}}$ is onto \mathcal{B} .

4° \Rightarrow 5°. Obvious.

5° \Rightarrow 4°. We need to prove only that $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.

4° \Rightarrow 3°. We have $\forall C \in \text{Base}(\mathcal{A}) : ((\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}})C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A})$ and consequently $\forall C \in \text{Base}(\mathcal{A}) : (\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A})$.

6° \Leftrightarrow 1°. From the last corollary.

1° \Leftrightarrow 7°. Obvious.

7° \Leftrightarrow 8°. Obvious. □

COROLLARY 1271. The following are equivalent for every filters \mathcal{A} and \mathcal{B} :

- 1°. \mathcal{A} is directly isomorphic to \mathcal{B} .
- 2°. There is a bijective **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that for every $C \in \mathcal{P} \text{Base}(\mathcal{B})$

$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}.$$

- 3°. There is a bijective **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that for every $C \in \mathcal{P} \text{Base}(\mathcal{B})$

$$\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$

- 4°. There is a bijective **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}}$ is a bijection from \mathcal{A} to \mathcal{B} .

- 5°. There is a bijective **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\langle \uparrow^{\text{FCD}} f \rangle|_{\mathcal{A}}$ is a function onto \mathcal{B} .

- 6°. There is a bijective **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$.

- 7°. There is a bijective morphism $f \in \text{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$.

- 8°. There is a bijective morphism $f \in \text{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$.

PROPOSITION 1272. **GreFunc**₁ and **GreFunc**₂ with function composition are categories.

PROOF. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ be morphisms of **GreFunc**₁. Then $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} \sqsubseteq \langle \uparrow^{\text{FCD}} g \rangle \mathcal{B}$. So

$$\langle \uparrow^{\text{FCD}} (g \circ f) \rangle \mathcal{A} = \langle \uparrow^{\text{FCD}} g \rangle \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} \sqsupseteq \langle \uparrow^{\text{FCD}} g \rangle \mathcal{B} \sqsupseteq \mathcal{C}.$$

Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\text{id}_{\text{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every filter \mathcal{A} .

Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ be morphisms of $\mathbf{GreFunc}_2$. Then $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} = \langle \uparrow^{\text{FCD}} g \rangle \mathcal{B}$. So

$$\langle \uparrow^{\text{FCD}} (g \circ f) \rangle \mathcal{A} = \langle \uparrow^{\text{FCD}} g \rangle \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} = \langle \uparrow^{\text{FCD}} g \rangle \mathcal{B} = \mathcal{C}.$$

Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_2$. Associativity law is evident. $\text{id}_{\text{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_2$ for every filter \mathcal{A} . \square

COROLLARY 1273. \leq_1 and \leq_2 are preorders.

THEOREM 1274. $\mathbf{FuncBij}$ is a groupoid.

PROOF. First let's prove it is a category. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ be morphisms of $\mathbf{FuncBij}$. Then $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ and $g : \text{Base}(\mathcal{B}) \rightarrow \text{Base}(\mathcal{C})$ are bijections and $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} = \langle \uparrow^{\text{FCD}} g \rangle \mathcal{B}$. Thus $g \circ f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{C})$ is a bijection and $\mathcal{C} = \langle \uparrow^{\text{FCD}} (g \circ f) \rangle \mathcal{A}$. Thus $g \circ f$ is a morphism of $\mathbf{FuncBij}$. $\text{id}_{\text{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{FuncBij}$ for every filter \mathcal{A} . Thus it is a category.

It remains to prove only that every morphism $f \in \text{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$ has a reverse (for every filters \mathcal{A}, \mathcal{B}). We have f is a bijection $\text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that for every $C \in \mathcal{P} \text{Base}(\mathcal{A})$

$$\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$

Then $f^{-1} : \text{Base}(\mathcal{B}) \rightarrow \text{Base}(\mathcal{A})$ is a bijection such that for every $C \in \mathcal{P} \text{Base}(\mathcal{B})$

$$\langle f^{-1} \rangle^* C \in \mathcal{A} \Leftrightarrow C \in \mathcal{B}.$$

Thus $f^{-1} \in \text{Hom}_{\mathbf{FuncBij}}(\mathcal{B}, \mathcal{A})$. \square

COROLLARY 1275. Being directly isomorphic is an equivalence relation.

Rudin-Keisler order of ultrafilters is considered in such a book as [40].

OBVIOUS 1276. For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

DEFINITION 1277. A filter \mathcal{A} is *isomorphic* to a filter \mathcal{B} iff there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$.

OBVIOUS 1278. Equivalent filters are isomorphic.

THEOREM 1279. Being isomorphic (for small filters) is an equivalence relation.

PROOF.

Reflexivity. Because every filter is directly isomorphic to itself.

Symmetry. If filter \mathcal{A} is isomorphic to \mathcal{B} then there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$ and thus $\mathcal{B} \div B$ is directly isomorphic to $\mathcal{A} \div A$. So \mathcal{B} is isomorphic to \mathcal{A} .

Transitivity. Let \mathcal{A} be isomorphic to \mathcal{B} and \mathcal{B} be isomorphic to \mathcal{C} . Then exist $A \in \mathcal{A}$, $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $C \in \mathcal{C}$ such that there are bijections $f : A \rightarrow B_1$ and $g : B_2 \rightarrow C$ such that

$$\forall X \in \mathcal{P} A : (X \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* X \in A) \quad \text{and} \quad \forall X \in \mathcal{P} B_1 : (X \in \mathcal{A} \Leftrightarrow \langle f \rangle^* X \in \mathcal{B})$$

$$\text{and also } \forall X \in \mathcal{P} B_2 : (X \in \mathcal{B} \Leftrightarrow \langle g \rangle^* X \in \mathcal{C}).$$

So $g \circ f$ is a bijection from $\langle f^{-1} \rangle^* (B_1 \cap B_2) \in \mathcal{A}$ to $\langle g \rangle^* (B_1 \cap B_2) \in \mathcal{C}$ such that

$$X \in \mathcal{A} \Leftrightarrow \langle f \rangle^* X \in \mathcal{B} \Leftrightarrow \langle g \rangle^* \langle f \rangle^* X \in \mathcal{C} \Leftrightarrow \langle g \circ f \rangle^* X \in \mathcal{C}.$$

Thus $g \circ f$ establishes a bijection which proves that \mathcal{A} is isomorphic to \mathcal{C} .

□

LEMMA 1280. Let $\text{card } X = \text{card } Y$, u be an ultrafilter on X and v be an ultrafilter on Y ; let $A \in u$ and $B \in v$. Let $u \div A$ and $v \div B$ be directly isomorphic. Then if $\text{card}(X \setminus A) = \text{card}(Y \setminus B)$ we have u and v directly isomorphic.

PROOF. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X \setminus A$ and $Y \setminus B$. □

THEOREM 1281. If $\text{card } X = \text{card } Y$ then being isomorphic and being directly isomorphic are the same for ultrafilters u on X and v on Y .

PROOF. That if two filters are isomorphic then they are directly isomorphic is obvious.

Let ultrafilters u and v be isomorphic that is there is a bijection $f : A \rightarrow B$ where $A \in u$, $B \in v$ witnessing isomorphism of u and v .

If one of the filters u or v is a trivial ultrafilter then the other is also a trivial ultrafilter and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial ultrafilters.

If $\text{card}(X \setminus A) = \text{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\text{card}(X \setminus A) < \text{card}(Y \setminus B)$.

$\text{card } B = \text{card } Y$ because otherwise $\text{card}(X \setminus A) = \text{card}(Y \setminus B)$.

It is easy to show that there exists $B' \supset B$ such that $\text{card}(X \setminus A) = \text{card}(Y \setminus B')$ and $\text{card } B' = \text{card } B$.

We will find a bijection g from B to B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin v$.

There exists a set $E \subseteq B$ such that $\text{card } E \geq \text{card } D$ and $E \notin v$.

We have $\text{card } E = \text{card}(D \cup E)$ and thus there exists a bijection $h : E \rightarrow D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\text{im}(g|_{B \setminus E}) = B \setminus E$; $\text{im}(g|_E) = \text{im } h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle^* v = v$. Indeed it follows from $B \setminus E \in v$. □

PROPOSITION 1282.

1°. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geq_2 \mathcal{B}$ iff $\mathcal{A} \div A \geq_2 \mathcal{B} \div B$.

2°. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geq_1 \mathcal{B}$ iff $\mathcal{A} \div A \geq_1 \mathcal{B} \div B$.

PROOF.

1°. $\mathcal{A} \geq_2 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing \mathcal{A} with $\mathcal{A} \div A$ and \mathcal{B} with $\mathcal{B} \div B$.

2°. $\mathcal{A} \geq_1 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} \subseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing \mathcal{A} with $\mathcal{A} \div A$ and \mathcal{B} with $\mathcal{B} \div B$. □

PROPOSITION 1283. For ultrafilters \geq_2 is the same as Rudin-Keisler ordering (as defined in [40]).

PROOF. $x \geq_2 y$ iff there exist sets $A \in x$ and $B \in y$ and a bijective **Set**-morphism $f : X \rightarrow Y$ such that

$$y \div B = \left\{ \frac{C \in \mathcal{P}Y}{\langle f^{-1} \rangle^* C \in x \div A} \right\}$$

that is when $C \in y \div B \Leftrightarrow \langle f^{-1} \rangle^* C \in x \div A$ what is equivalent to $C \in y \Leftrightarrow \langle f^{-1} \rangle^* C \in x$ what is the definition of Rudin-Keisler ordering. \square

REMARK 1284. The relation of being isomorphic for ultrafilters is traditionally called *Rudin-Keisler equivalence*.

OBVIOUS 1285. $(\geq_1) \supseteq (\geq_2)$.

DEFINITION 1286. Let Q and R be binary relations on the set of (small) filters. I will denote $\mathbf{MonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such monovalued reloids f that $(\text{dom } f) Q \mathcal{A}$ and $(\text{im } f) R \mathcal{B}$.

I will also denote $\mathbf{CoMonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such injective reloids f that $(\text{im } f) Q \mathcal{A}$ and $(\text{dom } f) R \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

LEMMA 1287. $\mathbf{CoMonRld}_{Q,R} \neq \emptyset \Leftrightarrow \mathbf{MonRld}_{Q,R} \neq \emptyset$.

PROOF.

$$\begin{aligned} f \in \mathbf{CoMonRld}_{Q,R} &\Leftrightarrow (\text{im } f) Q \mathcal{A} \wedge (\text{dom } f) R \mathcal{B} \Leftrightarrow \\ &(\text{dom } f^{-1}) Q \mathcal{A} \wedge (\text{im } f^{-1}) R \mathcal{B} \Leftrightarrow f^{-1} \in \mathbf{MonRld}_{Q,R} \end{aligned}$$

for every monovalued reloid f (or what is the same, injective reloid f^{-1}). \square

THEOREM 1288. For every filters \mathcal{A} and \mathcal{B} the following are equivalent:

- 1°. $\mathcal{A} \geq_1 \mathcal{B}$.
- 2°. $\text{Hom}_{\mathbf{MonRld}_{\sqsubseteq, \supseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
- 3°. $\text{Hom}_{\mathbf{MonRld}_{\sqsubseteq, \supseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
- 4°. $\text{Hom}_{\mathbf{MonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
- 5°. $\text{Hom}_{\mathbf{CoMonRld}_{\sqsubseteq, \supseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
- 6°. $\text{Hom}_{\mathbf{CoMonRld}_{\sqsubseteq, \supseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
- 7°. $\text{Hom}_{\mathbf{CoMonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. There exists a **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$. We have

$$\text{dom}(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \mathcal{A} \sqcap \top(\text{Base}(\mathcal{A})) = \mathcal{A}$$

and

$$\text{im}(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \text{im}(\text{FCD})(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \text{im}(\uparrow^{\text{FCD}} f)|_{\mathcal{A}} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} \supseteq \mathcal{B}.$$

Thus $(\uparrow^{\text{RLD}} f)|_{\mathcal{A}}$ is a monovalued reloid such that $\text{dom}(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and $\text{im}(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} \supseteq \mathcal{B}$.

$2^\circ \Rightarrow 3^\circ$, $4^\circ \Rightarrow 3^\circ$, $5^\circ \Rightarrow 6^\circ$, $7^\circ \Rightarrow 6^\circ$. Obvious.

- $3^\circ \Rightarrow 1^\circ$. We have $\mathcal{B} \sqsubseteq \langle (\text{FCD})f \rangle \mathcal{A}$ for a monovalued reloid $f \in \text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B}))$. Then there exists a **Set**-morphism $F : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} F \rangle \mathcal{A}$ that is $\mathcal{A} \geq_1 \mathcal{B}$.
- $6^\circ \Rightarrow 7^\circ$. Let f be an injective reloid such that $\text{im } f \sqsubseteq \mathcal{A}$ and $\text{dom } f \supseteq \mathcal{B}$. Then $\text{im } f|_{\mathcal{B}} \sqsubseteq \mathcal{A}$ and $\text{dom } f|_{\mathcal{B}} = \mathcal{B}$. So $f|_{\mathcal{B}} \in \text{Hom}_{\text{CoMonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{B})$.
- $2^\circ \Leftrightarrow 5^\circ$, $3^\circ \Leftrightarrow 6^\circ$, $4^\circ \Leftrightarrow 7^\circ$. By the lemma. □

THEOREM 1289. For every filters \mathcal{A} and \mathcal{B} the following are equivalent:

- 1°. $\mathcal{A} \geq_2 \mathcal{B}$.
- 2°. $\text{Hom}_{\text{MonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
- 3°. $\text{Hom}_{\text{CoMonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.

PROOF.

- $1^\circ \Rightarrow 2^\circ$. Let $\mathcal{A} \geq_2 \mathcal{B}$ that is $\mathcal{B} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ for some **Set**-morphism $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$. Then $\text{dom}(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and
- $$\text{im}(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \text{im}(\text{FCD})(\uparrow^{\text{RLD}} f)|_{\mathcal{A}} = \text{im}(\uparrow^{\text{FCD}} f)|_{\mathcal{A}} = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} = \mathcal{B}.$$
- So $(\uparrow^{\text{RLD}} f)|_{\mathcal{A}}$ is a sought for reloid.
- $2^\circ \Rightarrow 1^\circ$. There exists a monovalued reloid f with domain \mathcal{A} such that $\langle (\text{FCD})f \rangle \mathcal{A} = \mathcal{B}$. By corollary 1325 below, there exists a **Set**-morphism $F : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $f = (\uparrow^{\text{RLD}} F)|_{\mathcal{A}}$. Thus
- $$\langle \uparrow^{\text{FCD}} F \rangle \mathcal{A} = \text{im}(\uparrow^{\text{FCD}} F)|_{\mathcal{A}} = \text{im}(\text{FCD})(\uparrow^{\text{RLD}} F)|_{\mathcal{A}} = \text{im}(\text{FCD})f = \text{im } f = \mathcal{B}.$$
- Thus $\mathcal{A} \geq_2 \mathcal{B}$ is testified by the morphism F .
- $2^\circ \Leftrightarrow 3^\circ$. By the lemma. □

THEOREM 1290. The following are categories (with reloid composition):

- 1°. **MonRld** $_{\sqsubseteq, \supseteq}$;
- 2°. **MonRld** $_{\sqsubseteq, =}$;
- 3°. **MonRld** $_{=, =}$;
- 4°. **CoMonRld** $_{\sqsubseteq, \supseteq}$;
- 5°. **CoMonRld** $_{\sqsubseteq, =}$;
- 6°. **CoMonRld** $_{=, =}$.

PROOF. We will prove only the first three. The rest follow from duality. We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism are evident. We have:

- 1°. Let $f \in \text{Hom}_{\text{MonRld}_{\sqsubseteq, \supseteq}}(\mathcal{A}, \mathcal{B})$, $g \in \text{Hom}_{\text{MonRld}_{\sqsubseteq, \supseteq}}(\mathcal{B}, \mathcal{C})$. Then $\text{dom } f \sqsubseteq \mathcal{A}$, $\text{im } f \supseteq \mathcal{B}$, $\text{dom } g \sqsubseteq \mathcal{B}$, $\text{im } g \supseteq \mathcal{C}$. So $\text{dom}(g \circ f) \sqsubseteq \mathcal{A}$, $\text{im}(g \circ f) \supseteq \mathcal{C}$ that is $g \circ f \in \text{Hom}_{\text{MonRld}_{\sqsubseteq, \supseteq}}(\mathcal{A}, \mathcal{C})$.
- 2°. Let $f \in \text{Hom}_{\text{MonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{B})$, $g \in \text{Hom}_{\text{MonRld}_{\sqsubseteq, =}}(\mathcal{B}, \mathcal{C})$. Then $\text{dom } f \sqsubseteq \mathcal{A}$, $\text{im } f = \mathcal{B}$, $\text{dom } g \sqsubseteq \mathcal{B}$, $\text{im } g = \mathcal{C}$. So $\text{dom}(g \circ f) \sqsubseteq \mathcal{A}$, $\text{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \text{Hom}_{\text{MonRld}_{\sqsubseteq, =}}(\mathcal{A}, \mathcal{C})$.
- 3°. Let $f \in \text{Hom}_{\text{MonRld}_{=, =}}(\mathcal{A}, \mathcal{B})$, $g \in \text{Hom}_{\text{MonRld}_{=, =}}(\mathcal{B}, \mathcal{C})$. Then $\text{dom } f = \mathcal{A}$, $\text{im } f = \mathcal{B}$, $\text{dom } g = \mathcal{B}$, $\text{im } g = \mathcal{C}$. So $\text{dom}(g \circ f) = \mathcal{A}$, $\text{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \text{Hom}_{\text{MonRld}_{=, =}}(\mathcal{A}, \mathcal{C})$. □

DEFINITION 1291. Let **BijRld** be the groupoid of all bijections of the category of reloid triples. Its objects are filters and its morphisms from a filter \mathcal{A} to filter \mathcal{B} are monovalued injective reloids f such that $\text{dom } f = \mathcal{A}$ and $\text{im } f = \mathcal{B}$.

THEOREM 1292. Filters \mathcal{A} and \mathcal{B} are isomorphic iff $\text{Hom}_{\text{BijRld}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.

PROOF.

\Rightarrow . Let \mathcal{A} and \mathcal{B} be isomorphic. Then there are sets $A \in \mathcal{A}$, $B \in \mathcal{B}$ and a bijective **Set**-morphism $F : A \rightarrow B$ such that $\langle F \rangle^* : \mathcal{P}A \cap \mathcal{A} \rightarrow \mathcal{P}B \cap \mathcal{B}$ is a bijection.

Obviously $f = (\uparrow^{\text{RLD}} F)|_{\mathcal{A}}$ is monovalued and injective.

$$\begin{aligned} \text{im } f &= \\ \prod^{\tilde{\mathfrak{F}}} \left\{ \frac{\text{im } G}{G \in \text{up}(\uparrow^{\text{RLD}} F)|_{\mathcal{A}}} \right\} &= \\ \prod^{\tilde{\mathfrak{F}}} \left\{ \frac{\text{im}(H \cap F|_X)}{H \in \text{up}(\uparrow^{\text{RLD}} F)|_{\mathcal{A}}, X \in \mathcal{A}} \right\} &= \\ \prod^{\tilde{\mathfrak{F}}} \left\{ \frac{\text{im } F|_P}{P \in \mathcal{A}} \right\} &= \\ \prod^{\tilde{\mathfrak{F}}} \left\{ \frac{\langle F \rangle^* P}{P \in \mathcal{A}} \right\} &= \\ \prod^{\tilde{\mathfrak{F}}} \left\{ \frac{\langle F \rangle^* P}{P \in \mathcal{P}A \cap \mathcal{A}} \right\} &= \\ \prod^{\tilde{\mathfrak{F}}} (\mathcal{P}B \cap \mathcal{B}) &= \\ \prod^{\tilde{\mathfrak{F}}} \mathcal{B} &= \mathcal{B}. \end{aligned}$$

Thus $\text{dom } f = \mathcal{A}$ and $\text{im } f = \mathcal{B}$.

\Leftarrow . Let f be a monovalued injective reloid such that $\text{dom } f = \mathcal{A}$ and $\text{im } f = \mathcal{B}$. Then there exist a function F' and an injective binary relation F'' such that $F', F'' \in f$. Thus $F = F' \cap F''$ is an injection such that $F \in f$. The function F is a bijection from $A = \text{dom } F$ to $B = \text{im } F$. The function $\langle F \rangle^*$ is an injection on $\mathcal{P}A \cap \mathcal{A}$ (and moreover on $\mathcal{P}A$). It's simple to show that $\forall X \in \mathcal{P}A \cap \mathcal{A} : \langle F \rangle^* X \in \mathcal{P}B \cap \mathcal{B}$ and similarly

$$\forall Y \in \mathcal{P}B \cap \mathcal{B} : (\langle F \rangle^*)^{-1} Y = \langle F^{-1} \rangle^* Y \in \mathcal{P}A \cap \mathcal{A}.$$

Thus $\langle F \rangle^*|_{\mathcal{P}A \cap \mathcal{A}}$ is a bijection $\mathcal{P}A \cap \mathcal{A} \rightarrow \mathcal{P}B \cap \mathcal{B}$. So filters \mathcal{A} and \mathcal{B} are isomorphic. □

PROPOSITION 1293. $(\geq_1) = (\sqsupseteq) \circ (\geq_2)$ (when we limit to small filters).

PROOF. $\mathcal{A} \geq_1 \mathcal{B}$ iff exists a function $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$. But $\mathcal{B} \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$ is equivalent to $\exists \mathcal{B}' \in \mathcal{F} : (\mathcal{B}' \sqsupseteq \mathcal{B} \wedge \mathcal{B}' = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A})$. So $\mathcal{A} \geq_1 \mathcal{B}$ is equivalent to existence of $\mathcal{B}' \in \mathcal{F}$ such that $\mathcal{B}' \sqsupseteq \mathcal{B}$ and existence of a function $f : \text{Base}(\mathcal{A}) \rightarrow \text{Base}(\mathcal{B})$ such that $\mathcal{B}' = \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A}$. This is equivalent to $\mathcal{A} ((\sqsupseteq) \circ (\geq_2)) \mathcal{B}$. □

PROPOSITION 1294. If a and b are ultrafilters then $b \geq_1 a \Leftrightarrow b \geq_2 a$.

PROOF. We need to prove only $b \geq_1 a \Rightarrow b \geq_2 a$. If $b \geq_1 a$ then there exists a monovalued reloid $f : \text{Base}(b) \rightarrow \text{Base}(a)$ such that $\text{dom } f = b$ and $\text{im } f \sqsupseteq a$. Then $\text{im } f = \text{im}(\text{FCD})f \in \{\perp^{\mathcal{F}(\text{Base}(a))}\} \cup \text{atoms}^{\mathcal{F}(\text{Base}(a))}$ because $(\text{FCD})f$ is a monovalued funcoid. So $\text{im } f = a$ (taken into account $\text{im } f \neq \perp^{\mathcal{F}(\text{Base}(a))}$) and thus $b \geq_2 a$. □

COROLLARY 1295. For atomic filters \geq_1 is the same as \geq_2 .

Thus I will write simply \geq for atomic filters.

14.1.1. Existence of no more than one monovalued injective reloid for a given pair of ultrafilters.

14.1.1.1. *The lemmas.* The lemmas in this section were provided to me by ROBERT MARTIN SOLOVAY in [39]. They are based on WISTAR COMFORT's work.

In this section we will assume μ is an ultrafilter on a set I and function $f : I \rightarrow I$ has the property $X \in \mu \Leftrightarrow \langle f^{-1} \rangle^* X \in \mu$.

LEMMA 1296. If $X \in \mu$ then $X \cap \langle f \rangle^* X \in \mu$.

PROOF. If $\langle f \rangle^* X \notin \mu$ then $X \subseteq \langle f^{-1} \rangle^* \langle f \rangle^* X \notin \mu$ and so $X \notin \mu$. Thus $X \in \mu \wedge \langle f \rangle^* X \in \mu$ and consequently $X \cap \langle f \rangle^* X \in \mu$. \square

We will say that x is *periodic* when $f^n(x) = x$ for some positive integer x . The least such n is called *the period of x* .

Let's define $x \sim y$ iff there exist $i, j \in \mathbb{N}$ such that $f^i(x) = f^j(y)$. Trivially it is an equivalence relation. If x and y are periodic, then $x \sim y$ iff exists $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $A = \left\{ \frac{x \in I}{x \text{ is periodic with period} > 1} \right\}$.

We will show $A \notin \mu$. Let's assume $A \in \mu$.

Let a set $D \subseteq A$ contains (by the axiom of choice) exactly one element from each equivalence class of A defined by the relation \sim .

Let α be a function $A \rightarrow \mathbb{N}$ defined as follows. Let $x \in A$. Let y be the unique element of D such that $x \sim y$. Let $\alpha(x)$ be the least $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $B_0 = \left\{ \frac{x \in A}{\alpha(x) \text{ is even}} \right\}$ and $B_1 = \left\{ \frac{x \in A}{\alpha(x) \text{ is odd}} \right\}$.

Let $B_2 = \left\{ \frac{x \in A}{\alpha(x)=0} \right\}$.

LEMMA 1297. $B_0 \cap \langle f \rangle^* B_0 \subseteq B_2$.

PROOF. If $x \in B_0 \cap \langle f \rangle^* B_0$ then for a minimal even n and $x = f(x')$ where $f^m(y') = x'$ for a minimal even m . Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leq m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit; $f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geq m$ by minimality or $n = 0$.

Thus $n = m + 1$ what is impossible for even n and m . We have a contradiction what proves $B_0 \cap \langle f \rangle^* B_0 \subseteq \emptyset$.

Remained the case $n = 0$, then $x = f^0(y)$ and thus $\alpha(x) = 0$. \square

LEMMA 1298. $B_1 \cap \langle f \rangle^* B_1 = \emptyset$.

PROOF. Let $x \in B_1 \cap \langle f \rangle^* B_1$. Then $f^n(y) = x$ for an odd n and $x = f(x')$ where $f^m(y') = x'$ for an odd m . Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leq m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit;

$f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geq m$ by minimality ($n = 0$ is impossible because n is odd).

Thus $n = m + 1$ what is impossible for odd n and m . We have a contradiction what proves $B_1 \cap \langle f \rangle^* B_1 = \emptyset$. \square

LEMMA 1299. $B_2 \cap \langle f \rangle^* B_2 = \emptyset$.

PROOF. Let $x \in B_2 \cap \langle f \rangle^* B_2$. Then $x = y$ and $x' = y$ where $x = f(x')$. Thus $x = f(x)$ and so $x \notin A$ what is impossible. \square

LEMMA 1300. $A \notin \mu$.

PROOF. Suppose $A \in \mu$.

Since $A \in \mu$ we have $B_0 \in \mu$ or $B_1 \in \mu$.

So either $B_0 \cap \langle f \rangle^* B_0 \subseteq B_2$ or $B_1 \cap \langle f \rangle^* B_1 \subseteq B_2$. As such by the lemma 1296 we have $B_2 \in \mu$. This is incompatible with $B_2 \cap \langle f \rangle^* B_2 = \emptyset$. So we got a contradiction. \square

Let C be the set of points x which are not periodic but $f^n(x)$ is periodic for some positive n .

LEMMA 1301. $C \notin \mu$.

PROOF. Let β be a function $C \rightarrow \mathbb{N}$ such that $\beta(x)$ is the least $n \in \mathbb{N}$ such that $f^n(x)$ is periodic.

Let $C_0 = \left\{ \frac{x \in C}{\beta(x) \text{ is even}} \right\}$ and $C_1 = \left\{ \frac{x \in C}{\beta(x) \text{ is odd}} \right\}$.

Obviously $C_j \cap \langle f \rangle^* C_j = \emptyset$ for $j = 0, 1$. Hence by lemma 1296 we have $C_0, C_1 \notin \mu$ and thus $C = C_0 \cup C_1 \notin \mu$. \square

Let E be the set of $x \in I$ such that for no $n \in \mathbb{N}$ we have $f^n(x)$ periodic.

LEMMA 1302. Let $x, y \in E$ be such that $f^i(x) = f^j(y)$ and $f^{i'}(x) = f^{j'}(y)$ for some $i, j, i', j' \in \mathbb{N}$. Then $i - j = i' - j'$.

PROOF. $i \mapsto f^i(x)$ is a bijection.

So $y = f^{i-j}(y)$ and $y = f^{i'-j'}(y)$. Thus $f^{i-j}(y) = f^{i'-j'}(y)$ and so $i - j = i' - j'$. \square

LEMMA 1303. $E \notin \mu$.

PROOF. Let $D' \subseteq E$ be a subset of E with exactly one element from each equivalence class of the relation \sim on E .

Define the function $\gamma : E \rightarrow \mathbb{Z}$ as follows. Let $x \in E$. Let y be the unique element of D' such that $x \sim y$. Choose $i, j \in \mathbb{N}$ such that $f^i(y) = f^j(x)$. Let $\gamma(x) = i - j$. By the last lemma, γ is well-defined.

It is clear that if $x \in E$ then $f(x) \in E$ and moreover $\gamma(f(x)) = \gamma(x) + 1$.

Let $E_0 = \left\{ \frac{x \in E}{\gamma(x) \text{ is even}} \right\}$ and $E_1 = \left\{ \frac{x \in E}{\gamma(x) \text{ is odd}} \right\}$.

We have $E_0 \cap \langle f \rangle^* E_0 = \emptyset \notin \mu$ and hence $E_0 \notin \mu$.

Similarly $E_1 \notin \mu$.

Thus $E = E_0 \cup E_1 \notin \mu$. \square

LEMMA 1304. f is the identity function on a set in μ .

PROOF. We have shown $A, C, E \notin \mu$. But the points which lie in none of these sets are exactly points periodic with period 1 that is fixed points of f . Thus the set of fixed points of f belongs to the filter μ . \square

14.1.1.2. *The main theorem and its consequences.*

THEOREM 1305. For every ultrafilter a the morphism $(a, a, \text{id}_a^{\text{FCD}})$ is the only

- 1°. monovalued morphism of the category of reloid triples from a to a ;
- 2°. injective morphism of the category of reloid triples from a to a ;
- 3°. bijective morphism of the category of reloid triples from a to a .

PROOF. We will prove only 1° because the rest follow from it.

Let f be a monovalued morphism of reloid triples from a to a . Then it exists a **Set**-morphism F such that $F \in f$. Trivially $\langle \uparrow^{\text{FCD}} F \rangle a \sqsubseteq a$ and thus $\langle F \rangle^* A \in a$ for every $A \in a$. Thus by the lemma we have that F is the identity function on a set in a and so obviously f is an identity. \square

COROLLARY 1306. For every two atomic filters (with possibly different bases) \mathcal{A} and \mathcal{B} there exists at most one bijective reloid triple from \mathcal{A} to \mathcal{B} .

PROOF. Suppose that f and g are two different bijective reloids from \mathcal{A} to \mathcal{B} . Then $g^{-1} \circ f$ is not the identity reloid (otherwise $g^{-1} \circ f = \text{id}_{\text{dom } f}^{\text{RLD}}$ and so $f = g$ because f and g are isomorphisms). But $g^{-1} \circ f$ is a bijective reloid (as a composition of bijective reloids) from \mathcal{A} to \mathcal{A} what is impossible. \square

14.2. Rudin-Keisler equivalence and Rudin-Keisler order

THEOREM 1307. Atomic filters a and b (with possibly different bases) are isomorphic iff $a \geq b \wedge b \geq a$.

PROOF. Let $a \geq b \wedge b \geq a$. Then there are a monovalued reloids f and g such that $\text{dom } f = a$ and $\text{im } f = b$ and $\text{dom } g = b$ and $\text{im } g = a$. Thus $g \circ f$ and $f \circ g$ are monovalued morphisms from a to a and from b to b . By the above we have $g \circ f = \text{id}_a^{\text{RLD}}$ and $f \circ g = \text{id}_b^{\text{RLD}}$ so $g = f^{-1}$ and $f^{-1} \circ f = \text{id}_a^{\text{RLD}}$ and $f \circ f^{-1} = \text{id}_b^{\text{RLD}}$. Thus f is an injective monovalued reloid from a to b and thus a and b are isomorphic. \square

The last theorem cannot be generalized from atomic filters to arbitrary filters, as it's shown by the following example:

EXAMPLE 1308. $\mathcal{A} \geq_1 \mathcal{B} \wedge \mathcal{B} \geq_1 \mathcal{A}$ but \mathcal{A} is not isomorphic to \mathcal{B} for some filters \mathcal{A} and \mathcal{B} .

PROOF. Consider $\mathcal{A} = \uparrow^{\mathbb{R}} [0; 1]$ and $\mathcal{B} = \prod \left\{ \frac{\uparrow^{\mathbb{R}} [0; 1 + \epsilon]}{\epsilon > 0} \right\}$. Then the function $f = \lambda x \in \mathbb{R} : x/2$ witnesses both inequalities $\mathcal{A} \geq_1 \mathcal{B}$ and $\mathcal{B} \geq_1 \mathcal{A}$. But these filters cannot be isomorphic because only one of them is principal. \square

LEMMA 1309. Let f_0 and f_1 be **Set**-morphisms. Let $f(x, y) = (f_0 x, f_1 y)$ for a function f . Then

$$\left\langle \uparrow^{\text{FCD}(\text{Src } f_0 \times \text{Src } f_1, \text{Dst } f_0 \times \text{Dst } f_1)} f \right\rangle (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \langle \uparrow^{\text{FCD}} f_0 \rangle \mathcal{A} \times^{\text{RLD}} \langle \uparrow^{\text{FCD}} f_1 \rangle \mathcal{B}.$$

PROOF.

$$\begin{aligned} & \left\langle \uparrow^{\text{FCD}(\text{Src } f_0 \times \text{Src } f_1, \text{Dst } f_0 \times \text{Dst } f_1)} f \right\rangle (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \\ & \left\langle \uparrow^{\text{FCD}(\text{Src } f_0 \times \text{Src } f_1, \text{Dst } f_0 \times \text{Dst } f_1)} f \right\rangle \prod \left\{ \frac{\uparrow^{\text{Src } f_0 \times \text{Src } f_1} (A \times B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{Dst } f_0 \times \text{Dst } f_1} \langle f \rangle^* (A \times B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{Dst } f_0 \times \text{Dst } f_1} (\langle f_0 \rangle^* A \times \langle f_1 \rangle^* B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{Dst } f_0} \langle f_0 \rangle^* A \times \uparrow^{\text{Dst } f_1} \langle f_1 \rangle^* B}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \text{(theorem 891)} \\ & \prod \left\{ \frac{\uparrow^{\text{Dst } f_0} \langle f_0 \rangle^* A}{A \in \mathcal{A}} \right\} \times^{\text{RLD}} \prod \left\{ \frac{\uparrow^{\text{Dst } f_1} \langle f_1 \rangle^* B}{B \in \mathcal{B}} \right\} = \\ & \langle \uparrow^{\text{FCD}} f_0 \rangle \mathcal{A} \times^{\text{RLD}} \langle \uparrow^{\text{FCD}} f_1 \rangle \mathcal{B}. \end{aligned}$$

□

THEOREM 1310. Let f be a monovalued reloid. Then $\text{GR } f$ is isomorphic to the filter $\text{dom } f$.

PROOF. Let f be a monovalued reloid. There exists a function $F \in \text{GR } f$. Consider the bijective function $p = \lambda x \in \text{dom } F : (x, Fx)$.

$\langle p \rangle^* \text{dom } F = F$ and consequently

$$\begin{aligned} \langle p \rangle \text{dom } f &= \\ \bigsqcup_{K \in \text{up } f}^{\text{RLD}} \langle p \rangle^* \text{dom } K &= \\ \bigsqcup_{K \in \text{up } f}^{\text{RLD}} \langle p \rangle^* \text{dom}(K \cap F) &= \\ \bigsqcup_{K \in \text{up } f}^{\text{RLD}} (K \cap F) &= \\ \bigsqcup_{K \in \text{up } f}^{\text{RLD}} K &= f. \end{aligned}$$

Thus p witnesses that f is isomorphic to the filter $\text{dom } f$. □

COROLLARY 1311. The graph of a monovalued reloid with atomic domain is atomic.

COROLLARY 1312. $\text{id}_{\mathcal{A}}^{\text{RLD}}$ is isomorphic to \mathcal{A} for every filter \mathcal{A} .

THEOREM 1313. There are atomic filters incomparable by Rudin-Keisler order. (Elements a and b are *incomparable* when $a \not\sqsubseteq b \wedge b \not\sqsubseteq a$.)

PROOF. See [13]. □

THEOREM 1314. \geq_1 and \geq_2 are different relations.

PROOF. Consider a is an arbitrary non-empty filter. Then $a \geq_1 \perp^{\mathcal{F}(\text{Base}(a))}$ but not $a \geq_2 \perp^{\mathcal{F}(\text{Base}(a))}$. □

PROPOSITION 1315. If $a \geq_2 b$ where a is an ultrafilter then b is also an ultrafilter.

PROOF. $b = \langle \uparrow^{\text{FCD}} f \rangle a$ for some $f : \text{Base}(a) \rightarrow \text{Base}(b)$. So b is an ultrafilter since f is monovalued. □

COROLLARY 1316. If $a \geq_1 b$ where a is an ultrafilter then b is also an ultrafilter or $\perp^{\mathcal{F}(\text{Base}(a))}$.

PROOF. $b \sqsubseteq \langle \uparrow^{\text{FCD}} f \rangle a$ for some $f : \text{Base}(a) \rightarrow \text{Base}(b)$. Therefore $b' = \langle \uparrow^{\text{FCD}} f \rangle a$ is an ultrafilter. From this our statement follows. □

PROPOSITION 1317. Principal filters, generated by sets of the same cardinality, are isomorphic.

PROOF. Let A and B be sets of the same cardinality. Then there are a bijection f from A to B . We have $\langle f \rangle^* A = B$ and thus A and B are isomorphic. □

PROPOSITION 1318. If a filter is isomorphic to a principal filter, then it is also a principal filter induced by a set with the same cardinality.

PROOF. Let A be a principal filter and B is a filter isomorphic to A . Then there are sets $X \in A$ and $Y \in B$ such that there are a bijection $f : X \rightarrow Y$ such that $\langle f \rangle^* A = B$.

So $\min B$ exists and $\min B = \langle f \rangle^* \min A$ and thus B is a principal filter (of the same cardinality as A). \square

PROPOSITION 1319. A filter isomorphic to a non-trivial ultrafilter is a non-trivial ultrafilter.

PROOF. Let a be a non-trivial ultrafilter and a be isomorphic to b . Then $a \geq_2 b$ and thus b is an ultrafilter. The filter b cannot be trivial because otherwise a would be also trivial. \square

THEOREM 1320. For an infinite set U there exist $2^{2^{\text{card } U}}$ equivalence classes of isomorphic ultrafilters.

PROOF. The number of bijections between any two given subsets of U is no more than $(\text{card } U)^{\text{card } U} = 2^{\text{card } U}$. The number of bijections between all pairs of subsets of U is no more than $2^{\text{card } U} \cdot 2^{\text{card } U} = 2^{\text{card } U}$. Therefore each isomorphism class contains at most $2^{\text{card } U}$ ultrafilters. But there are $2^{2^{\text{card } U}}$ ultrafilters. So there are $2^{2^{\text{card } U}}$ classes. \square

REMARK 1321. One of the above mentioned equivalence classes contains trivial ultrafilters.

COROLLARY 1322. There exist non-isomorphic nontrivial ultrafilters on any infinite set.

14.3. Consequences

THEOREM 1323. The graph of reloid $\mathcal{F} \times^{\text{RLD}} \uparrow^A \{a\}$ is isomorphic to the filter \mathcal{F} for every set A and $a \in A$.

PROOF. From 1310. \square

THEOREM 1324. If f, g are reloids, $f \sqsubseteq g$ and g is monovalued then $g|_{\text{dom } f} = f$.

PROOF. It's simple to show that $f = \bigsqcup \left\{ \frac{f|_a}{a \in \text{atoms } \mathcal{F}(\text{Src } f)} \right\}$ (use the fact that $k \sqsubseteq f|_a$ for some $a \in \text{atoms } \mathcal{F}(\text{Src } f)$ for every $k \in \text{atoms } f$ and the fact that $\text{RLD}(\text{Src } f, \text{Dst } f)$ is atomistic).

Suppose that $g|_{\text{dom } f} \neq f$. Then there exists $a \in \text{atoms dom } f$ such that $g|_a \neq f|_a$.

Obviously $g|_a \sqsupseteq f|_a$.

If $g|_a \sqsupset f|_a$ then $g|_a$ is not atomic (because $f|_a \neq \perp^{\text{RLD}(\text{Src } f, \text{Dst } f)}$) what contradicts to a theorem above. So $g|_a = f|_a$ what is a contradiction and thus $g|_{\text{dom } f} = f$. \square

COROLLARY 1325. Every monovalued reloid is a restricted principal monovalued reloid.

PROOF. Let f be a monovalued reloid. Then there exists a function $F \in \text{GR } f$. So we have

$$(\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} F)|_{\text{dom } f} = f.$$

\square

COROLLARY 1326. Every monovalued injective reloid is a restricted injective monovalued principal reloid.

PROOF. Let f be a monovalued injective reloid. There exists a function F such that $f = (\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} F)|_{\text{dom } f}$. Also there exists an injection $G \in \text{up } f$.

Thus

$$\begin{aligned} f &= f \sqcap (\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} G)|_{\text{dom } f} = \\ &(\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} F)|_{\text{dom } f} \sqcap (\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} G)|_{\text{dom } f} = \\ &(\uparrow^{\text{RLD}(\text{Src } f, \text{Dst } f)} (F \sqcap G))|_{\text{dom } f}. \end{aligned}$$

Obviously $F \sqcap G$ is an injection. \square

THEOREM 1327. If a reloid f is monovalued and $\text{dom } f$ is an principal filter then f is principal.

PROOF. f is a restricted principal monovalued reloid. Thus $f = F|_{\text{dom } f}$ where F is a principal monovalued reloid. Thus f is principal. \square

LEMMA 1328. If a filter \mathcal{A} is isomorphic to a filter \mathcal{B} then if X is a typed set then there exists a typed set Y such that $\uparrow^{\text{Base}(\mathcal{A})} X \sqcap \mathcal{A}$ is a filter isomorphic to $\uparrow^{\text{Base}(\mathcal{B})} Y \sqcap \mathcal{B}$.

PROOF. Let f be a monovalued injective reloid such that $\text{dom } f = \mathcal{A}$, $\text{im } f = \mathcal{B}$.

By proposition 629 we have: $\uparrow^{\text{Base}(\mathcal{A})} X \sqcap \mathcal{A} = \mathcal{X}$ where \mathcal{X} is a filter complementary to \mathcal{A} . Let $\mathcal{Y} = \mathcal{A} \setminus \mathcal{X}$.

$\langle\langle \text{FCD} \rangle f \rangle \mathcal{X} \sqcap \langle\langle \text{FCD} \rangle f \rangle \mathcal{Y} = \langle\langle \text{FCD} \rangle f \rangle (\mathcal{X} \sqcap \mathcal{Y}) = \perp$ by injectivity of f .

$\langle\langle \text{FCD} \rangle f \rangle \mathcal{X} \sqcup \langle\langle \text{FCD} \rangle f \rangle \mathcal{Y} = \langle\langle \text{FCD} \rangle f \rangle (\mathcal{X} \sqcup \mathcal{Y}) = \langle\langle \text{FCD} \rangle f \rangle \mathcal{A} = \mathcal{B}$. So $\langle\langle \text{FCD} \rangle f \rangle \mathcal{X}$ is a filter complementary to \mathcal{B} . So by proposition 629 there exists a set Y such that $\langle\langle \text{FCD} \rangle f \rangle \mathcal{X} = \uparrow Y \sqcap \mathcal{B}$.

$f|_{\mathcal{X}}$ is obviously a monovalued injective reloid with $\text{dom}(f|_{\mathcal{X}}) = \uparrow X \sqcap \mathcal{A}$ and $\text{im}(f|_{\mathcal{X}}) = \uparrow Y \sqcap \mathcal{B}$. So $\uparrow X \sqcap \mathcal{A}$ is isomorphic to $\uparrow Y \sqcap \mathcal{B}$. \square

EXAMPLE 1329. $\mathcal{A} \geq_2 \mathcal{B} \wedge \mathcal{B} \geq_2 \mathcal{A}$ but \mathcal{A} is not isomorphic to \mathcal{B} for some filters \mathcal{A} and \mathcal{B} .

PROOF. (proof idea by ANDREAS BLASS, rewritten using reloids by me)

Let u_n, h_n with n ranging over the set \mathbb{Z} be sequences of ultrafilters on \mathbb{N} and functions $\mathbb{N} \rightarrow \mathbb{N}$ such that $\langle \uparrow^{\text{FCD}(\mathbb{N}, \mathbb{N})} h_n \rangle u_{n+1} = u_n$ and u_n are pairwise non-isomorphic. (See [6] for a proof that such ultrafilters and functions exist.)

$\mathcal{A} \stackrel{\text{def}}{=} \bigsqcup_{n \in \mathbb{Z}} (\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1})$; $\mathcal{B} \stackrel{\text{def}}{=} \bigsqcup_{n \in \mathbb{Z}} (\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n})$.

Let the **Set**-morphisms $f, g : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Z} \times \mathbb{N}$ be defined by the formulas $f(n, x) = (n, h_{2n}x)$ and $g(n, x) = (n-1, h_{2n-1}x)$.

Using the fact that every function induces a complete funcoid and a lemma above we get:

$$\begin{aligned}
& \langle \uparrow^{\text{FCD}} f \rangle \mathcal{A} = \\
& \bigsqcup \langle \langle \uparrow^{\text{FCD}} f \rangle \rangle * \left\{ \frac{\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1}}{n \in \mathbb{Z}} \right\} = \\
& \bigsqcup \left\{ \frac{\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n}}{n \in \mathbb{Z}} \right\} = \\
& \mathcal{B}. \\
& \langle \uparrow^{\text{FCD}} g \rangle \mathcal{B} = \\
& \bigsqcup \langle \langle \uparrow^{\text{FCD}} g \rangle \rangle * \left\{ \frac{\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n}}{n \in \mathbb{Z}} \right\} = \\
& \bigsqcup \left\{ \frac{\uparrow^{\mathbb{Z}} \{n-1\} \times^{\text{RLD}} u_{2n-1}}{n \in \mathbb{Z}} \right\} = \\
& \bigsqcup \left\{ \frac{\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1}}{n \in \mathbb{Z}} \right\} = \\
& \mathcal{A}.
\end{aligned}$$

It remains to show that \mathcal{A} and \mathcal{B} are not isomorphic.

Let $X \in \text{up}(\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1})$ for some $n \in \mathbb{Z}$. Then if $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \sqcap \mathcal{A}$ is an ultrafilter we have $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \sqcap \mathcal{A} = \uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1}$ and thus by the theorem 1323 is isomorphic to u_{2n+1} .

If $X \notin \text{up}(\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1})$ for every $n \in \mathbb{Z}$ then $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \text{up}(\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n+1})$ and thus $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \text{up} \mathcal{A}$ and thus $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \sqcap \mathcal{A} = \perp^{\mathbb{Z} \times \mathbb{N}}$.

We have also

$$\begin{aligned}
(\uparrow^{\mathbb{Z}} \{0\} \times^{\text{RLD}} \mathbb{N}) \sqcap \mathcal{B} &= (\uparrow^{\mathbb{Z}} \{0\} \times^{\text{RLD}} \mathbb{N}) \sqcap \bigsqcup \left\{ \frac{\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n}}{n \in \mathbb{Z}} \right\} = \\
& \bigsqcup \left\{ \frac{(\uparrow^{\mathbb{Z}} \{0\} \times^{\text{RLD}} \mathbb{N}) \sqcap (\uparrow^{\mathbb{Z}} \{n\} \times^{\text{RLD}} u_{2n})}{n \in \mathbb{Z}} \right\} = \uparrow^{\mathbb{Z}} \{0\} \times^{\text{RLD}} u_0 \text{ (an ultrafilter)}.
\end{aligned}$$

Thus every ultrafilter generated as intersecting \mathcal{A} with a principal filter $\uparrow^{\mathbb{Z} \times \mathbb{N}} X$ is isomorphic to some u_{2n+1} and thus is not isomorphic to u_0 . By the lemma it follows that \mathcal{A} and \mathcal{B} are non-isomorphic. \square

14.3.1. Metamonovalued reolds.

PROPOSITION 1330. $(\bigcap G) \circ f = \bigcap_{g \in G} (g \circ f)$ for every function f and a set G of binary relations.

PROOF.

$$\begin{aligned}
(x, z) \in \left(\bigcap G \right) \circ f &\Leftrightarrow \\
\exists y : (fx = y \wedge (y, z) \in \bigcap G) &\Leftrightarrow \\
(fx, z) \in \bigcap G &\Leftrightarrow \\
\forall g \in G : (fx, z) \in g &\Leftrightarrow \\
\forall g \in G \exists y : (fx = y \wedge (y, z) \in g) &\Leftrightarrow \\
\forall g \in G : (x, z) \in g \circ f &\Leftrightarrow \\
(x, z) \in \bigcap_{g \in G} (g \circ f). &
\end{aligned}$$

\square

LEMMA 1331. $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$ if f is a monovalued principal reloid and G is a set of reloids (with matching sources and destinations).

PROOF. Let $f = \uparrow^{\text{RLD}} \varphi$ for some monovalued **Rel**-morphism φ .

$$(\prod G) \circ f = \prod_{g \in \text{up} \prod G}^{\text{RLD}} (g \circ \varphi);$$

$$\begin{aligned} & \text{up} \prod_{g \in G} (g \circ f) = \\ & \text{up} \prod_{g \in G} \prod_{\Gamma \in \text{up} g}^{\text{RLD}} (\Gamma \circ \varphi) = \\ & \text{up} \prod_{g \in G} \bigcup \left\{ \frac{\uparrow^{\text{RLD}} (\Gamma \circ \varphi)}{\Gamma \in \text{up} g} \right\} = \\ & \text{up} \prod_{\Gamma \in \text{up} \prod G}^{\text{RLD}} (\Gamma \circ \varphi) = \\ & \text{up} \prod \left\{ \frac{(\Gamma_0 \circ \varphi) \sqcap \dots \sqcap (\Gamma_n \circ \varphi)}{\Gamma_i \in \text{up} \prod G \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \text{(proposition above)} \\ & \text{up} \prod \left\{ \frac{(\Gamma_0 \sqcap \dots \sqcap \Gamma_n) \circ \varphi}{\Gamma_i \in \text{up} \prod G \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \\ & \text{up} \prod \left\{ \frac{\Gamma \circ \varphi}{\Gamma \in \text{up} \prod G} \right\}. \end{aligned}$$

Thus $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$. □

THEOREM 1332.

- 1°. Monovalued reloids are metamonovalued.
- 2°. Injective reloids are metainjective.

PROOF. We will prove only the first, as the second is dual.

Let G be a set of reloids and f be a monovalued reloid.

Let f' be a principal monovalued continuation of f (so that $f = f'|_{\text{dom } f}$).

By the lemma $(\prod G) \circ f' = \prod_{g \in G} (g \circ f')$. Restricting this equality to $\text{dom } f$ we get: $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$. □

CONJECTURE 1333. Every metamonovalued reloid is monovalued.

Counter-examples about funcoids and reloids

For further examples we will use the filter defined by the formula

$$\Delta = \prod^{\mathcal{F}(\mathbb{R})} \left\{ \frac{]-\epsilon; \epsilon[}{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}.$$

I will denote $\Omega(A)$ the Fréchet filter on a set A .

EXAMPLE 1334. There exist a funcoid f and a set S of funcoids such that $f \circ \bigsqcup S \neq \bigsqcup \langle f \circ \rangle^* S$.

PROOF. Let $f = \Delta \times^{\text{FCD}} \uparrow^{\mathcal{F}(\mathbb{R})} \{0\}$ and $S = \left\{ \frac{ \uparrow^{\text{FCD}(\mathbb{R}, \mathbb{R})} (] \epsilon; +\infty[\times \{0\}) }{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}$. Then

$$\begin{aligned} f \circ \bigsqcup S &= (\Delta \times^{\text{FCD}} \uparrow^{\mathcal{F}(\mathbb{R})} \{0\}) \circ \uparrow^{\text{FCD}(\mathbb{R}, \mathbb{R})} (]0; +\infty[\times \{0\}) = \\ &= (\Delta \circ \uparrow^{\mathcal{F}(\mathbb{R})}]0; +\infty[) \times^{\text{FCD}} \uparrow^{\mathcal{F}(\mathbb{R})} \{0\} \neq \perp^{\text{FCD}(\mathbb{R}, \mathbb{R})} \end{aligned}$$

while $\bigsqcup \langle f \circ \rangle^* S = \bigsqcup \{ \perp^{\text{FCD}(\mathbb{R}, \mathbb{R})} \} = \perp^{\text{FCD}(\mathbb{R}, \mathbb{R})}$. \square

EXAMPLE 1335. There exist a set R of funcoids and a funcoid f such that $f \circ \bigsqcup R \neq \bigsqcup \langle f \circ \rangle^* R$.

PROOF. Let $f = \Delta \times^{\text{FCD}} \uparrow^{\mathcal{F}(\mathbb{R})} \{0\}$, $R = \left\{ \frac{ \uparrow^{\mathbb{R}} \{0\} \times^{\text{FCD}} \uparrow^{\mathbb{R}}] \epsilon; +\infty[}{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}$.

We have $\bigsqcup R = \uparrow^{\mathbb{R}} \{0\} \times^{\text{FCD}} \uparrow^{\mathbb{R}}]0; +\infty[$; $f \circ \bigsqcup R = \uparrow^{\text{FCD}(\mathbb{R}, \mathbb{R})} (\{0\} \times \{0\}) \neq \perp^{\text{FCD}(\mathbb{R}, \mathbb{R})}$ and $\bigsqcup \langle f \circ \rangle^* R = \bigsqcup \{ \perp^{\text{FCD}(\mathbb{R}, \mathbb{R})} \} = \perp^{\text{FCD}(\mathbb{R}, \mathbb{R})}$. \square

EXAMPLE 1336. There exist a set R of reloids and a reloid f such that $f \circ \bigsqcup R \neq \bigsqcup \langle f \circ \rangle^* R$.

PROOF. Let $f = \Delta \times^{\text{RLD}} \uparrow^{\mathcal{F}(\mathbb{R})} \{0\}$, $R = \left\{ \frac{ \uparrow^{\mathbb{R}} \{0\} \times^{\text{RLD}} \uparrow^{\mathbb{R}}] \epsilon; +\infty[}{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}$.

We have $\bigsqcup R = \uparrow^{\mathbb{R}} \{0\} \times^{\text{RLD}} \uparrow^{\mathbb{R}}]0; +\infty[$; $f \circ \bigsqcup R = \uparrow^{\text{RLD}(\mathbb{R}, \mathbb{R})} (\{0\} \times \{0\}) \neq \perp^{\text{RLD}(\mathbb{R}, \mathbb{R})}$ and $\bigsqcup \langle f \circ \rangle^* R = \bigsqcup \{ \perp^{\text{RLD}(\mathbb{R}, \mathbb{R})} \} = \perp^{\text{RLD}(\mathbb{R}, \mathbb{R})}$. \square

EXAMPLE 1337. There exist a set R of funcoids and filters \mathcal{X} and \mathcal{Y} such that

- 1°. $\mathcal{X} \llbracket \bigsqcup R \rrbracket \mathcal{Y} \wedge \nexists f \in R : \mathcal{X} \llbracket f \rrbracket \mathcal{Y}$;
- 2°. $\langle \bigsqcup R \rangle \mathcal{X} \sqsubset \bigsqcup \left\{ \frac{ \langle f \rangle \mathcal{X} }{ f \in R } \right\}$.

PROOF.

1°. Take $\mathcal{X} = \Delta$ and $\mathcal{Y} = \top^{\mathcal{F}(\mathbb{R})}$, $R = \left\{ \frac{ \uparrow^{\text{FCD}(\mathbb{R}, \mathbb{R})} (] \epsilon; +\infty[\times \mathbb{R}) }{ \epsilon \in \mathbb{R}, \epsilon > 0 } \right\}$. Then $\bigsqcup R = \uparrow^{\text{FCD}(\mathbb{R}, \mathbb{R})} (]0; +\infty[\times \mathbb{R})$. So $\mathcal{X} \llbracket \bigsqcup R \rrbracket \mathcal{Y}$ and $\forall f \in R : \neg(\mathcal{X} \llbracket f \rrbracket \mathcal{Y})$.

2°. With the same \mathcal{X} and R we have $\langle \bigsqcup R \rangle \mathcal{X} = \top^{\mathcal{F}(\mathbb{R})}$ and $\langle f \rangle \mathcal{X} = \perp^{\mathcal{F}(\mathbb{R})}$ for every $f \in R$, thus $\bigsqcup \left\{ \frac{ \langle f \rangle \mathcal{X} }{ f \in R } \right\} = \perp^{\mathcal{F}(\mathbb{R})}$. \square

EXAMPLE 1338. $\bigsqcup_{\mathcal{B} \in T} (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \neq \mathcal{A} \times^{\text{RLD}} \bigsqcup T$ for some filter \mathcal{A} and set of filters T (with a common base).

PROOF. Take $\mathbb{R}_+ = \left\{ \frac{x \in \mathbb{R}}{x > 0} \right\}$, $\mathcal{A} = \Delta$, $T = \left\{ \frac{\uparrow \{x\}}{x \in \mathbb{R}_+} \right\}$ where $\uparrow = \uparrow^{\mathbb{R}}$.

$$\bigsqcup T = \uparrow \mathbb{R}_+; \mathcal{A} \times^{\text{RLD}} \bigsqcup T = \Delta \times^{\text{RLD}} \uparrow \mathbb{R}_+.$$

$$\bigsqcup_{\mathcal{B} \in T} (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \bigsqcup_{x \in \mathbb{R}_+} (\Delta \times^{\text{RLD}} \uparrow \{x\}).$$

We'll prove that $\bigsqcup_{x \in \mathbb{R}_+} (\Delta \times^{\text{RLD}} \uparrow \{x\}) \neq \Delta \times^{\text{RLD}} \uparrow \mathbb{R}_+$.

Consider $K = \bigcup_{x \in \mathbb{R}_+} (\{x\} \times]-1/x; 1/x[)$.

$K \in \text{up}(\Delta \times^{\text{RLD}} \uparrow \{x\})$ and thus $K \in \text{up} \bigsqcup_{x \in \mathbb{R}_+} (\Delta \times^{\text{RLD}} \uparrow \{x\})$. But $K \notin \text{up}(\Delta \times^{\text{RLD}} \uparrow \mathbb{R}_+)$. \square

THEOREM 1339. For a filter a we have $a \times^{\text{RLD}} a \sqsubseteq 1_{\text{Base}(a)}^{\text{RLD}}$ only in the case if $a = \perp^{\mathcal{F}(\text{Base}(a))}$ or a is a trivial ultrafilter.

PROOF. If $a \times^{\text{RLD}} a \sqsubseteq 1_{\text{Base}(a)}^{\text{RLD}}$ then there exists $m \in \text{up}(a \times^{\text{RLD}} a)$ such that $m \sqsubseteq 1_{\text{Base}(a)}^{\text{Rel}}$. Consequently there exist $A, B \in \text{up} a$ such that $A \times B \sqsubseteq 1_{\text{Base}(a)}^{\text{Rel}}$ what is possible only in the case when $\uparrow A = \uparrow B = a$ is trivial a ultrafilter or the least filter. \square

COROLLARY 1340. Reloidal product of a non-trivial atomic filter with itself is non-atomic.

PROOF. Obviously $(a \times^{\text{RLD}} a) \cap 1_{\text{Base}(a)}^{\text{RLD}} \neq \perp^{\text{RLD}}$ and $(a \times^{\text{RLD}} a) \cap 1_{\text{Base}(a)}^{\text{RLD}} \sqsubset a \times^{\text{RLD}} a$. \square

EXAMPLE 1341. There exist two atomic reloids whose composition is non-atomic and non-empty.

PROOF. Let a be a non-trivial ultrafilter on \mathbb{N} and $x \in \mathbb{N}$. Then

$$\begin{aligned} (a \times^{\text{RLD}} \uparrow^{\mathbb{N}} \{x\}) \circ (\uparrow^{\mathbb{N}} \{x\} \times^{\text{RLD}} a) &= \prod_{A \in a}^{\text{RLD}(\mathbb{N}, \mathbb{N})} ((A \times \{x\}) \circ (\{x\} \times A)) = \\ &= \prod_{A \in a}^{\text{RLD}(\mathbb{N}, \mathbb{N})} (A \times A) = a \times^{\text{RLD}} a \end{aligned}$$

is non-atomic despite of $a \times^{\text{RLD}} \uparrow^{\mathbb{N}} \{x\}$ and $\uparrow^{\mathbb{N}} \{x\} \times^{\text{RLD}} a$ are atomic. \square

EXAMPLE 1342. There exists non-monovalued atomic reloid.

PROOF. From the previous example it follows that the atomic reloid $\uparrow^{\mathbb{N}} \{x\} \times^{\text{RLD}} a$ is not monovalued. \square

EXAMPLE 1343. Non-convex reloids exist.

PROOF. Let a be a non-trivial ultrafilter. Then id_a^{RLD} is non-convex. This follows from the fact that only reloidal products which are below $1_{\text{Base}(a)}^{\text{RLD}}$ are reloidal products of ultrafilters and id_a^{RLD} is not their join. \square

EXAMPLE 1344. There exists (atomic) composable funcoids f and g such that

$$H \in \text{up}(g \circ f) \not\Rightarrow \exists F \in \text{up} f, G \in \text{up} g : H \sqsupseteq G \circ F.$$

PROOF. Let a be a nontrivial ultrafilter and p be an arbitrary point, $f = a \times^{\text{FCD}} \{p\}$, $g = \{p\} \times^{\text{FCD}} a$. Then $g \circ f = a \times^{\text{FCD}} a$. Take $H = 1$. Let $F \in \text{up} f$ and $G \in \text{up} g$. We have $F \in \text{up}(A_0 \times^{\text{FCD}} \{p\})$, $G \in \text{up}(\{p\} \times^{\text{FCD}} A_1)$ where $A_0, A_1 \in \text{up} a$ (take $A_0 = \langle F \rangle^* @ \{p\}$ and similarly for A_1). Thus $G \circ F \sqsupseteq A_0 \times A_1$ and so $H \notin \text{up}(G \circ F)$. \square

EXAMPLE 1345. $(\text{RLD})_{\text{in}} f \neq (\text{RLD})_{\text{out}} f$ for a funcoid f .

PROOF. Let $f = 1_{\mathbb{N}}^{\text{FCD}}$. Then $(\text{RLD})_{\text{in}}f = \bigsqcup_{a \in \text{atoms}_{\mathcal{F}(\mathbb{N})}}(a \times^{\text{RLD}} a)$ and $(\text{RLD})_{\text{out}}f = 1_{\mathbb{N}}^{\text{RLD}}$. But we have shown above $a \times^{\text{RLD}} a \not\sqsubseteq 1_{\mathbb{N}}^{\text{RLD}}$ for non-trivial ultrafilter a , and so $(\text{RLD})_{\text{in}}f \not\sqsubseteq (\text{RLD})_{\text{out}}f$. \square

PROPOSITION 1346. $1_{\mathfrak{U}}^{\text{FCD}} \cap \uparrow^{\text{FCD}(\mathfrak{U}, \mathfrak{U})}((\mathfrak{U} \times \mathfrak{U}) \setminus \text{id}_{\mathfrak{U}}) = \text{id}_{\Omega(\mathfrak{U})}^{\text{FCD}} \neq \perp^{\text{FCD}(\mathfrak{U}, \mathfrak{U})}$ for every infinite set \mathfrak{U} .

PROOF. Note that $\langle \text{id}_{\Omega(\mathfrak{U})}^{\text{FCD}} \rangle \mathcal{X} = \mathcal{X} \cap \Omega(\mathfrak{U})$ for every filter \mathcal{X} on \mathfrak{U} .

Let $f = 1_{\mathfrak{U}}^{\text{FCD}}$, $g = \uparrow^{\text{FCD}(\mathfrak{U}, \mathfrak{U})}((\mathfrak{U} \times \mathfrak{U}) \setminus \text{id}_{\mathfrak{U}})$.

Let x be a non-trivial ultrafilter on \mathfrak{U} . If $X \in \text{up } x$ then $\text{card } X \geq 2$ (In fact, X is infinite but we don't need this.) and consequently $\langle g \rangle^* X = \top^{\mathcal{F}(\mathfrak{U})}$. Thus $\langle g \rangle x = \top^{\mathcal{F}(\mathfrak{U})}$. Consequently

$$\langle f \sqcap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap \top^{\mathcal{F}(\mathfrak{U})} = x.$$

Also $\langle \text{id}_{\Omega(\mathfrak{U})}^{\text{FCD}} \rangle x = x \cap \Omega(\mathfrak{U}) = x$.

Let now x be a trivial ultrafilter. Then $\langle f \rangle x = x$ and $\langle g \rangle x = \top^{\mathcal{F}(\mathfrak{U})} \setminus x$. So

$$\langle f \sqcap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap (\top^{\mathcal{F}(\mathfrak{U})} \setminus x) = \perp^{\mathcal{F}(\mathfrak{U})}.$$

Also $\langle \text{id}_{\Omega(\mathfrak{U})}^{\text{FCD}} \rangle x = x \cap \Omega(\mathfrak{U}) = \perp^{\mathcal{F}(\mathfrak{U})}$.

So $\langle f \sqcap g \rangle x = \langle \text{id}_{\Omega(\mathfrak{U})}^{\text{FCD}} \rangle x$ for every ultrafilter x on \mathfrak{U} . Thus $f \sqcap g = \text{id}_{\Omega(\mathfrak{U})}^{\text{FCD}}$. \square

EXAMPLE 1347. There exist binary relations f and g such that $\uparrow^{\text{FCD}(A, B)} f \sqcap \uparrow^{\text{FCD}(A, B)} g \neq \uparrow^{\text{FCD}(A, B)}(f \sqcap g)$ for some sets A, B such that $f, g \subseteq A \times B$.

PROOF. From the proposition above. \square

EXAMPLE 1348. There exists a principal funcoid which is not a complemented element of the lattice of funcoids.

PROOF. I will prove that quasi-complement of the funcoid $1_{\mathbb{N}}^{\text{FCD}}$ is not its complement (it is enough by proposition 145). We have:

$$\begin{aligned} (1_{\mathbb{N}}^{\text{FCD}})^* &= \\ & \bigsqcup \left\{ \frac{c \in \text{FCD}(\mathbb{N}, \mathbb{N})}{c \asymp 1_{\mathbb{N}}^{\text{FCD}}} \right\} \sqsupseteq \\ & \bigsqcup \left\{ \frac{\uparrow^{\mathbb{N}} \{\alpha\} \times^{\text{FCD}} \uparrow^{\mathbb{N}} \{\beta\}}{\alpha, \beta \in \mathbb{N}, \uparrow^{\mathbb{N}} \{\alpha\} \times^{\text{FCD}} \uparrow^{\mathbb{N}} \{\beta\} \asymp 1_{\mathbb{N}}^{\text{FCD}}} \right\} = \\ & \bigsqcup \left\{ \frac{\uparrow^{\mathbb{N}} \{\alpha\} \times^{\text{FCD}} \uparrow^{\mathbb{N}} \{\beta\}}{\alpha, \beta \in \mathbb{N}, \alpha \neq \beta} \right\} = \\ & \uparrow^{\text{FCD}(\mathbb{N}, \mathbb{N})} \bigcup \left\{ \frac{\{\alpha\} \times \{\beta\}}{\alpha, \beta \in \mathbb{N}, \alpha \neq \beta} \right\} = \\ & \uparrow^{\text{FCD}(\mathbb{N}, \mathbb{N})} (\mathbb{N} \times \mathbb{N} \setminus \text{id}_{\mathbb{N}}) \end{aligned}$$

(used corollary 923). But by proved above $(1_{\mathbb{N}}^{\text{FCD}})^* \cap 1_{\mathbb{N}}^{\text{FCD}} \neq \perp^{\mathcal{F}(\mathbb{N})}$. \square

EXAMPLE 1349. There exists a funcoid h such that $\text{up } h$ is not a filter.

PROOF. Consider the funcoid $h = \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}}$. We have (from the proof of proposition 1346) that $f \in \text{up } h$ and $g \in \text{up } h$, but $f \sqcap g \notin \text{up } h$. \square

EXAMPLE 1350. There exists a funcoid $h \neq \perp^{\text{FCD}(A, B)}$ such that $(\text{RLD})_{\text{out}}h = \perp^{\text{RLD}(A, B)}$.

PROOF. Consider $h = \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}}$. By proved above $h = f \sqcap g$ where $f = 1_{\mathbb{N}}^{\text{FCD}} = \uparrow^{\text{FCD}(\mathbb{N}, \mathbb{N})} \text{id}_{\mathbb{N}}$, $g = \uparrow^{\text{FCD}(\mathbb{N}, \mathbb{N})} (\mathbb{N} \times \mathbb{N} \setminus \text{id}_{\mathbb{N}})$.

We have $\text{id}_{\mathbb{N}}, \mathbb{N} \times \mathbb{N} \setminus \text{id}_{\mathbb{N}} \in \text{GR } h$.

So

$$(\text{RLD})_{\text{out}} h = \bigsqcap^{\text{RLD}} \text{up } h = \bigsqcap^{\text{RLD}(\mathbb{N}, \mathbb{N})} \text{GR } h \sqsubseteq \uparrow^{\text{RLD}(\mathbb{N}, \mathbb{N})} (\text{id}_{\mathbb{N}} \cap (\mathbb{N} \times \mathbb{N} \setminus \text{id}_{\mathbb{N}})) = \perp^{\text{RLD}(\mathbb{N}, \mathbb{N})};$$

and thus $(\text{RLD})_{\text{out}} h = \perp^{\text{RLD}(\mathbb{N}, \mathbb{N})}$. \square

EXAMPLE 1351. There exists a funcoid h such that $(\text{FCD})(\text{RLD})_{\text{out}} h \neq h$.

PROOF. It follows from the previous example. \square

EXAMPLE 1352. $(\text{RLD})_{\text{in}}(\text{FCD})f \neq f$ for some convex reloid f .

PROOF. Let $f = 1_{\mathbb{N}}^{\text{RLD}}$. Then $(\text{FCD})f = 1_{\mathbb{N}}^{\text{FCD}}$. Let a be some non-trivial ultrafilter on \mathbb{N} . Then $(\text{RLD})_{\text{in}}(\text{FCD})f \sqsupseteq a \times^{\text{RLD}} a \not\sqsubseteq 1_{\mathbb{N}}^{\text{RLD}}$ and thus $(\text{RLD})_{\text{in}}(\text{FCD})f \not\sqsubseteq f$. \square

EXAMPLE 1353. There exist composable funcoids f and g such that

$$(\text{RLD})_{\text{out}}(g \circ f) \sqsupset (\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f.$$

PROOF. $f = \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}}$ and $g = \top^{\mathcal{F}(\mathbb{N})} \times^{\text{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}$ for some $\alpha \in \mathbb{N}$. Then $(\text{RLD})_{\text{out}}f = \perp^{\text{RLD}(\mathbb{N}, \mathbb{N})}$ and thus $(\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f = \perp^{\text{RLD}(\mathbb{N}, \mathbb{N})}$.

We have $g \circ f = \Omega(\mathbb{N}) \times^{\text{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}$.

$(\text{RLD})_{\text{out}}(\Omega(\mathbb{N}) \times^{\text{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}) = \Omega(\mathbb{N}) \times^{\text{RLD}} \uparrow^{\mathbb{N}} \{\alpha\}$ by properties of funcoidal reloids.

Thus $(\text{RLD})_{\text{out}}(g \circ f) = \Omega(\mathbb{N}) \times^{\text{RLD}} \uparrow^{\mathbb{N}} \{\alpha\} \neq \perp^{\text{RLD}(\mathbb{N}, \mathbb{N})}$. \square

CONJECTURE 1354. For every composable funcoids f and g

$$(\text{RLD})_{\text{out}}(g \circ f) \sqsupseteq (\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f.$$

EXAMPLE 1355. (FCD) does not preserve binary meets.

PROOF. $(\text{FCD})(1_{\mathbb{N}}^{\text{RLD}} \sqcap (\top^{\text{RLD}(\mathbb{N}, \mathbb{N})} \setminus 1_{\mathbb{N}}^{\text{RLD}})) = (\text{FCD})\perp^{\text{RLD}(\mathbb{N}, \mathbb{N})} = \perp^{\text{FCD}(\mathbb{N}, \mathbb{N})}$.

On the other hand,

$$\begin{aligned} (\text{FCD})1_{\mathbb{N}}^{\text{RLD}} \sqcap (\text{FCD})(\top^{\text{RLD}(\mathbb{N}, \mathbb{N})} \setminus 1_{\mathbb{N}}^{\text{RLD}}) = \\ 1_{\mathbb{N}}^{\text{FCD}} \sqcap \uparrow^{\text{FCD}(\mathbb{N}, \mathbb{N})} (\mathbb{N} \times \mathbb{N} \setminus \text{id}_{\mathbb{N}}) = \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}} \neq \perp^{\text{FCD}(\mathbb{N}, \mathbb{N})} \end{aligned}$$

(used proposition 1059). \square

COROLLARY 1356. (FCD) is not an upper adjoint (in general).

Considering restricting polynomials (considered as reloids) to ultrafilters, it is simple to prove that each that restriction is injective if not restricting a constant polynomial. Does this hold in general? No, see the following example:

EXAMPLE 1357. There exists a monovalued reloid with atomic domain which is neither injective nor constant (that is not a restriction of a constant function).

PROOF. (based on [31]) Consider the function $F \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}}$ defined by the formula $(x, y) \mapsto x$.

Let ω_x be a non-trivial ultrafilter on the vertical line $\{x\} \times \mathbb{N}$ for every $x \in \mathbb{N}$.

Let T be the collection of such sets Y that $Y \cap (\{x\} \times \mathbb{N}) \in \omega_x$ for all but finitely many vertical lines. Obviously T is a filter.

Let $\omega \in \text{atoms } T$.

For every $x \in \mathbb{N}$ we have some $Y \in T$ for which $(\{x\} \times \mathbb{N}) \cap Y = \emptyset$ and thus $\uparrow^{\mathbb{N} \times \mathbb{N}} (\{x\} \times \mathbb{N}) \sqcap \omega = \perp^{\mathcal{F}(\mathbb{N} \times \mathbb{N})}$.

Let $g = (\uparrow^{\text{RLD}(\mathbb{N}, \mathbb{N})} F)|_{\omega}$. If g is constant, then there exist a constant function $G \in \text{up } g$ and $F \sqcap G$ is also constant. Obviously $\text{dom } \uparrow^{\text{RLD}(\mathbb{N} \times \mathbb{N}, \mathbb{N})} (F \sqcap G) \supseteq \omega$. The function $F \sqcap G$ cannot be constant because otherwise $\omega \subseteq \text{dom } \uparrow^{\text{RLD}(\mathbb{N} \times \mathbb{N}, \mathbb{N})} (F \sqcap G) \subseteq \uparrow^{\mathbb{N} \times \mathbb{N}} (\{x\} \times \mathbb{N})$ for some $x \in \mathbb{N}$ what is impossible by proved above. So g is not constant.

Suppose that g is injective. Then there exists an injection $G \in \text{up } g$. $F \sqcap G \in \text{up } g$ is an injection which depends only on the first argument. So $\text{dom}(F \sqcap G)$ intersects each vertical line by atmost one element that is $\overline{\text{dom}(F \sqcap G)}$ intersects every vertical line by the whole line or the line without one element. Thus $\text{dom}(F \sqcap G) \in T \supseteq \omega$ and consequently $\text{dom}(F \sqcap G) \not\subseteq \omega$ what is impossible.

Thus g is neither injective nor constant. \square

15.1. Second product. Oblique product

DEFINITION 1358. $\mathcal{A} \times_F^{\text{RLD}} \mathcal{B} = (\text{RLD})_{\text{out}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B})$ for every filters \mathcal{A} and \mathcal{B} . I will call it *second product* of filters \mathcal{A} and \mathcal{B} .

REMARK 1359. The letter F in the above definition is from the word ‘‘funcoid’’. It signifies that it seems to be impossible to define $\mathcal{A} \times_F^{\text{RLD}} \mathcal{B}$ directly without referring to funcoidal product.

DEFINITION 1360. *Oblique products* of filters \mathcal{A} and \mathcal{B} are defined as

$$\mathcal{A} \times \mathcal{B} = \prod \left\{ \frac{\uparrow^{\text{RLD}} f}{f \in \mathbf{Rel}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B})), \exists B \in \text{up } \mathcal{B} : \uparrow^{\text{FCD}} f \supseteq \mathcal{A} \times^{\text{FCD}} \uparrow B} \right\};$$

$$\mathcal{A} \times \mathcal{B} = \prod \left\{ \frac{\uparrow^{\text{RLD}} f}{f \in \mathbf{Rel}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B})), \exists A \in \text{up } \mathcal{A} : \uparrow^{\text{FCD}} f \supseteq \uparrow A \times^{\text{FCD}} \mathcal{B}} \right\}.$$

PROPOSITION 1361.

- 1°. $\mathcal{A} \times \mathcal{B} = \mathcal{A} \times_F^{\text{RLD}} \mathcal{B}$ if \mathcal{A} and \mathcal{B} are filters and \mathcal{B} is principal.
- 2°. $\mathcal{A} \times \mathcal{B} = \mathcal{A} \times_F^{\text{RLD}} \mathcal{B}$ if \mathcal{A} and \mathcal{B} are filters and \mathcal{A} is principal.

PROOF. $\mathcal{A} \times \mathcal{B} = \prod^{\text{RLD}} \left\{ \frac{f}{f \in \mathbf{Rel}, f \supseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}} \right\} = \mathcal{A} \times_F^{\text{RLD}} \mathcal{B}$. The other is analogous. \square

PROPOSITION 1362. $\mathcal{A} \times_F^{\text{RLD}} \mathcal{B} \subseteq \mathcal{A} \times \mathcal{B} \subseteq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for every filters \mathcal{A}, \mathcal{B} .

PROOF.

$$\begin{aligned} & \mathcal{A} \times \mathcal{B} \subseteq \\ \prod \left\{ \frac{\uparrow^{\text{RLD}} f}{f \in \mathbf{Rel}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B})), \exists A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B} : \uparrow^{\text{FCD}} f \supseteq \uparrow A \times^{\text{FCD}} \uparrow B} \right\} & \subseteq \\ \prod \left\{ \frac{\uparrow A \times^{\text{RLD}} \uparrow B}{A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\} & = \\ \mathcal{A} \times^{\text{RLD}} \mathcal{B}. & \\ \mathcal{A} \times \mathcal{B} \supseteq & \\ \prod \left\{ \frac{\uparrow^{\text{RLD}} f}{f \in \mathbf{Rel}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B})), \uparrow^{\text{FCD}} f \supseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}} \right\} & = \\ \prod \left\{ \frac{\uparrow^{\text{RLD}} f}{f \in \text{up}(\mathcal{A} \times^{\text{FCD}} \mathcal{B})} \right\} & = \\ (\text{RLD})_{\text{out}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) & = \\ \mathcal{A} \times_F^{\text{RLD}} \mathcal{B}. & \\ & \square \end{aligned}$$

CONJECTURE 1363. $\mathcal{A} \times_F^{\text{RLD}} \mathcal{B} \sqsubset \mathcal{A} \times \mathcal{B}$ for some filters \mathcal{A}, \mathcal{B} .

A stronger conjecture:

CONJECTURE 1364. $\mathcal{A} \times_F^{\text{RLD}} \mathcal{B} \sqsubset \mathcal{A} \times \mathcal{B} \sqsubset \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for some filters \mathcal{A}, \mathcal{B} . Particularly, is this formula true for $\mathcal{A} = \mathcal{B} = \Delta \cap \uparrow^{\mathbb{R}}]0; +\infty[$?

The above conjecture is similar to Fermat Last Theorem as having no value by itself but being somehow challenging to prove it (not expected to be as hard as FLT however).

EXAMPLE 1365. $\mathcal{A} \times \mathcal{B} \sqsubset \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for some filters \mathcal{A}, \mathcal{B} .

PROOF. It's enough to prove $\mathcal{A} \times \mathcal{B} \neq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$.

Let $\Delta_+ = \Delta \cap \uparrow^{\mathbb{R}}]0; +\infty[$. Let $\mathcal{A} = \mathcal{B} = \Delta_+$.

Let $K = (\leq)|_{\mathbb{R} \times \mathbb{R}}$.

Obviously $K \notin \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$.

$\mathcal{A} \times \mathcal{B} \sqsupseteq \uparrow^{\text{RLD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B}))} K$ and thus $K \in \text{up}(\mathcal{A} \times \mathcal{B})$ because

$$\uparrow^{\text{FCD}(\text{Base}(\mathcal{A}), \text{Base}(\mathcal{B}))} K \sqsupseteq \Delta_+ \times^{\text{FCD}} \uparrow B = \mathcal{A} \times^{\text{FCD}} \uparrow B$$

for $B =]0; +\infty[$ because for every $X \in \partial \Delta_+$ there is $x \in X$ such that $x \in]0; \epsilon[$ (for every positive ϵ) and thus $] \epsilon; +\infty[\subseteq \langle K \rangle^* \{x\}$ so having

$$\langle K \rangle^* X =]0; +\infty[\in \text{GR} \langle \Delta_+ \times^{\text{FCD}} \uparrow B \rangle^* X.$$

Thus $\mathcal{A} \times \mathcal{B} \neq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$. □

EXAMPLE 1366. $\mathcal{A} \times_F^{\text{RLD}} \mathcal{B} \sqsubset \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for some filters \mathcal{A}, \mathcal{B} .

PROOF. This follows from the above example. □

CONJECTURE 1367. $(\mathcal{A} \times \mathcal{B}) \sqcap (\mathcal{A} \times \mathcal{B}) \neq \mathcal{A} \times_F^{\text{RLD}} \mathcal{B}$ for some filters \mathcal{A}, \mathcal{B} .

(Earlier I presented a proof of the negation of this conjecture, but it was in error.)

EXAMPLE 1368. $(\mathcal{A} \times \mathcal{B}) \sqcup (\mathcal{A} \times \mathcal{B}) \sqsubset \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for some filters \mathcal{A}, \mathcal{B} .

PROOF. (based on [8]) Let $\mathcal{A} = \mathcal{B} = \Omega(\mathbb{N})$. It's enough to prove $(\mathcal{A} \times \mathcal{B}) \sqcup (\mathcal{A} \times \mathcal{B}) \neq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$.

Let $X \in \text{up} \mathcal{A}, Y \in \text{up} \mathcal{B}$ that is $X \in \Omega(\mathbb{N}), Y \in \Omega(\mathbb{N})$.

Removing one element x from X produces a set P . Removing one element y from Y produces a set Q . Obviously $P \in \Omega(\mathbb{N}), Q \in \Omega(\mathbb{N})$.

Obviously $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \in \text{up}((\mathcal{A} \times \mathcal{B}) \sqcup (\mathcal{A} \times \mathcal{B}))$.

$(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \not\subseteq X \times Y$ because $(x, y) \in X \times Y$ but $(x, y) \notin (P \times \mathbb{N}) \cup (\mathbb{N} \times Q)$ for every $X \in \text{up} \mathcal{A}, Y \in \text{up} \mathcal{B}$.

Thus some $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \notin \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$ by properties of filter bases. □

EXAMPLE 1369. $(\text{RLD})_{\text{out}}(\text{FCD})f \neq f$ for some convex reloid f .

PROOF. Let $f = \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ where \mathcal{A} and \mathcal{B} are from example 1366.

$(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ by proposition 1069.

So $(\text{RLD})_{\text{out}}(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = (\text{RLD})_{\text{out}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathcal{A} \times_F^{\text{RLD}} \mathcal{B} \neq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$. □

Funcoids are filters

The motto of this chapter is: “Funcoids are filters on a (boolean) lattice.”

16.1. Rearrangement of collections of sets

Let Q be a set of sets.

Let \equiv be the relation on $\bigcup Q$ defined by the formula

$$a \equiv b \Leftrightarrow \forall X \in Q : (a \in X \Leftrightarrow b \in X).$$

PROPOSITION 1370. \equiv is an equivalence relation on $\bigcup Q$.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $a \equiv b \wedge b \equiv c$. Then $a \in X \Leftrightarrow b \in X \Leftrightarrow c \in X$ for every $X \in Q$.

Thus $a \equiv c$.

□

DEFINITION 1371. *Rearrangement* $\mathfrak{R}(Q)$ of Q is the set of equivalence classes of $\bigcup Q$ for \equiv .

OBVIOUS 1372. $\bigcup \mathfrak{R}(Q) = \bigcup Q$.

OBVIOUS 1373. $\emptyset \notin \mathfrak{R}(Q)$.

LEMMA 1374. $\text{card } \mathfrak{R}(Q) \leq 2^{\text{card } Q}$.

PROOF. Having an equivalence class C , we can find the set $f \in \mathcal{P}Q$ of all $X \in Q$ such that $a \in X$, for every $a \in C$.

$$b \equiv a \Leftrightarrow \forall X \in Q : (a \in X \Leftrightarrow b \in X) \Leftrightarrow \forall X \in Q : (X \in f \Leftrightarrow b \in X).$$

So $C = \left\{ \frac{b \in \bigcup Q}{b \equiv a} \right\}$ can be restored knowing f . Consequently there are no more than $\text{card } \mathcal{P}Q = 2^{\text{card } Q}$ classes. □

COROLLARY 1375. If Q is finite, then $\mathfrak{R}(Q)$ is finite.

PROPOSITION 1376. If $X \in Q$, $Y \in \mathfrak{R}(Q)$ then $X \cap Y \neq \emptyset \Leftrightarrow Y \subseteq X$.

PROOF. Let $X \cap Y \neq \emptyset$ and $x \in X \cap Y$. Then

$$y \in Y \Leftrightarrow x \equiv y \Leftrightarrow \forall X' \in Q : (x \in X' \Leftrightarrow y \in X') \Rightarrow (x \in X \Leftrightarrow y \in X) \Leftrightarrow y \in X$$

for every y . Thus $Y \subseteq X$.

$Y \subseteq X \Rightarrow X \cap Y \neq \emptyset$ because $Y \neq \emptyset$. □

PROPOSITION 1377. If $\emptyset \neq X \in Q$ then there exists $Y \in \mathfrak{R}(Q)$ such that $Y \subseteq X \wedge X \cap Y \neq \emptyset$.

PROOF. Let $a \in X$. Then

$$[a] = \left\{ \frac{b \in \bigcup Q}{\forall X' \in Q : (a \in X' \Leftrightarrow b \in X')} \right\} \subseteq \left\{ \frac{b \in \bigcup Q}{a \in X \Leftrightarrow b \in X} \right\} = \left\{ \frac{b \in \bigcup Q}{b \in X} \right\} = X.$$

But $[a] \in \mathfrak{R}(Q)$.

$X \cap Y \neq \emptyset$ follows from $Y \subseteq X$ by the previous proposition. \square

PROPOSITION 1378. If $X \in Q$ then $X = \bigcup(\mathfrak{R}(Q) \cap \mathcal{P}X)$.

PROOF. $\bigcup(\mathfrak{R}(Q) \cap \mathcal{P}X) \subseteq X$ is obvious.

Let $x \in X$. Then there is $Y \in \mathfrak{R}(Q)$ such that $x \in Y$. We have $Y \subseteq X$ that is $Y \in \mathcal{P}X$ by a proposition above. So $x \in Y$ where $Y \in \mathfrak{R}(Q) \cap \mathcal{P}X$ and thus $x \in \bigcup(\mathfrak{R}(Q) \cap \mathcal{P}X)$. We have $X \subseteq \bigcup(\mathfrak{R}(Q) \cap \mathcal{P}X)$. \square

16.2. Finite unions of Cartesian products

Let A, B be sets.

I will denote $\overline{X} = A \setminus X$.

Let denote $\Gamma(A, B)$ the set of all finite unions $X_0 \times Y_0 \cup \dots \cup X_{n-1} \times Y_{n-1}$ of Cartesian products, where $n \in \mathbb{N}$ and $X_i \in \mathcal{P}A, Y_i \in \mathcal{P}B$ for every $i = 0, \dots, n-1$.

PROPOSITION 1379. The following sets are pairwise equal:

- 1°. $\Gamma(A, B)$;
- 2°. the set of all sets of the form $\bigcup_{X \in S} (X \times Y_X)$ where S are finite collections on A and $Y_X \in \mathcal{P}B$ for every $X \in S$;
- 3°. the set of all sets of the form $\bigcup_{X \in S} (X \times Y_X)$ where S are finite partitions of A and $Y_X \in \mathcal{P}B$ for every $X \in S$;
- 4°. the set of all finite unions $\bigcup_{(X,Y) \in \sigma} (X \times Y)$ where σ is a relation between a partition of A and a partition of B (that is $\text{dom } \sigma$ is a partition of A and $\text{im } \sigma$ is a partition of B).
- 5°. the set of all finite intersections $\bigcap_{i=0, \dots, n-1} (X_i \times Y_i \cup \overline{X}_i \times B)$ where $n \in \mathbb{N}$ and $X_i \in \mathcal{P}A, Y_i \in \mathcal{P}B$ for every $i = 0, \dots, n-1$.

PROOF.

1° \supseteq 2°, 2° \supseteq 3°. Obvious.

1° \subseteq 2°. Let $Q \in \Gamma(A, B)$. Then $Q = X_0 \times Y_0 \cup \dots \cup X_{n-1} \times Y_{n-1}$. Denote $S = \{X_0, \dots, X_{n-1}\}$. We have $Q = \bigcup_{X' \in S} \left(X' \times \bigcup_{i=0, \dots, n-1} \left\{ \frac{Y_i}{X_i = X'} \right\} \right) \in 2^\circ$.

2° \subseteq 3°. Let $Q = \bigcup_{X \in S} (X \times Y_X)$ where S is a finite collection on A and $Y_X \in \mathcal{P}B$ for every $X \in S$. Let

$$P = \bigcup_{X' \in \mathfrak{R}(S)} \left(X' \times \bigcup_{X \in S} \left\{ \frac{Y_X}{\exists X \in S : X' \subseteq X} \right\} \right).$$

To finish the proof let's show $P = Q$.

$$\langle P \rangle^* \{x\} = \bigcup_{X \in S} \left\{ \frac{Y_X}{\exists X \in S : X' \subseteq X} \right\} \text{ where } x \in X'.$$

$$\text{Thus } \langle P \rangle^* \{x\} = \bigcup \left\{ \frac{Y_X}{\exists X \in S : x \in X} \right\} = \langle Q \rangle^* \{x\}. \text{ So } P = Q.$$

4° \subseteq 3°. $\bigcup_{(X,Y) \in \sigma} (X \times Y) = \bigcup_{X \in \text{dom } \sigma} \left(X \times \bigcup_{\left(\frac{Y \in \mathcal{P}B}{(X,Y) \in \sigma} \right)} \right) \in 3^\circ$.

3° ⊆ 4°.

$$\begin{aligned} \bigcup_{X \in S} (X \times Y_X) &= \bigcup_{X \in S} \left(X \times \bigcup \left(\mathfrak{R} \left(\left\{ \frac{Y_X}{X \in S} \right\} \right) \cap \mathcal{P}Y_X \right) \right) = \\ &= \bigcup_{X \in S} \left(X \times \bigcup \left\{ \frac{Y' \in \mathfrak{R}(\{\frac{Y_X}{X \in S}\})}{Y' \subseteq Y_X} \right\} \right) = \\ &= \bigcup_{X \in S} \left(X \times \bigcup \left\{ \frac{Y' \in \mathfrak{R}(\{\frac{Y_X}{X \in S}\})}{(X, Y') \in \sigma} \right\} \right) = \bigcup_{(X, Y) \in \sigma} (X \times Y) \end{aligned}$$

where σ is a relation between S and $\mathfrak{R}(\{\frac{Y_X}{X \in S}\})$, and $(X, Y') \in \sigma \Leftrightarrow Y' \subseteq Y_X$.

5° ⊆ 1°. Obvious.

3° ⊆ 5°. Let $Q = \bigcup_{X \in S} (X \times Y_X) = \bigcup_{i=0, \dots, n-1} (X_i \times Y_i)$ for a partition $S = \{X_0, \dots, X_{n-1}\}$ of A . Then $Q = \bigcap_{i=0, \dots, n-1} (X_i \times Y_i \cup \overline{X_i} \times B)$. \square

EXERCISE 1380. Formulate the duals of these sets.

PROPOSITION 1381. $\Gamma(A, B)$ is a boolean lattice, a sublattice of the lattice $\mathcal{P}(A \times B)$.

PROOF. That it's a sublattice is obvious. That it has complement, is also obvious. Distributivity follows from distributivity of $\mathcal{P}(A \times B)$. \square

16.3. Before the diagram

Next we will prove the below theorem 1397 (the theorem with a diagram). First we will present parts of this theorem as several lemmas, and then then state a statement about the diagram which concisely summarizes the lemmas (and their easy consequences).

Below for simplicity we will equate reloids with their graphs (that is with filters on binary cartesian products).

OBVIOUS 1382. $\text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f = (\text{up } f) \cap \Gamma$ for every reloid f .

CONJECTURE 1383. $\uparrow\uparrow^{\mathfrak{F}(\mathfrak{B})} \text{up}^{\mathfrak{A}} \mathcal{X}$ is not a filter for some filter $\mathcal{X} \in \mathfrak{F}\Gamma(A, B)$ for some sets A, B .

REMARK 1384. About this conjecture see also:

- <http://goo.gl/DHyuuU>
- <http://goo.gl/4a6wY6>

LEMMA 1385. Let A, B be sets. The following are mutually inverse order isomorphisms between $\mathfrak{F}\Gamma(A, B)$ and $\text{FCD}(A, B)$:

- 1°. $\mathcal{A} \mapsto \prod^{\text{FCD}} \text{up } \mathcal{A}$;
- 2°. $f \mapsto \text{up}^{\Gamma(A, B)} f$.

PROOF. Let's prove that $\text{up}^{\Gamma(A, B)} f$ is a filter for every funcooid f . We need to prove that $P \cap Q \in \text{up } f$ whenever

$$P = \bigcap_{i=0, \dots, n-1} (X_i \times Y_i \cup \overline{X_i} \times B) \quad \text{and} \quad Q = \bigcap_{j=0, \dots, m-1} (X'_j \times Y'_j \cup \overline{X'_j} \times B).$$

This follows from $P \in \text{up } f \Leftrightarrow \forall i \in 0, \dots, n-1 : \langle f \rangle X_i \subseteq Y_i$ and likewise for Q , so having $\langle f \rangle (X_i \cap X'_j) \subseteq Y_i \cap Y'_j$ for every $i = 0, \dots, n-1$ and $j = 0, \dots, m-1$. From this it follows

$$((X_i \cap X'_j) \times (Y_i \cap Y'_j)) \cup (\overline{X_i \cap X'_j} \times B) \supseteq f$$

and thus $P \cap Q \in \text{up } f$.

Let \mathcal{A}, \mathcal{B} be filters on Γ . Let $\prod^{\text{FCD}} \text{up } \mathcal{A} = \prod^{\text{FCD}} \text{up } \mathcal{B}$. We need to prove $\mathcal{A} = \mathcal{B}$. (The rest follows from proof of the lemma 924). We have:

$$\begin{aligned}
\mathcal{A} &= \prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B \in \text{up } \mathcal{A}}{X \in \mathcal{P}A, Y \in \mathcal{P}B} \right\} = \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \exists P \in \text{up } \mathcal{A} : P \subseteq X \times Y \cup \bar{X} \times B} \right\} = \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \exists P \in \text{up } \mathcal{A} : \langle P \rangle^* X \subseteq Y} \right\} = (*) \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \prod \left\{ \frac{\langle P \rangle^* X}{X \in \text{up } \mathcal{A}} \right\} \subseteq Y} \right\} = \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \prod \left\{ \frac{\langle P \rangle^* X}{X \in \text{up } \prod^{\text{RLD}} \text{up } \mathcal{A}} \right\} \subseteq Y} \right\} = \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \langle \prod^{\text{FCD}} \prod^{\text{RLD}} \text{up } \mathcal{A} \rangle X \subseteq Y} \right\} = (**) \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \langle \prod^{\text{FCD}} \text{up } \prod^{\text{RLD}} \text{up } \mathcal{A} \rangle X \subseteq Y} \right\} = \\
&\prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \langle \prod^{\text{FCD}} \text{up } \mathcal{A} \rangle X \subseteq Y} \right\}.
\end{aligned}$$

(*) by properties of generalized filter bases, because $\left\{ \frac{\langle P \rangle^* X}{P \in \text{up } \mathcal{A}} \right\}$ is a filter base.

(**) by theorem 1061.

Similarly

$$\mathcal{B} = \prod^{\text{FCD}} \left\{ \frac{X \times Y \cup \bar{X} \times B}{X \in \mathcal{P}A, Y \in \mathcal{P}B, \langle \prod^{\text{FCD}} \text{up } \mathcal{B} \rangle X \subseteq Y} \right\}.$$

Thus $\mathcal{A} = \mathcal{B}$. □

PROPOSITION 1386. $g \circ f \in \Gamma(A, C)$ if $f \in \Gamma(A, B)$ and $g \in \Gamma(B, C)$ for some sets A, B, C .

PROOF. Because composition of Cartesian products is a Cartesian product. □

DEFINITION 1387. $g \circ f = \prod^{\mathfrak{F}\Gamma(A, C)} \left\{ \frac{G \circ F}{F \in \text{up } f, G \in \text{up } g} \right\}$ for $f \in \mathfrak{F}\Gamma(A, B)$ and $g \in \mathfrak{F}\Gamma(B, C)$ (for every sets A, B, C).

We define f^{-1} for $f \in \mathfrak{F}\Gamma(A, B)$ similarly to f^{-1} for reloids and similarly derive the formulas:

- 1°. $(f^{-1})^{-1} = f$;
- 2°. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

16.4. Associativity over composition

LEMMA 1388. $\prod^{\text{RLD}} \text{up}^{\Gamma(A,C)}(g \circ f) = \left(\prod^{\text{RLD}} \text{up}^{\Gamma(B,C)} g \right) \circ \left(\prod^{\text{RLD}} \text{up}^{\Gamma(A,B)} f \right)$
for every $f \in \mathfrak{F}(\Gamma(A,B))$, $g \in \mathfrak{F}(\Gamma(B,C))$ (for every sets A, B, C).

PROOF. If $K \in \text{up} \prod^{\text{RLD}} \text{up}^{\Gamma(A,C)}(g \circ f)$ then $K \supseteq G \circ F$ for some $F \in f$, $G \in g$.
But $F \in \text{up}^{\Gamma(A,B)} f$, thus

$$F \in \prod^{\text{RLD}} \text{up}^{\Gamma(A,B)} f$$

and similarly

$$G \in \prod^{\text{RLD}} \text{up}^{\Gamma(B,C)} g.$$

So we have

$$K \supseteq G \circ F \in \text{up} \left(\left(\prod^{\text{RLD}} \text{up}^{\Gamma(B,C)} g \right) \circ \left(\prod^{\text{RLD}} \text{up}^{\Gamma(A,B)} f \right) \right).$$

Let now

$$K \in \text{up} \left(\left(\prod^{\text{RLD}} \text{up}^{\Gamma(B,C)} g \right) \circ \left(\prod^{\text{RLD}} \text{up}^{\Gamma(A,B)} f \right) \right).$$

Then there exist $F \in \text{up} \prod^{\text{RLD}} \text{up}^{\Gamma(A,B)} f$ and $G \in \text{up} \prod^{\text{RLD}} \text{up}^{\Gamma(B,C)} g$ such that $K \supseteq G \circ F$. By properties of generalized filter bases we can take $F \in \text{up}^{\Gamma(A,B)} f$ and $G \in \text{up}^{\Gamma(B,C)} g$. Thus $K \in \text{up}^{\Gamma(A,C)}(g \circ f)$ and so $K \in \text{up} \prod^{\text{RLD}} \text{up}^{\Gamma(A,C)}(g \circ f)$. \square

LEMMA 1389. $(\text{RLD})_{\text{in}} X = X$ for $X \in \Gamma(A,B)$.

PROOF. $X = X_0 \times Y_0 \cup \dots \cup X_n \times Y_n = (X_0 \times^{\text{FCD}} Y_0) \sqcup^{\text{FCD}} \dots \sqcup^{\text{FCD}} (X_n \times^{\text{FCD}} Y_n)$.

$(\text{RLD})_{\text{in}} X =$

$$\begin{aligned} & (\text{RLD})_{\text{in}} (X_0 \times^{\text{FCD}} Y_0) \sqcup^{\text{RLD}} \dots \sqcup^{\text{RLD}} (\text{RLD})_{\text{in}} (X_n \times^{\text{FCD}} Y_n) = \\ & (X_0 \times^{\text{RLD}} Y_0) \sqcup^{\text{RLD}} \dots \sqcup^{\text{RLD}} (X_n \times^{\text{RLD}} Y_n) = \\ & X_0 \times Y_0 \cup \dots \cup X_n \times Y_n = X. \end{aligned}$$

\square

LEMMA 1390. $\prod^{\text{RLD}} f = (\text{RLD})_{\text{in}} \prod^{\text{FCD}} f$ for every filter $f \in \mathfrak{F}\Gamma(A,B)$.

PROOF.

$$(\text{RLD})_{\text{in}} \prod^{\text{FCD}} f = \prod^{\text{RLD}} \langle (\text{RLD})_{\text{in}} \rangle^* f = (\text{by the previous lemma}) = \prod^{\text{RLD}} f.$$

\square

LEMMA 1391.

- 1°. $f \mapsto \prod^{\text{RLD}} \text{up} f$ and $\mathcal{A} \mapsto \Gamma(A,B) \cap \text{up} \mathcal{A}$ are mutually inverse bijections between $\mathfrak{F}\Gamma(A,B)$ and a subset of reloids.
- 2°. These bijections preserve composition.

PROOF.

1°. That they are mutually inverse bijections is obvious.

2°.

$$\begin{aligned} \left(\prod^{\text{RLD}} \text{up } g \right) \circ \left(\prod^{\text{RLD}} \text{up } f \right) &= \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in \prod^{\text{RLD}} f, G \in \prod^{\text{RLD}} g} \right\} = \\ &= \prod^{\text{RLD}} \left\{ \frac{G \circ F}{F \in f, G \in g} \right\} = \prod^{\text{RLD}} \mathfrak{F}^{\Gamma(\text{Src } f, \text{Dst } g)} \left\{ \frac{G \circ F}{F \in f, G \in g} \right\} = \prod^{\text{RLD}} (g \circ f). \end{aligned}$$

So \prod^{RLD} preserves composition. That $\mathcal{A} \mapsto \Gamma(A, B) \cap \text{up } \mathcal{A}$ preserves composition follows from properties of bijections. \square

LEMMA 1392. Let A, B, C be sets.

- 1°. $\left(\prod^{\text{FCD}} \text{up } g \right) \circ \left(\prod^{\text{FCD}} \text{up } f \right) = \prod^{\text{FCD}} \text{up}(g \circ f)$ for every $f \in \mathfrak{F}\Gamma(A, B)$, $g \in \mathfrak{F}\Gamma(B, C)$;
- 2°. $\left(\text{up}^{\Gamma(B, C)} g \right) \circ \left(\text{up}^{\Gamma(A, B)} f \right) = \text{up}^{\Gamma(A, B)}(g \circ f)$ for every functors $f \in \text{FCD}(A, B)$ and $g \in \text{FCD}(B : C)$.

PROOF. It's enough to prove only the first formula, because of the bijection from lemma 1385.

Really:

$$\begin{aligned} \prod^{\text{FCD}} \text{up}(g \circ f) &= \prod^{\text{FCD}} \text{up} \prod^{\text{RLD}} \text{up}(g \circ f) = \\ &= \prod^{\text{FCD}} \text{up} \left(\prod^{\text{RLD}} \text{up } g \circ \prod^{\text{RLD}} \text{up } f \right) = (\text{FCD}) \left(\prod^{\text{RLD}} \text{up } g \circ \prod^{\text{RLD}} \text{up } f \right) = \\ &= \left((\text{FCD}) \prod^{\text{RLD}} \text{up } g \right) \circ \left((\text{FCD}) \prod^{\text{RLD}} \text{up } f \right) = \\ &= \left(\prod^{\text{FCD}} \text{up} \prod^{\text{RLD}} \text{up } g \right) \circ \left(\prod^{\text{FCD}} \text{up} \prod^{\text{RLD}} \text{up } f \right) = \\ &= \left(\prod^{\text{FCD}} \text{up } g \right) \circ \left(\prod^{\text{FCD}} \text{up } f \right). \end{aligned}$$

\square

COROLLARY 1393. $(h \circ g) \circ f = h \circ (g \circ f)$ for every $f \in \mathfrak{F}(\Gamma(A, B))$, $g \in \mathfrak{F}(\Gamma(B, C))$, $h \in \mathfrak{F}(\Gamma(C, D))$ for every sets A, B, C, D .

LEMMA 1394. $\Gamma(A, B) \cap \text{GR } f$ is a filter on the lattice $\Gamma(A, B)$ for every reloid $f \in \text{RLD}(A, B)$.

PROOF. That it is an upper set, is obvious. If $A, B \in \Gamma(A, B) \cap \text{GR } f$ then $A, B \in \Gamma(A, B)$ and $A, B \in \text{GR } f$. Thus $A \cap B \in \Gamma(A, B) \cap \text{GR } f$. \square

PROPOSITION 1395. If $Y \in \text{up}\langle f \rangle \mathcal{X}$ for a functor f then there exists $A \in \text{up } \mathcal{X}$ such that $Y \in \text{up}\langle f \rangle A$.

PROOF. $Y \in \text{up} \prod_{A \in \text{up } a}^{\mathcal{F}} \langle f \rangle A$. So by properties of generalized filter bases, there exists $A \in \text{up } a$ such that $Y \in \text{up}\langle f \rangle A$. \square

LEMMA 1396. $(\text{FCD})f = \prod^{\text{FCD}}(\Gamma(A, B) \cap \text{GR } f)$ for every reloid $f \in \text{RLD}(A, B)$.

PROOF. Let a be an atomic filter object. We need to prove

$$\langle (\text{FCD})f \rangle a = \left\langle \prod^{\text{FCD}} (\Gamma(A, B) \cap \text{GR } f) \right\rangle a$$

that is

$$\left\langle \prod^{\text{FCD}} \text{up } f \right\rangle a = \left\langle \prod^{\text{FCD}} (\Gamma(A, B) \cap \text{GR } f) \right\rangle a$$

that is

$$\prod_{F \in \text{up } f} \langle F \rangle a = \prod_{F \in \Gamma(A, B) \cap \text{up } f} \langle F \rangle a.$$

For this it's enough to prove that $Y \in \text{up}\langle F \rangle a$ for some $F \in \text{up } f$ implies $Y \in \text{up}\langle F' \rangle a$ for some $F' \in \Gamma(A, B) \cap \text{GR } f$.

Let $Y \in \text{up}\langle F \rangle a$. Then (proposition above) there exists $A \in \text{up } a$ such that $Y \in \text{up}\langle F \rangle A$.

$Y \in \text{up}\langle A \times^{\text{FCD}} Y \sqcup \bar{A} \times^{\text{FCD}} \top \rangle a$; $\langle A \times^{\text{FCD}} Y \sqcup \bar{A} \times^{\text{FCD}} \top \rangle \mathcal{X} = Y \in \text{up}\langle F \rangle \mathcal{X}$ if $\perp \neq \mathcal{X} \sqsubseteq A$ and $\langle A \times^{\text{FCD}} Y \sqcup \bar{A} \times^{\text{FCD}} \top \rangle \mathcal{X} = \top \in \text{up}\langle F \rangle \mathcal{X}$ if $\mathcal{X} \not\sqsubseteq A$.

Thus $A \times^{\text{FCD}} Y \sqcup \bar{A} \times^{\text{FCD}} \top \sqsupseteq F$. So $A \times^{\text{FCD}} Y \sqcup \bar{A} \times^{\text{FCD}} \top$ is the sought for F' . \square

16.5. The diagram

THEOREM 1397. The diagram at the figure 10 is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore “parallel” arrows are mutually inverse). The arrows preserve order, composition, and reversal ($f \mapsto f^{-1}$).

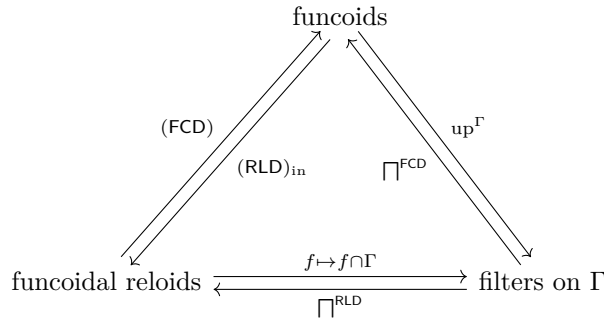


FIGURE 10

PROOF. First we need to show that $\prod^{\text{RLD}} f$ is a funcoidal reloid. But it follows from lemma 1390.

Next, we need to show that all morphisms depicted on the diagram are bijections and the depicted “opposite” morphisms are mutually inverse.

That (FCD) and (RLD)_{in} are mutually inverse was proved above in the book.

That \prod^{RLD} and $f \mapsto f \cap \Gamma$ are mutually inverse was proved above.

That \prod^{FCD} and up^Γ are mutually inverse was proved above.

That the morphisms preserve order and composition was proved above. That they preserve reversal is obvious.

So it remains to apply lemma 196 (taking into account lemma 1390). \square

Another proof that $(\text{FCD})(\text{RLD})_{\text{in}} f = f$ for every funcoid f :

PROOF. For every filter $\mathcal{X} \in \mathcal{F}(\text{Src } f)$ we have $\langle (\text{FCD})(\text{RLD})_{\text{in}f} \rangle \mathcal{X} = \prod_{F \in \text{up}(\text{RLD})_{\text{in}f}} \langle F \rangle \mathcal{X} = \prod_{F \in \text{up}^{\Gamma}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{X}$.

Obviously $\prod_{F \in \text{up}^{\Gamma}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{X} \supseteq \langle f \rangle \mathcal{X}$. So $(\text{FCD})(\text{RLD})_{\text{in}f} \supseteq f$.

Let $Y \in \text{up} \langle f \rangle \mathcal{X}$. Then (proposition above) there exists $A \in \text{up } \mathcal{X}$ such that $Y \in \text{up} \langle f \rangle A$.

Thus $A \times Y \sqcup \bar{A} \times \top \in \text{up } f$. So $\langle (\text{FCD})(\text{RLD})_{\text{in}f} \rangle \mathcal{X} = \prod_{F \in \text{up}^{\Gamma}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{X} \sqsubseteq \langle A \times Y \sqcup \bar{A} \times \top \rangle \mathcal{X} = Y$. So $Y \in \text{up} \langle (\text{FCD})(\text{RLD})_{\text{in}f} \rangle \mathcal{X}$ that is $\langle f \rangle \mathcal{X} \supseteq \langle (\text{FCD})(\text{RLD})_{\text{in}f} \rangle \mathcal{X}$ that is $f \supseteq (\text{FCD})(\text{RLD})_{\text{in}f}$. \square

16.6. Some additional properties

PROPOSITION 1398. For every funcoid $f \in \text{FCD}(A, B)$ (for sets A, B):

- 1°. $\text{dom } f = \prod_{\mathcal{F}(A)} \langle \text{dom} \rangle^* \text{up}^{\Gamma(A, B)} f$;
- 2°. $\text{im } f = \prod_{\mathcal{F}(B)} \langle \text{im} \rangle^* \text{up}^{\Gamma(A, B)} f$.

PROOF. Take $\left\{ \frac{X \times Y}{X \in \mathcal{P}A, Y \in \mathcal{P}B, X \times Y \supseteq f} \right\} \subseteq \text{up}^{\Gamma(A, B)} f$. I leave the rest reasoning as an exercise. \square

THEOREM 1399. For every reloid f and $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$:

- 1°. $\mathcal{X} [(\text{FCD})f] \mathcal{Y} \Leftrightarrow \forall F \in \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f : \mathcal{X} [F] \mathcal{Y}$;
- 2°. $\langle (\text{FCD})f \rangle \mathcal{X} = \prod_{F \in \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f} \langle F \rangle \mathcal{X}$.

PROOF.

1°.

$$\begin{aligned} \forall F \in \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f : \mathcal{X} [F] \mathcal{Y} &\Leftrightarrow \\ \forall F \in \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f : (\mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \sqcap F &\neq \perp \Leftrightarrow (*) \\ (\mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \sqcap \prod_{\text{FCD}} \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f &\neq \perp \Leftrightarrow \\ \mathcal{X} \left[\prod_{\text{FCD}} \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f \right] \mathcal{Y} &\Leftrightarrow \mathcal{X} [(\text{FCD})f] \mathcal{Y}. \end{aligned}$$

(*) by properties of generalized filter bases, taking into account that funcoids are isomorphic to filters.

2°. $\prod_{F \in \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f} \langle F \rangle a = \left\langle \prod_{\text{FCD}} \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f \right\rangle a = \langle (\text{FCD})f \rangle a$ for every ultrafilter a .

It remains to prove that the function

$$\varphi = \lambda \mathcal{X} \in \mathcal{F}(\text{Src } f) : \prod_{F \in \text{up}^{\Gamma(\text{Src } f, \text{Dst } f)} f} \langle F \rangle \mathcal{X}$$

is a component of a funcoid (from what follows that $\varphi = \langle (\text{FCD})f \rangle$). To prove this, it's enough to show that it preserves finite joins and filtered meets.

$\varphi \perp = \perp$ is obvious. $\varphi(\mathcal{I} \sqcup \mathcal{J}) = \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{I} \sqcup \langle F \rangle \mathcal{J} = \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{I} \sqcup \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{J} = \varphi \mathcal{I} \sqcup \varphi \mathcal{J}$. If S is a generalized filter base of $\text{Src } f$, then

$$\begin{aligned} \varphi \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} S &= \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle F \rangle \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} S = \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle \langle F \rangle \rangle^* S = \\ &= \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{X} = \prod_{\mathcal{X} \in S} \prod_{F \in \text{up}^\mathcal{F}(\text{Src } f, \text{Dst } f)} \langle F \rangle \mathcal{X} = \prod_{\mathcal{X} \in S} \varphi \mathcal{X} = \prod_{\mathcal{X} \in S} \langle \varphi \rangle^* S. \end{aligned}$$

So φ is a component of a functor.

□

DEFINITION 1400. $\boxtimes f = \prod^{\text{RLD}} \text{up}^\Gamma(\text{Src } f, \text{Dst } f) f$ for reloid f .

CONJECTURE 1401. $\boxtimes f = (\text{RLD})_{\text{in}}(\text{FCD})f$ for every reloid f .

OBVIOUS 1402. $\boxtimes f \sqsupseteq f$ for every reloid f .

EXAMPLE 1403. $(\text{RLD})_{\text{in}}f \neq \boxtimes(\text{RLD})_{\text{out}}f$ for some functor f .

PROOF. Take $f = \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}}$. Then, as it was shown above, $(\text{RLD})_{\text{out}}f = \perp$ and thus $\boxtimes(\text{RLD})_{\text{out}}f = \perp$. But $(\text{RLD})_{\text{in}}f \sqsupseteq (\text{RLD})_{\text{in}}f \neq \perp$. So $(\text{RLD})_{\text{in}}f \neq \boxtimes(\text{RLD})_{\text{out}}f$. □

Another proof of the theorem “ $\text{dom}(\text{RLD})_{\text{in}}f = \text{dom } f$ and $\text{im}(\text{RLD})_{\text{in}}f = \text{im } f$ for every functor f ”:

PROOF. We have for every filter $\mathcal{X} \in \mathcal{F}(\text{Src } f)$:

$$\begin{aligned} \mathcal{X} \sqsupseteq \text{dom}(\text{RLD})_{\text{in}}f &\Leftrightarrow \mathcal{X} \times^{\text{RLD}} \top \sqsupseteq (\text{RLD})_{\text{in}}f \Leftrightarrow \\ &\forall a \in \mathcal{F}(\text{Src } f), b \in \mathcal{F}(\text{Dst } f) : (a \times^{\text{FCD}} b \sqsubseteq f \Rightarrow a \times^{\text{RLD}} b \sqsubseteq \mathcal{X} \times^{\text{RLD}} \top) \Leftrightarrow \\ &\forall a \in \mathcal{F}(\text{Src } f), b \in \mathcal{F}(\text{Dst } f) : (a \times^{\text{FCD}} b \sqsubseteq f \Rightarrow a \sqsubseteq \mathcal{X}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{X} \sqsupseteq \text{dom } f &\Leftrightarrow \mathcal{X} \times^{\text{FCD}} \top \sqsupseteq f \Leftrightarrow \\ &\forall a \in \mathcal{F}(\text{Src } f), b \in \mathcal{F}(\text{Dst } f) : (a \times^{\text{FCD}} b \sqsubseteq f \Rightarrow a \times^{\text{FCD}} b \sqsubseteq \mathcal{X} \times^{\text{FCD}} \top) \Leftrightarrow \\ &\forall a \in \mathcal{F}(\text{Src } f), b \in \mathcal{F}(\text{Dst } f) : (a \times^{\text{FCD}} b \sqsubseteq f \Rightarrow a \sqsubseteq \mathcal{X}). \end{aligned}$$

Thus $\text{dom}(\text{RLD})_{\text{in}}f = \text{dom } f$. The rest follows from symmetry. □

Another proof that $\text{dom}(\text{RLD})_{\text{in}}f = \text{dom } f$ and $\text{im}(\text{RLD})_{\text{in}}f = \text{im } f$ for every functor f :

PROOF. $\text{dom}(\text{RLD})_{\text{in}}f \sqsupseteq \text{dom } f$ and $\text{im}(\text{RLD})_{\text{in}}f \sqsupseteq \text{im } f$ because $(\text{RLD})_{\text{in}}f \sqsupseteq (\text{RLD})_{\text{in}}$ and $\text{dom}(\text{RLD})_{\text{in}}f = \text{dom } f$ and $\text{im}(\text{RLD})_{\text{in}}f = \text{im } f$.

It remains to prove (as the rest follows from symmetry) that $\text{dom}(\text{RLD})_{\text{in}}f \sqsubseteq \text{dom } f$.

Really,

$$\begin{aligned} \text{dom}(\text{RLD})_{\text{in}}f \sqsubseteq \prod_{\mathcal{X} \in S} \left\{ \begin{array}{l} X \in \text{up } \text{dom } f \\ X \times \top \in \text{up } f \end{array} \right\} &= \\ &= \prod_{\mathcal{X} \in S} \left\{ \begin{array}{l} X \in \text{up } \text{dom } f \\ X \in \text{up } \text{dom } f \end{array} \right\} = \prod_{\mathcal{X} \in S} \text{up } \text{dom } f = \text{dom } f. \end{aligned}$$

□

16.7. More on properties of funcoids

PROPOSITION 1404. $\Gamma(A, B)$ is the center of lattice $\text{FCD}(A, B)$.

PROOF. Theorem 613. □

PROPOSITION 1405. $\text{up}^{\Gamma(A, B)}(\mathcal{A} \times^{\text{FCD}} \mathcal{B})$ is defined by the filter base $\left\{ \frac{A \times B}{A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\}$ on the lattice $\Gamma(A, B)$.

PROOF. It follows from the fact that $\mathcal{A} \times^{\text{FCD}} \mathcal{B} = \prod^{\text{FCD}} \left\{ \frac{A \times B}{A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\}$. □

PROPOSITION 1406. $\text{up}^{\Gamma(A, B)}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathfrak{F}(\Gamma(A, B)) \cap \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$.

PROOF. It follows from the fact that $\mathcal{A} \times^{\text{FCD}} \mathcal{B} = \prod^{\text{FCD}} \left\{ \frac{A \times B}{A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}} \right\}$. □

PROPOSITION 1407. For every $f \in \mathfrak{F}(\Gamma(A, B))$:

- 1°. $f \circ f$ is defined by the filter base $\left\{ \frac{F \circ F}{F \in \text{up } f} \right\}$ (if $A = B$);
- 2°. $f^{-1} \circ f$ is defined by the filter base $\left\{ \frac{F^{-1} \circ F}{F \in \text{up } f} \right\}$;
- 3°. $f \circ f^{-1}$ is defined by the filter base $\left\{ \frac{F \circ F^{-1}}{F \in \text{up } f} \right\}$.

PROOF. I will prove only 1° and 2° because 3° is analogous to 2°.

1°. It's enough to show that $\forall F, G \in \text{up } f \exists H \in \text{up } f : H \circ H \sqsubseteq G \circ F$. To prove it take $H = F \sqcap G$.

2°. It's enough to show that $\forall F, G \in \text{up } f \exists H \in \text{up } f : H^{-1} \circ H \sqsubseteq G^{-1} \circ F$. To prove it take $H = F \sqcap G$. Then $H^{-1} \circ H = (F \sqcap G)^{-1} \circ (F \sqcap G) \sqsubseteq G^{-1} \circ F$. □

THEOREM 1408. For every sets A, B, C if $g, h \in \mathfrak{F}\Gamma(A, B)$ then

- 1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$;
- 2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$.

PROOF. It follows from the order isomorphism above, which preserves composition. □

THEOREM 1409. $f \sqcap g = f \sqcap^{\text{FCD}} g$ if $f, g \in \Gamma(A, B)$.

PROOF. Let $f = X_0 \times Y_0 \cup \dots \cup X_n \times Y_n$ and $g = X'_0 \times Y'_0 \cup \dots \cup X'_m \times Y'_m$. Then

$$f \sqcap g = \bigcup_{i=0, \dots, n, j=0, \dots, m} ((X_i \times Y_i) \cap (X'_j \times Y'_j)) = \bigcup_{i=0, \dots, n, j=0, \dots, m} ((X_i \cap X'_j) \times (Y_i \cap Y'_j)).$$

But $f = X_0 \times Y_0 \sqcup^{\text{FCD}} \dots \sqcup^{\text{FCD}} X_n \times Y_n$ and $g = X'_0 \times Y'_0 \sqcup^{\text{FCD}} \dots \sqcup^{\text{FCD}} X'_m \times Y'_m$;

$$f \sqcap^{\text{FCD}} g = \bigsqcup_{i=0, \dots, n, j=0, \dots, m} ((X_i \times Y_i) \sqcap^{\text{FCD}} (X'_j \times Y'_j)) = \bigsqcup_{i=0, \dots, n, j=0, \dots, m} ((X_i \sqcap X'_j) \times^{\text{FCD}} (Y_i \sqcap Y'_j)).$$

COROLLARY 1410. If X and Y are finite binary relations, then

- 1°. $X \sqcap^{\text{FCD}} Y = X \sqcap Y$;
- 2°. $(\top \setminus X) \sqcap^{\text{FCD}} (\top \setminus Y) = (\top \setminus X) \sqcap (\top \setminus Y)$;
- 3°. $X \sqcap^{\text{FCD}} (\top \setminus Y) = X \sqcap (\top \setminus Y)$.

Now it's obvious that $f \cap g = f \sqcap^{\text{FCD}} g$. \square

THEOREM 1411. The set of funcroids (from a given set A to a given set B) is with separable core.

PROOF. Let $f, g \in \text{FCD}(A, B)$ (for some sets A, B).

Because filters on distributive lattices are with separable core, there exist $F, G \in \Gamma(A, B)$ such that $F \cap G = \emptyset$. Then by the previous theorem $F \sqcap^{\text{FCD}} G = \perp$. \square

THEOREM 1412. The coatoms of funcroids from a set A to a set B are exactly $(A \times B) \setminus (\{x\} \times \{y\})$ for $x \in A, y \in B$.

PROOF. That coatoms of $\Gamma(A, B)$ are exactly $(A \times B) \setminus (\{x\} \times \{y\})$ for $x \in A, y \in B$, is obvious. To show that coatoms of funcroids are the same, it remains to apply proposition 560. \square

THEOREM 1413. The set of funcroids (for given A and B) is coatomic.

PROOF. Proposition 562. \square

EXERCISE 1414. Prove that in general funcroids are not coatomic.

16.8. Funcroid bases

This section will present mainly a counter-example against a statement you have not thought about anyway.

LEMMA 1415. If S is an upper set of principal funcroids, then $\prod^{\text{FCD}}(S \cap \Gamma) = \prod^{\text{FCD}} S$.

PROOF. $\prod^{\text{FCD}}(S \cap \Gamma) \supseteq \prod^{\text{FCD}} S$ is obvious.

$\prod^{\text{FCD}} S = \prod^{\text{FCD}} \prod_{K \in S} T_K \supseteq \prod^{\text{FCD}}(S \cap \Gamma)$, where $T_K \in \mathcal{P}(S \cap \Gamma)$. So $\prod^{\text{FCD}}(S \cap \Gamma) = \prod^{\text{FCD}} S$. \square

THEOREM 1416. If S is a filter base on the set of binary relations then S is a base of $\prod^{\text{FCD}} S$.

First prove a special case of our theorem to get the idea:

EXAMPLE 1417. Take the filter base $S = \left\{ \left\{ \frac{(x, y)}{\varepsilon > 0} \right\} \right\}$ and $K = \left\{ \frac{(x, y)}{|x - y| < \exp x} \right\}$

where x and y range real numbers. Then $K \notin \text{up} \prod^{\text{FCD}} S$.

PROOF. Take a nontrivial ultrafilter x on \mathbb{R} . We can for simplicity assume $x \sqsubseteq \mathbb{Z}$.

$$\left\langle \prod^{\text{FCD}} S \right\rangle x = \prod_{L \in S} \langle L \rangle x = \prod_{L \in S, X \in \text{up } x} \langle L \rangle^* X = \prod_{\varepsilon > 0, X \in \text{up } x} \bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[.$$

$$\langle K \rangle x = \prod_{X \in \text{up } x} \langle K \rangle^* X = \prod_{X \in \text{up } x} \bigsqcup_{\alpha \in X}]\alpha - \exp \alpha; \alpha + \exp \alpha[.$$

Suppose for the contrary that $\langle K \rangle x \supseteq \left\langle \prod^{\text{FCD}} S \right\rangle x$.

Then

$$\bigsqcup_{\alpha \in X}]\alpha - \exp \alpha; \alpha + \exp \alpha[\supseteq \prod_{\varepsilon > 0, X \in \text{up } x} \bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[\text{ for every } X \in \text{up } x;$$

thus by properties of generalized filter bases $\left\{ \left\{ \frac{\bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[}{\varepsilon > 0} \right\} \right\}$ is a filter base and even a chain)

$\bigsqcup_{\alpha \in X}]\alpha - \exp \alpha; \alpha + \exp \alpha[\supseteq \prod_{X \in \text{up } x} \bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[$ for some $\varepsilon > 0$ and thus by properties of generalized filter bases ($\left\{ \frac{\bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[}{X \in \text{up } x} \right\}$ is a filter base) for some $X' \in \text{up } x$

$$\bigsqcup_{\alpha \in X}]\alpha - \exp \alpha; \alpha + \exp \alpha[\supseteq \bigsqcup_{\alpha \in X'}]\alpha - \varepsilon; \alpha + \varepsilon[$$

what is impossible by the fact that $\exp \alpha$ goes infinitely small as $\alpha \rightarrow -\infty$ and the fact that we can take $X = \mathbb{Z}$ for some x . \square

Now prove the general case:

PROOF. Suppose that $K \in \text{up} \prod^{\text{FCD}} S$ and thus $\langle K \rangle x \supseteq \langle \prod^{\text{FCD}} S \rangle x$. We need to prove that there is some $L \in S$ such that $K \supseteq L$.

Take an ultrafilter x .

$$\langle \prod^{\text{FCD}} S \rangle x = \prod_{L \in S} \langle L \rangle x = \prod_{L \in S, X \in \text{up } x} \langle L \rangle^* X.$$

$$\langle K \rangle x = \prod_{X \in \text{up } x} \langle K \rangle^* X.$$

Then $\langle K \rangle^* X \supseteq \prod_{L \in S, X \in \text{up } x} \langle L \rangle^* X$ for every $X \in \text{up } x$; thus by properties of generalized filter bases ($\left\{ \frac{\langle L \rangle^* X}{L \in S} \right\}$ is a filter base);

$\langle K \rangle^* X \supseteq \prod_{X \in \text{up } x} \langle L \rangle^* X$ for some $L \in S$ and thus by properties of generalized filter bases ($\left\{ \frac{\langle L \rangle^* X}{X \in \text{up } x} \right\}$ is a filter base) for some $X' \in \text{up } x$

$$\langle K \rangle^* X \supseteq \langle L \rangle^* X' \supseteq \langle L \rangle x.$$

So $\langle K \rangle x \supseteq \langle L \rangle x$ because this equality holds for every $X \in \text{up } x$. Therefore $K \supseteq L$. \square

EXAMPLE 1418. A base of a funcoid which is not a filter base.

PROOF. Consider $f = \text{id}_{\Omega}^{\text{FCD}}$. We know that $\text{up } f$ is not a filter base. But it is a base of a funcoid. \square

EXERCISE 1419. Prove that a set S is a filter (on some set) iff

$$\forall X_0, \dots, X_n \in S : \text{up}(X_0 \sqcap \dots \sqcap X_n) \subseteq S$$

for every natural n .

A similar statement does *not* hold for funcoids:

EXAMPLE 1420. For a set S of binary relations

$$\forall X_0, \dots, X_n \in S : \text{up}(X_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} X_n) \subseteq S$$

does not imply that there exists funcoid f such that $S = \text{up } f$.

PROOF. Take $S_0 = \text{up } 1^{\text{FCD}}$ (where 1^{FCD} is the identity funcoid on any infinite set) and $S_1 = \bigcup_{F \in S_0} \left\{ \frac{\text{up } G}{G \in \text{up}^{\Gamma} F} \right\}$ (that is $S_1 = \bigcup_{F \in \text{up}^{\Gamma} 1^{\text{FCD}}} \text{up } F$).

Both S_0 and S_1 are upper sets. $S_0 \neq S_1$ because $1^{\text{FCD}} \in S_0$ and $1^{\text{FCD}} \notin S_1$.

The formula in the example works for $S = S_0$ because $X_0, \dots, X_n \in \text{up } 1^{\text{FCD}}$. It also holds for $S = S_1$ by the following reason:

Suppose $X_0, \dots, X_n \in S_1$. Then $X_i \supseteq F_i$ where $F_i \in S_0$. Consequently (take into account that Γ is a sublattice of FCD) $X_0, \dots, X_n \supseteq F_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} F_n$ and so $X_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} X_n = X_0 \sqcap \dots \sqcap X_n \supseteq F_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} F_n \supseteq 1^{\text{FCD}}$. Thus $X_0 \sqcap \dots \sqcap X_n \in \text{up}^{\Gamma} 1^{\text{FCD}} \subseteq S_1$; $\text{up}(X_0 \sqcap \dots \sqcap X_n) \subseteq S_1$ as S_1 is an upper set.

To finish the proof suppose for the contrary that $\text{up } f_0 = S_0$ and $\text{up } f_1 = S_1$ for some funcoids f_0 and f_1 . In this case $f_0 = \prod^{\text{FCD}} S_0 = 1^{\text{FCD}} = \prod^{\text{FCD}} \text{up}^{\Gamma} 1^{\text{FCD}} = \prod^{\text{FCD}} S_1 = f_1$ and thus $S_0 = S_1$, contradiction. \square

PROPOSITION 1421. For a set S of binary relations

$$\forall X_0, \dots, X_n \in S : \text{up}(X_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} X_n) \subseteq S$$

does not imply that S is a funcooid base.

PROOF. Suppose for the contrary that it does imply. Then, because S is an upper set (as follows from the condition, taking $n = 0$), it implies that $S = \text{up } f$ for a funcooid f , what contradicts to the above example. \square

CONJECTURE 1422. Let $\forall X, Y \in S : \text{up}(X \sqcap^{\text{FCD}} Y) \subseteq S$.

Then

$$\forall X_0, \dots, X_n \in S : \text{up}(X_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} X_n) \subseteq S.$$

EXERCISE 1423. $\text{up}(f_0 \sqcap^{\text{FCD}} \dots \sqcap^{\text{FCD}} f_n) \subseteq \left\{ \frac{F_0 \sqcap \dots \sqcap F_n}{F_0 \in \text{up } f_0 \wedge \dots \wedge F_n \in \text{up } f_n} \right\}$ for every funcooids f_0, \dots, f_n ($n \in \mathbb{N}$).

16.9. Some (example) values

I will do some calculations of particular funcooids and reloids.

First note that \sqcap^{FCD} can be decomposed (see below for a short easy proof):

$$f \sqcap^{\text{FCD}} g = (\text{FCD})((\text{RLD})_{\text{in}} f \sqcap (\text{RLD})_{\text{in}} g).$$

The above is a more understandable decomposition of the operation \sqcap^{FCD} which behaves in strange way, mapping meet of two binary relations into a funcooid which is not a binary relation ($1^{\text{FCD}} \sqcap^{\text{FCD}} (\top \setminus 1^{\text{FCD}}) = 1_{\Omega}^{\text{FCD}}$).

The last formula is easy to prove (and proved above in the book) but the result is counter-intuitive.

More generally:

$$\bigsqcap^{\text{FCD}} S = (\text{FCD}) \bigsqcap^{\text{RLD}} \langle (\text{RLD})_{\text{in}} \rangle^* S.$$

The above formulas follow from the fact that (FCD) is an upper adjoint and that $(\text{FCD})(\text{RLD})_{\text{in}} f = f$ for every funcooid f .

Let FCD denote funcooids on a set U .

Consider a special case of the above formulas:

$$1^{\text{FCD}} \sqcap^{\text{FCD}} (\top \setminus 1^{\text{FCD}}) = (\text{FCD})((\text{RLD})_{\text{in}} 1^{\text{FCD}} \sqcap (\text{RLD})_{\text{in}} (\top \setminus 1^{\text{FCD}})). \quad (17)$$

We want to calculate terms of the formula (17) and more generally do some (probably useless) calculations for particular funcooids and reloids related to the above formula.

The left side is already calculated. The term $(\text{RLD})_{\text{in}} 1^{\text{FCD}}$ which I call “thick equality” above is well understood. Let’s compute $(\text{RLD})_{\text{in}} (\top \setminus 1^{\text{FCD}})$.

PROPOSITION 1424. $(\text{RLD})_{\text{in}} (\top \setminus 1^{\text{FCD}}) = \top \setminus 1^{\text{FCD}}$.

PROOF. Consider funcooids on a set U . For any filters x and y (or without loss of generality ultrafilters x and y) we have:

$$\begin{aligned} x \times^{\text{FCD}} y \sqsubseteq \top \setminus 1^{\text{FCD}} &\Leftrightarrow \\ (\text{theorem 574 and the fact that funcooids are filters}) &\Leftrightarrow \\ x \times^{\text{FCD}} y \asymp 1^{\text{FCD}} &\Leftrightarrow \neg(x [1^{\text{FCD}}] y) \Leftrightarrow x \asymp y \Rightarrow \\ &\exists X \in \text{up } x, Y \in \text{up } y : X \asymp Y. \end{aligned}$$

Thus $(\text{RLD})_{\text{in}} (\top \setminus 1^{\text{FCD}}) = \bigsqcup \left\{ \frac{X \times Y}{X, Y \in \mathcal{F}U, X \asymp Y} \right\} = \top \setminus 1^{\text{FCD}}$. \square

So, we have:

$$1_{\Omega}^{\text{FCD}} = 1^{\text{FCD}} \sqcap^{\text{FCD}} (\top \setminus 1^{\text{FCD}}) = (\text{RLD})_{\text{in}} 1^{\text{FCD}} \sqcap^{\text{FCD}} (\top \setminus 1^{\text{FCD}}).$$

PROPOSITION 1425. If $X_0 \sqcup \dots \sqcup X_n = \top$ then $(X_0 \times X_0) \sqcup \dots \sqcup (X_n \times X_n) \in \text{up}(\text{RLD})_{\text{in}} 1^{\text{FCD}}$.

PROOF. It's enough to prove $(X_0 \times X_0) \sqcup \dots \sqcup (X_n \times X_n) \in \text{up}(x \times x)$ for every ultrafilter x , what follows from the fact that $x \sqsubseteq X_i$ for some i and thus $x \times x \sqsubseteq X_i \times X_i$. \square

PROPOSITION 1426. For finite tuples X, Y of typed sets

$$(X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n) \sqsupseteq 1 \Leftrightarrow (X_0 \sqcap Y_0) \sqcup \dots \sqcup (X_n \sqcap Y_n) = \top.$$

PROOF.

$$\begin{aligned} (X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n) \sqsupseteq 1 &\Leftrightarrow \\ ((X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n)) \sqcap 1 = 1 &\Leftrightarrow \\ ((X_0 \times Y_0) \sqcap 1) \sqcup \dots \sqcup ((X_n \times Y_n) \sqcap 1) = 1 &\Leftrightarrow \\ \text{id}_{X_0 \sqcap Y_0} \sqcup \dots \sqcup \text{id}_{X_n \sqcap Y_n} = 1 &\Leftrightarrow \\ \text{id}_{(X_0 \sqcap Y_0) \sqcup \dots \sqcup (X_n \sqcap Y_n)} = 1 &\Leftrightarrow \\ (X_0 \sqcap Y_0) \sqcup \dots \sqcup (X_n \sqcap Y_n) = \top. & \end{aligned}$$

\square

COROLLARY 1427.

$$\text{up}^{\Gamma} 1 = \left\{ \frac{(X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n)}{n \in \mathbb{N}, \forall i \in n : X_i, Y_i \in \mathcal{S}U, (X_0 \sqcap Y_0) \sqcup \dots \sqcup (X_n \sqcap Y_n) = \top} \right\}.$$

COROLLARY 1428. The predicate $(X_0 \sqcap Y_0) \sqcup \dots \sqcup (X_n \sqcap Y_n) = \top$ for an element $(X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n)$ of Γ does not depend on its representation $(X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n)$.

PROPOSITION 1429.

$$\text{up}^{\Gamma} 1 = \bigcup \left\{ \frac{\text{up}^{\Gamma}((X_0 \times X_0) \sqcup \dots \sqcup (X_n \times X_n))}{n \in \mathbb{N}, \forall i \in n : X_i \in \mathcal{S}U, X_0 \sqcup \dots \sqcup X_n = \top} \right\}.$$

PROOF. If $(X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n) \in \text{up}^{\Gamma} 1$ then we have

$$\begin{aligned} (X_0 \times Y_0) \sqcup \dots \sqcup (X_n \times Y_n) &\sqsupseteq \\ ((X_0 \sqcap Y_0) \times (X_0 \sqcap Y_0)) \sqcup \dots \sqcup ((X_n \sqcap Y_n) \times (X_n \sqcap Y_n)) &\in \text{up}^{\Gamma} 1. \end{aligned}$$

Thus

$$\text{up}^{\Gamma} 1 \subseteq \bigcup \left\{ \frac{\text{up}^{\Gamma}((X_0 \times X_0) \sqcup \dots \sqcup (X_n \times X_n))}{n \in \mathbb{N}, \forall i \in n : X_i \in \mathcal{S}U, X_0 \sqcup \dots \sqcup X_n = \top} \right\}.$$

The reverse inclusion is obvious. \square

PROPOSITION 1430.

$$(\text{RLD})_{\text{in}} 1^{\text{FCD}} = \prod^{\text{RLD}} \left\{ \frac{(X_0 \times X_0) \sqcup \dots \sqcup (X_n \times X_n)}{n \in \mathbb{N}, \forall i \in n : X_i \in \mathcal{S}U, X_0 \sqcup \dots \sqcup X_n = \top} \right\}.$$

PROOF. By the diagram we have $(\text{RLD})_{\text{in}} 1^{\text{FCD}} = \prod^{\text{RLD}} \text{up}^{\Gamma} 1$. So it follows from the previous proposition. \square

PROPOSITION 1431. $\text{up}^{\Gamma}(\text{RLD})_{\text{in}} 1^{\text{FCD}} = \text{up}^{\Gamma} 1$.

PROOF. If $K \in \text{up}^\Gamma 1$ then $K \in \text{up}^\Gamma((X_0 \times X_0) \sqcup \dots \sqcup (X_n \times X_n))$ and thus $K \in \text{up}^\Gamma(\text{RLD})_{\text{in}1}^{\text{FCD}}$ (see proposition 1425). Thus $\text{up}^\Gamma 1 \subseteq \text{up}^\Gamma(\text{RLD})_{\text{in}1}^{\text{FCD}}$. But $\text{up}^\Gamma(\text{RLD})_{\text{in}1}^{\text{FCD}} \subseteq \text{up}^\Gamma 1$ is obvious. \square

Generalized cofinite filters

The following is a straightforward generalization of cofinite filter.

DEFINITION 1432. $\Omega_{1a} = \prod_{X \in \text{coatoms}^3}^{\mathfrak{A}} X$; $\Omega_{1b} = \prod_{X \in \text{coatoms}^{\mathfrak{A}}}^{\mathfrak{A}} X$.

PROPOSITION 1433. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
- 3°. $\Omega_{1a} = \Omega_{1b}$ for this filtrator.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Proposition 560. □

PROPOSITION 1434. Let $(\mathfrak{A}, \mathfrak{Z})$ be a primary filtrator. Let \mathfrak{Z} be a subset of $\mathcal{P}U$. Let it be a meet-semilattice with greatest element. Let also every non-coempty cofinite set lies in \mathfrak{Z} . Then

$$\partial\Omega = \left\{ \frac{Y \in \mathfrak{Z}}{\text{card atoms}^3 Y \geq \omega} \right\}. \quad (18)$$

PROOF. Ω exists by corollary 518.

$$\begin{aligned} Y \in \partial\Omega &\Leftrightarrow Y \not\prec^{\mathfrak{A}} \prod_{X \in \text{coatoms}^3}^{\mathfrak{A}} X \Leftrightarrow \\ &\text{(by properties of filter bases)} \Leftrightarrow \\ &\forall S \in \mathcal{P}_{\text{fin}} \text{coatoms}^3 : Y \not\prec^{\mathfrak{A}} \prod^{\mathfrak{A}} S \Leftrightarrow \\ &\text{(corollary 536)} \Leftrightarrow \forall S \in \mathcal{P}_{\text{fin}} \text{coatoms}^3 : Y \not\prec \prod S \Leftrightarrow \\ &\forall K \in \mathcal{P}_{\text{fin}} U : Y \setminus K \neq \emptyset \Leftrightarrow \\ &\text{card } Y \geq \omega \Leftrightarrow \text{card atoms}^3 Y \geq \omega. \end{aligned}$$

(Here \mathcal{P}_{fin} denotes the set of finite subsets.) □

COROLLARY 1435. Formula (18) holds for both reloids and funcoids.

PROOF. For reloids it's straightforward, for funcoids take that they are isomorphic to filters on lattice Γ . □

COROLLARY 1436. $\Omega^{\text{FCD}} \neq \perp^{\text{FCD}}$ (for $\text{FCD}(A, B)$ where $A \times B$ is an infinite set).

PROPOSITION 1437. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an atomic ideal base and $\forall \alpha \in \text{atoms}^3 \exists X \in \text{coatoms}^3 : \alpha \not\sqsubseteq X$.

- 3°. Ω_{1a} and $\text{Cor } \Omega_{1a}$ are defined, $\forall \alpha \in \text{atoms}^3 \exists X \in \text{coatoms}^3 : a \not\sqsubseteq X$ and \mathfrak{J} is an atomic poset.
 4°. $\text{Cor } \Omega_{1a} = \perp^3$.

PROOF.

- 1° \Rightarrow 2°. Obvious.
 2° \Rightarrow 3°. Obvious.
 3° \Rightarrow 4°. Suppose $\alpha \in \text{atoms}^3 \text{Cor } \Omega$. Then $\exists X \in \text{up } \Omega : \alpha \not\sqsubseteq X$. Therefore $\alpha \notin \text{atoms}^3 \text{Cor } \Omega$. So $\text{atoms}^3 \text{Cor } \Omega_{1a} = \emptyset$ and thus by atomicity $\text{Cor } \Omega_{1a} = \perp^3$.

□

COROLLARY 1438. $\text{Cor } \Omega^{\text{FCD}} = \perp$.

PROPOSITION 1439. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over an atomic meet-semilattice with greatest element such that $\forall \alpha \in \text{atoms}^3 \exists X \in \text{coatoms}^3 : a \not\sqsubseteq X$.
 3°. \mathfrak{A} is a complete lattice, $\forall \alpha \in \text{atoms}^3 \exists X \in \text{coatoms}^3 : a \not\sqsubseteq X$ and $(\mathfrak{A}; \mathfrak{J})$ is a filtered filtrator over an atomic poset.
 4°. $\Omega_{1a} = \max \left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\text{Cor } \mathcal{X} = \perp^3} \right\}$

PROOF.

- 1° \Rightarrow 2°. Obvious.
 2° \Rightarrow 3°. Obvious.
 3° \Rightarrow 4°. Due the last proposition, it is enough to show that $\text{Cor } \mathcal{X} = \perp^3 \Rightarrow \mathcal{X} \sqsubseteq \Omega_{1a}$ for every $\mathcal{X} \in \mathfrak{A}$.

Let $\text{Cor } \mathcal{X} = \perp^3$ for some $\mathcal{X} \in \mathfrak{A}$. Because of our filtrator being filtered, it's enough to show $X \in \text{up } \mathcal{X}$ for every $X \in \text{up } \Omega_{1a}$. $X = a_0 \sqcap \dots \sqcap a_n$ for a_i being coatoms of \mathfrak{J} . $a_i \sqsupseteq \mathcal{X}$ because otherwise $a_i \not\sqsupseteq \text{Cor } \mathcal{X}$. So $X \in \text{up } \mathcal{X}$.

□

PROPOSITION 1440. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice.
 3°. $\text{up } \Omega_{1a} = \left\{ \frac{\sqcap S}{S \in \mathcal{P}_{\text{fin}} \text{coatoms}^3} \right\}$

PROOF.

- 1° \Rightarrow 2°. Obvious.
 2° \Rightarrow 3°. Because $\left\{ \frac{\sqcap S}{S \in \mathcal{P}_{\text{fin}} \text{coatoms}^3} \right\}$ is a filter.

□

COROLLARY 1441. $\text{up } \Omega^{\text{FCD}} = \text{up } \Omega^{\text{RLD}}$.

DEFINITION 1442. $\Omega_{1c} = \bigsqcup (\text{atoms}^{\mathfrak{A}} \setminus \mathfrak{J})$.

PROPOSITION 1443. The following is an implications tuple:

- 1°. $(\mathfrak{A}; \mathfrak{J})$ is a powerset filtrator.
 2°. $(\mathfrak{A}; \mathfrak{J})$ is a down-aligned filtered complete lattice filtrator over an atomistic poset and $\forall \alpha \in \text{atoms}^3 \exists X \in \text{coatoms}^3 : a \not\sqsubseteq X$.
 3°. $\Omega_{1c} = \Omega_{1a}$.

PROOF.

- 1° \Rightarrow 2°. Obvious.

$2^\circ \Rightarrow 3^\circ$. For $x \in \text{atoms}^{\mathfrak{A}} \setminus \mathfrak{B}$ we have $\text{Cor } x = \perp$ because otherwise $\perp \neq \text{Cor } x \sqsubseteq x$.

Thus by previous $x \sqsubseteq \Omega_{1a}$ and so $\Omega_{1c} = \bigsqcup(\text{atoms}^{\mathfrak{A}} \setminus \mathfrak{B}) \sqsubseteq \Omega_{1a}$.

If $x \in \text{atoms } \Omega_{1a}$ then $x \notin \mathfrak{B}$ because otherwise $\text{Cor } x \neq \perp$. So

$$\Omega_{1a} = \bigsqcup \text{atoms } \Omega_{1a} = \bigsqcup(\text{atoms } \Omega_{1a} \setminus \mathfrak{B}) \sqsubseteq \bigsqcup(\text{atoms}^{\mathfrak{A}} \setminus \mathfrak{B}) = \Omega_{1c}.$$

□

THEOREM 1444. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete atomic boolean lattice.
- 3°. All of the following:
 - (a) \mathfrak{A} is atomistic complete starrish lattice.
 - (b) \mathfrak{B} is a complete atomistic lattice.
 - (c) $(\mathfrak{A}, \mathfrak{B})$ is a filtered down-aligned filtrator with binarily meet-closed core.
- 4°. Cor' is the lower adjoint of $\Omega_{1c} \sqcup^{\mathfrak{A}} -$.

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. Obvious.

$3^\circ \Rightarrow 4^\circ$. It with join-closed core by theorem 534.

We will prove $\text{Cor}' \mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \Omega_{1c} \sqcup \mathcal{Y}$.

By atomisticity it is equivalent to: $\text{atoms}^{\mathfrak{A}} \text{Cor}' \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}}(\Omega_{1c} \sqcup \mathcal{Y})$; (theorem 603) $\text{atoms}^{\mathfrak{A}} \text{Cor}' \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}} \Omega_{1c} \cup \text{atoms}^{\mathfrak{A}} \mathcal{Y}$; what by below is equivalent to: $\text{atoms}^{\mathfrak{B}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{B}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}} \Omega_{1c} \cup \text{atoms}^{\mathfrak{A}} \mathcal{Y}$.

$\text{Cor}' \mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{A}} \text{Cor}' \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}} \mathcal{Y} \Rightarrow \text{atoms}^{\mathfrak{B}} \text{Cor}' \mathcal{X} \subseteq \text{atoms}^{\mathfrak{B}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{B}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{B}} \mathcal{Y}$;

$\text{atoms}^{\mathfrak{B}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{B}} \mathcal{Y} \Rightarrow$ (theorem 599) $\Rightarrow \text{Cor}' \mathcal{X} \sqsubseteq \text{Cor}' \mathcal{Y} \Rightarrow$ (theorem 543) $\Rightarrow \text{Cor}' \mathcal{X} \sqsubseteq \mathcal{Y}$.

Finishing the proof $\text{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{A}} \Omega_{1c} \cup \text{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq (\text{atoms}^{\mathfrak{A}} \setminus \mathfrak{B}) \cup \text{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{B}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{B}} \mathcal{Y} \Leftrightarrow \text{atoms}^{\mathfrak{B}} \mathcal{X} \subseteq \text{atoms}^{\mathfrak{B}} \mathcal{Y}$.

□

Next there is an alternative proof of the above theorem. This alternative proof requires additional condition $\forall \alpha \in \text{atoms}^{\mathfrak{B}} \exists X \in \text{coatoms}^{\mathfrak{B}} : \alpha \not\sqsubseteq X$ however.

PROOF. Define $\Omega = \Omega_{1a} = \Omega_{1c}$.

It with join-closed core by theorem 534.

It's enough to prove that

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{A}} \text{Cor}' \mathcal{X} \quad \text{and} \quad \text{Cor}'(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$\text{Cor}'(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) =$ (theorem 603) $= \text{Cor}' \Omega \sqcup^{\mathfrak{B}} \text{Cor}' \mathcal{Y} =$ (proposition 1437) $= \perp^{\mathfrak{B}} \sqcup^{\mathfrak{B}} \text{Cor}' \mathcal{Y} \sqsubseteq$ (theorem 543) $\sqsubseteq \mathcal{Y}$.

$\Omega \sqcup^{\mathfrak{A}} \text{Cor}' \mathcal{X} = \bigsqcup \text{atoms}(\Omega \sqcup^{\mathfrak{A}} \text{Cor}' \mathcal{X}) = \bigsqcup(\text{atoms } \Omega \cup \text{Cor}' \mathcal{X}) = \bigsqcup \text{atoms } \Omega \sqcup \bigsqcup \text{atoms } \mathcal{X} \supseteq \bigsqcup(\text{atoms } \mathcal{X} \setminus \mathfrak{B}) \sqcup \bigsqcup(\text{atoms } \mathcal{X} \cap \mathfrak{B}) = \bigsqcup((\text{atoms } \mathcal{X} \setminus \mathfrak{B}) \cup (\text{atoms } \mathcal{X} \cap \mathfrak{B})) = \bigsqcup \text{atoms } \mathcal{X} = \mathcal{X}$. □

COROLLARY 1445. Under conditions of the last theorem $\text{Cor}' \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle \text{Cor}' \rangle^* S$.

PROPOSITION 1446. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{B})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{B})$ is a primary filtrator over a complete atomic boolean lattice.
- 3°. All of the following:

- (a) \mathfrak{A} is atomistic complete co-brouwerian lattice.
- (b) \mathfrak{B} is a complete atomistic lattice.
- (c) $(\mathfrak{A}, \mathfrak{B})$ is a filtered down-aligned filtrator with binarily meet-closed core.

4°. $\text{Cor}' \mathcal{X} = \mathcal{X} \setminus * \Omega_{1c}$

PROOF.

1° \Rightarrow 2° Obvious.

2° \Rightarrow 3° Because complete atomic boolean lattice is isomorphic to a powerset.

3° \Rightarrow 4° Theorems 1444 and 154. □

PROPOSITION 1447.

1°. $\langle \Omega^{\text{FCD}} \rangle \{x\} = \Omega^U$;

2°. $\langle \Omega^{\text{FCD}} \rangle p = \top$ for every nontrivial atomic filter p .

PROOF. $\langle \Omega^{\text{FCD}} \rangle \{x\} = \prod_{y \in U}^{\mathfrak{A}} (U \setminus \{y\}) = \Omega^U$; $\langle \Omega^{\text{FCD}} \rangle p = \prod_{y \in U}^{\mathfrak{A}} \top = \top$. □

PROPOSITION 1448. $(\text{FCD})\Omega^{\text{RLD}} = \Omega^{\text{FCD}}$.

PROOF. $(\text{FCD})\Omega^{\text{RLD}} = \prod^{\text{FCD}} \text{up} \Omega^{\text{RLD}} = \Omega^{\text{FCD}}$. □

PROPOSITION 1449. $(\text{RLD})_{\text{out}} \Omega^{\text{FCD}} = \Omega^{\text{RLD}}$.

PROOF. $(\text{RLD})_{\text{out}} \Omega^{\text{FCD}} = \prod^{\text{RLD}} \text{up} \Omega^{\text{FCD}} = \prod^{\text{RLD}} \text{up} \Omega^{\text{RLD}} = \Omega^{\text{RLD}}$. □

PROPOSITION 1450. $(\text{RLD})_{\text{in}} \Omega^{\text{FCD}} = \Omega^{\text{RLD}}$.

PROOF.

$$\begin{aligned}
 (\text{RLD})_{\text{in}} \Omega^{\text{FCD}} &= \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, a \times^{\text{FCD}} b \sqsubseteq \Omega^{\text{FCD}}} \right\} = \\
 &\bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, \text{not } a \text{ and } b \text{ both trivial}} \right\} = \\
 &\bigsqcup \left\{ \frac{\bigsqcup \text{atoms}(a \times^{\text{RLD}} b)}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, \text{not } a \text{ and } b \text{ both trivial}} \right\} = \\
 &\bigsqcup \bigsqcup \left\{ \frac{\text{atoms}(a \times^{\text{RLD}} b)}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, \text{not } a \text{ and } b \text{ both trivial}} \right\} = \\
 &\bigsqcup (\text{nontrivial atomic reloids under } A \times B) = \Omega^{\text{RLD}}.
 \end{aligned}$$

□

Convergence of funcoids

18.1. Convergence

The following generalizes the well-known notion of a filter convergent to a point or to a set:

DEFINITION 1451. A filter $\mathcal{F} \in \mathcal{F}(\text{Dst } \mu)$ converges to a filter $\mathcal{A} \in \mathcal{F}(\text{Src } \mu)$ regarding a funcoid μ ($\mathcal{F} \xrightarrow{\mu} \mathcal{A}$) iff $\mathcal{F} \sqsubseteq \langle \mu \rangle \mathcal{A}$.

DEFINITION 1452. A funcoid f converges to a filter $\mathcal{A} \in \mathcal{F}(\text{Src } \mu)$ regarding a funcoid μ where $\text{Dst } f = \text{Dst } \mu$ (denoted $f \xrightarrow{\mu} \mathcal{A}$) iff $\text{im } f \sqsubseteq \langle \mu \rangle \mathcal{A}$ that is iff $\text{im } f \xrightarrow{\mu} \mathcal{A}$.

DEFINITION 1453. A funcoid f converges to a filter $\mathcal{A} \in \mathcal{F}(\text{Src } \mu)$ on a filter $\mathcal{B} \in \mathcal{F}(\text{Src } f)$ regarding a funcoid μ where $\text{Dst } f = \text{Dst } \mu$ iff $f|_{\mathcal{B}} \xrightarrow{\mu} \mathcal{A}$.

OBVIOUS 1454. A funcoid f converges to a filter $\mathcal{A} \in \mathcal{F}(\text{Src } \mu)$ on a filter $\mathcal{B} \in \mathcal{F}(\text{Src } f)$ regarding a funcoid μ iff $\langle f \rangle \mathcal{B} \sqsubseteq \langle \mu \rangle \mathcal{A}$.

REMARK 1455. We can define also convergence for a reloid f : $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \text{im } f \sqsubseteq \langle \mu \rangle \mathcal{A}$ or what is the same $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow (\text{FCD})f \xrightarrow{\mu} \mathcal{A}$.

THEOREM 1456. Let f, g be funcoids, μ, ν be endofuncoids, $\text{Dst } f = \text{Src } g = \text{Ob } \mu$, $\text{Dst } g = \text{Ob } \nu$, $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$. If $f \xrightarrow{\mu} \mathcal{A}$,

$$g|_{\langle \mu \rangle \mathcal{A}} \in C(\mu \sqcap (\langle \mu \rangle \mathcal{A} \times^{\text{FCD}} \langle \mu \rangle \mathcal{A}), \nu),$$

and $\langle \mu \rangle \mathcal{A} \sqsupseteq \mathcal{A}$, then $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$.

PROOF.

$$\begin{aligned} \text{im } f &\sqsubseteq \langle \mu \rangle \mathcal{A}; \\ \langle g \rangle \text{im } f &\sqsubseteq \langle g \rangle \langle \mu \rangle \mathcal{A}; \\ \text{im}(g \circ f) &\sqsubseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \rangle \langle \mu \rangle \mathcal{A}; \\ \text{im}(g \circ f) &\sqsubseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \rangle \langle \mu \sqcap (\langle \mu \rangle \mathcal{A} \times^{\text{FCD}} \langle \mu \rangle \mathcal{A}) \rangle \mathcal{A}; \\ \text{im}(g \circ f) &\sqsubseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \circ (\mu \sqcap (\langle \mu \rangle \mathcal{A} \times^{\text{FCD}} \langle \mu \rangle \mathcal{A})) \rangle \mathcal{A}; \\ \text{im}(g \circ f) &\sqsubseteq \langle \nu \circ g|_{\langle \mu \rangle \mathcal{A}} \rangle \mathcal{A}; \\ \text{im}(g \circ f) &\sqsubseteq \langle \nu \circ g \rangle \mathcal{A}; \\ g \circ f &\xrightarrow{\nu} \langle g \rangle \mathcal{A}. \end{aligned}$$

□

COROLLARY 1457. Let f, g be funcoids, μ, ν be endofuncoids, $\text{Dst } f = \text{Src } g = \text{Ob } \mu$, $\text{Dst } g = \text{Ob } \nu$, $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$. If $f \xrightarrow{\mu} \mathcal{A}$, $g \in C(\mu, \nu)$, and $\langle \mu \rangle \mathcal{A} \sqsupseteq \mathcal{A}$ then $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$.

PROOF. From the last theorem and theorem 1182.

□

18.2. Relationships between convergence and continuity

LEMMA 1458. Let μ, ν be endofunctors, $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$, $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$, $\text{Src } f = \text{Ob } \mu$, $\text{Dst } f = \text{Ob } \nu$. If $f \in \text{C}(\mu|_{\mathcal{A}}, \nu)$ then

$$f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A}.$$

PROOF.

$$\begin{aligned} f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} &\Leftrightarrow \text{im } f|_{\langle \mu \rangle \mathcal{A}} \sqsubseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \Leftrightarrow \\ &\langle f \rangle \langle \mu \rangle \mathcal{A} \sqsubseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A}. \end{aligned}$$

□

THEOREM 1459. Let μ, ν be endofunctors, $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$, $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$, $\text{Src } f = \text{Ob } \mu$, $\text{Dst } f = \text{Ob } \nu$. If $f \in \text{C}(\mu|_{\mathcal{A}}, \nu)$ then $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A}$.

PROOF.

$$\begin{aligned} f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} &\Leftrightarrow (\text{by the lemma}) \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A} \Leftrightarrow \\ &f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f \Leftrightarrow f \in \text{C}(\mu|_{\mathcal{A}}, \nu). \end{aligned}$$

□

COROLLARY 1460. Let μ, ν be endofunctors, $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$, $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$, $\text{Src } f = \text{Ob } \mu$, $\text{Dst } f = \text{Ob } \nu$. If $f \in \text{C}(\mu, \nu)$ then $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A}$.

THEOREM 1461. Let μ, ν be endofunctors, $f \in \text{FCD}(\text{Ob } \mu, \text{Ob } \nu)$, $\mathcal{A} \in \mathcal{F}(\text{Ob } \mu)$ be an ultrafilter, $\text{Src } f = \text{Ob } \mu$, $\text{Dst } f = \text{Ob } \nu$. $f \in \text{C}(\mu|_{\mathcal{A}}, \nu)$ iff $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A}$.

PROOF.

$$\begin{aligned} f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} &\Leftrightarrow (\text{by the lemma}) \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A} \Leftrightarrow \\ &(\text{used the fact that } \mathcal{A} \text{ is an ultrafilter}) \\ &f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f|_{\mathcal{A}} \Leftrightarrow f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f \Leftrightarrow f \in \text{C}(\mu|_{\mathcal{A}}, \nu). \end{aligned}$$

□

18.3. Convergence of join

PROPOSITION 1462. $\bigsqcup S \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \xrightarrow{\mu} \mathcal{A}$ for every collection S of filters on $\text{Dst } \mu$ and filter \mathcal{A} on $\text{Src } \mu$, for every functor μ .

PROOF.

$$\bigsqcup S \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \bigsqcup S \sqsubseteq \langle \mu \rangle \mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \sqsubseteq \langle \mu \rangle \mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \xrightarrow{\mu} \mathcal{A}.$$

□

COROLLARY 1463. $\bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : f \xrightarrow{\mu} \mathcal{A}$ for every collection F of functors f such that $\text{Dst } f = \text{Dst } \mu$ and filter \mathcal{A} on $\text{Src } \mu$, for every functor μ .

PROOF. By corollary 896 we have

$$\begin{aligned} \bigsqcup F \xrightarrow{\mu} \mathcal{A} &\Leftrightarrow \text{im } \bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \bigsqcup (\text{im})^* F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \\ &\forall f \in (\text{im})^* F : \mathcal{F} \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : \text{im } f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : f \xrightarrow{\mu} \mathcal{A}. \end{aligned}$$

□

THEOREM 1464. $f|_{\mathcal{B}_0 \sqcup \mathcal{B}_1} \xrightarrow{\mu} \mathcal{A} \Leftrightarrow f|_{\mathcal{B}_0} \xrightarrow{\mu} \mathcal{A} \wedge f|_{\mathcal{B}_1} \xrightarrow{\mu} \mathcal{A}$. for all filters \mathcal{A} , \mathcal{B}_0 , \mathcal{B}_1 and functors μ, f and g on suitable sets.

PROOF. As easily follows from distributivity of the lattices of functors we have $f|_{\mathcal{B}_0 \sqcup \mathcal{B}_1} = f|_{\mathcal{B}_0} \sqcup f|_{\mathcal{B}_1}$. Thus our theorem follows from the previous corollary. □

18.4. Limit

DEFINITION 1465. $\lim^\mu f = a$ iff $f \xrightarrow{\mu} \uparrow^{\text{Src } \mu} \{a\}$ for a T_2 -separable funcoid μ and a non-empty funcoid f such that $\text{Dst } f = \text{Dst } \mu$.

It is defined correctly, that is f has no more than one limit.

PROOF. Let $\lim^\mu f = a$ and $\lim^\mu f = b$. Then $\text{im } f \sqsubseteq \langle \mu \rangle^* @ \{a\}$ and $\text{im } f \sqsubseteq \langle \mu \rangle^* @ \{b\}$.

Because $f \neq \perp^{\text{FCD}(\text{Src } f, \text{Dst } f)}$ we have $\text{im } f \neq \perp^{\mathcal{F}(\text{Dst } f)}; \langle \mu \rangle^* @ \{a\} \cap \langle \mu \rangle^* @ \{b\} \neq \perp^{\mathcal{F}(\text{Dst } f)}; \uparrow^{\text{Src } \mu} \{b\} \cap \langle \mu^{-1} \rangle \langle \mu \rangle^* @ \{a\} \neq \perp^{\mathcal{F}(\text{Src } \mu)}; \uparrow^{\text{Src } \mu} \{b\} \cap \langle \mu^{-1} \circ \mu \rangle @ \{a\} \neq \perp^{\mathcal{F}(\text{Src } \mu)}; @ \{a\} [\mu^{-1} \circ \mu] @ \{b\}$. Because μ is T_2 -separable we have $a = b$. \square

DEFINITION 1466. $\lim_{\mathcal{B}}^\mu f = \lim^\mu (f|_{\mathcal{B}})$.

REMARK 1467. We can also in an obvious way define limit of a reloid.

18.5. Generalized limit

18.5.1. Definition. Let μ and ν be endofuncoids. Let G be a transitive permutation group on $\text{Ob } \mu$.

For an element $r \in G$ we will denote $\uparrow r = \uparrow^{\text{FCD}(\text{Ob } \mu, \text{Ob } \mu)} r$.

We require that μ and every $r \in G$ commute, that is

$$\mu \circ \uparrow r = \uparrow r \circ \mu.$$

We require for every $y \in \text{Ob } \nu$

$$\nu \sqsupseteq \langle \nu \rangle^* @ \{y\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}. \quad (19)$$

PROPOSITION 1468. Formula (19) follows from $\nu \sqsupseteq \nu \circ \nu^{-1}$.

PROOF. Let $\nu \sqsupseteq \nu \circ \nu^{-1}$. Then

$$\begin{aligned} \langle \nu \rangle^* @ \{y\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} &= \\ \langle \nu \rangle @ \{y\} \times^{\text{FCD}} \langle \nu \rangle @ \{y\} &= \\ \nu \circ (\uparrow^{\text{Ob } \nu} \{y\} \times^{\text{FCD}} \uparrow^{\text{Ob } \nu} \{y\}) \circ \nu^{-1} &= \\ \nu \circ \uparrow^{\text{FCD}(\text{Ob } \nu, \text{Ob } \nu)} (\{y\} \times \{y\}) \circ \nu^{-1} &\sqsubseteq \\ \nu \circ 1_{\text{Ob } \nu}^{\text{FCD}} \circ \nu^{-1} &= \\ \nu \circ \nu^{-1} &\sqsubseteq \nu. \end{aligned}$$

\square

REMARK 1469. The formula (19) usually works if ν is a proximity. It does not work if μ is a pretopology or preclosure.

We are going to consider (generalized) limits of arbitrary functions acting from $\text{Ob } \mu$ to $\text{Ob } \nu$. (The functions in consideration are not required to be continuous.)

REMARK 1470. Most typically G is the group of translations of some topological vector space.

Generalized limit is defined by the following formula:

DEFINITION 1471. $\text{xlim } f \stackrel{\text{def}}{=} \left\{ \frac{\nu \circ f \circ \uparrow r}{r \in G} \right\}$ for any funcoid f .

REMARK 1472. Generalized limit technically is a set of funcoids.

We will assume that $\text{dom } f \sqsupseteq \langle \mu \rangle^* @ \{x\}$.

DEFINITION 1473. $\text{xlim}_x f = \text{xlim } f|_{\langle \mu \rangle^* @ \{x\}}$.

OBVIOUS 1474. $\text{xlim}_x f = \left\{ \frac{\nu \circ f|_{\langle \mu \rangle^* @ \{x\}} \circ \uparrow r}{r \in G} \right\}$.

REMARK 1475. $\text{xlim}_x f$ is the same for functors μ and $\text{Compl } \mu$.

The function τ will define an injection from the set of points of the space ν (“numbers”, “points”, or “vectors”) to the set of all (generalized) limits (i.e. values which $\text{xlim}_x f$ may take).

DEFINITION 1476. $\tau(y) \stackrel{\text{def}}{=} \left\{ \frac{\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}}{x \in D} \right\}$.

PROPOSITION 1477. $\tau(y) = \left\{ \frac{\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} \circ \uparrow r}{r \in G} \right\}$ for every (fixed) $x \in D$.

PROOF.

$$\begin{aligned} & \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} \circ \uparrow r = \\ & \langle \uparrow r^{-1} \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} = \\ & \langle \mu \rangle \langle \uparrow r^{-1} \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} = \\ & \langle \mu \rangle^* @ \{r^{-1}x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} \in \left\{ \frac{\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}}{x \in D} \right\}. \end{aligned}$$

Reversely $\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} = \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} \circ \uparrow e$ where e is the identify element of G . \square

PROPOSITION 1478. $\tau(y) = \text{xlim}(\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \uparrow^{\text{Base}(\text{Ob } \nu)} \{y\})$ (for every x). Informally: Every $\tau(y)$ is a generalized limit of a constant functor.

PROOF.

$$\begin{aligned} & \text{xlim}(\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \uparrow^{\text{Base}(\text{Ob } \nu)} \{y\}) = \\ & \left\{ \frac{\nu \circ (\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \uparrow^{\text{Base}(\text{Ob } \nu)} \{y\}) \circ \uparrow r}{r \in G} \right\} = \\ & \left\{ \frac{\langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} \circ \uparrow r}{r \in G} \right\} = \tau(y). \end{aligned}$$

\square

THEOREM 1479. If f is a function and $f|_{\langle \mu \rangle^* @ \{x\}} \in C(\mu, \nu)$ and $\langle \mu \rangle^* @ \{x\} \sqsupseteq \uparrow^{\text{Ob } \mu} \{x\}$ then $\text{xlim}_x f = \tau(fx)$.

PROOF. $f|_{\langle \mu \rangle^* @ \{x\}} \circ \mu \sqsubseteq \nu \circ f|_{\langle \mu \rangle^* @ \{x\}} \sqsubseteq \nu \circ f$; thus $\langle f \rangle \langle \mu \rangle^* @ \{x\} \sqsubseteq \langle \nu \rangle \langle f \rangle^* @ \{x\}$; consequently we have

$$\begin{aligned} & \nu \sqsupseteq \langle \nu \rangle \langle f \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\} \sqsupseteq \langle f \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\}. \\ & \nu \circ f|_{\langle \mu \rangle^* @ \{x\}} \sqsupseteq \\ & \langle f \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\} \circ f|_{\langle \mu \rangle^* @ \{x\}} = \\ & (f|_{\langle \mu \rangle^* @ \{x\}})^{-1} \langle f \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\} \sqsupseteq \\ & \left\langle \text{id}_{\text{dom } f|_{\langle \mu \rangle^* @ \{x\}}}^{\text{FCD}} \right\rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\} \sqsupseteq \\ & \text{dom } f|_{\langle \mu \rangle^* @ \{x\}} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\} = \\ & \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\}. \end{aligned}$$

$$\begin{aligned} \text{im}(\nu \circ f|_{\langle \mu \rangle^* @ \{x\}}) &= \langle \nu \rangle \langle f \rangle^* @ \{x\}; \\ \nu \circ f|_{\langle \mu \rangle^* @ \{x\}} &\sqsubseteq \\ \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \text{im}(\nu \circ f|_{\langle \mu \rangle^* @ \{x\}}) &= \\ \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\}. & \end{aligned}$$

So $\nu \circ f|_{\langle \mu \rangle^* @ \{x\}} = \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\}$.

$$\text{Thus } \text{xlim}_x f = \left\{ \frac{\langle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle^* @ \{x\} \rangle \circ \uparrow r}{r \in G} \right\} = \tau(fx). \quad \square$$

REMARK 1480. Without the requirement of $\langle \mu \rangle^* @ \{x\} \sqsupseteq \uparrow^{\text{Ob } \mu} \{x\}$ the last theorem would not work in the case of removable singularity.

THEOREM 1481. Let $\nu \sqsubseteq \nu \circ \nu$. If $f|_{\langle \mu \rangle^* @ \{x\}} \xrightarrow{\nu} \uparrow^{\text{Ob } \mu} \{y\}$ then $\text{xlim}_x f = \tau(y)$.

PROOF. $\text{im } f|_{\langle \mu \rangle^* @ \{x\}} \sqsubseteq \langle \nu \rangle^* @ \{y\}$; $\langle f \rangle \langle \mu \rangle^* @ \{x\} \sqsubseteq \langle \nu \rangle^* @ \{y\}$;

$$\begin{aligned} \nu \circ f|_{\langle \mu \rangle^* @ \{x\}} &\sqsubseteq \\ (\langle \nu \rangle^* @ \{y\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}) \circ f|_{\langle \mu \rangle^* @ \{x\}} &= \\ \langle (f|_{\langle \mu \rangle^* @ \{x\}})^{-1} \rangle \langle \nu \rangle^* @ \{y\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} &= \\ \langle \text{id}_{\langle \mu \rangle^* @ \{x\}}^{\text{FCD}} \circ f^{-1} \rangle \langle \nu \rangle^* @ \{y\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} &\sqsupseteq \\ \langle \text{id}_{\langle \mu \rangle^* @ \{x\}}^{\text{FCD}} \circ f^{-1} \rangle \langle f \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} &= \\ \langle \text{id}_{\langle \mu \rangle^* @ \{x\}}^{\text{FCD}} \rangle \langle f^{-1} \circ f \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} &\sqsupseteq \\ \langle \text{id}_{\langle \mu \rangle^* @ \{x\}}^{\text{FCD}} \rangle \langle \text{id}_{\langle \mu \rangle^* @ \{x\}}^{\text{FCD}} \rangle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} &= \\ \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}. & \end{aligned}$$

On the other hand, $f|_{\langle \mu \rangle^* @ \{x\}} \sqsubseteq \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}$;

$$\nu \circ f|_{\langle \mu \rangle^* @ \{x\}} \sqsubseteq \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle \nu \rangle^* @ \{y\} \sqsubseteq \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}.$$

So $\nu \circ f|_{\langle \mu \rangle^* @ \{x\}} = \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\}$.

$$\text{xlim}_x f = \left\{ \frac{\nu \circ f|_{\langle \mu \rangle^* @ \{x\}} \circ \uparrow r}{r \in G} \right\} = \left\{ \frac{\langle \langle \mu \rangle^* @ \{x\} \times^{\text{FCD}} \langle \nu \rangle^* @ \{y\} \rangle \circ \uparrow r}{r \in G} \right\} = \tau(y). \quad \square$$

COROLLARY 1482. If $\lim_{\nu}^{\nu} f|_{\langle \mu \rangle^* @ \{x\}} = y$ then $\text{xlim}_x f = \tau(y)$ (provided that $\nu \sqsubseteq \nu \circ \nu$).

We have injective τ if $\langle \nu \rangle^* @ \{y_1\} \cap \langle \nu \rangle^* @ \{y_2\} = \perp^{\mathcal{F}(\text{Ob } \nu)}$ for every distinct $y_1, y_2 \in \text{Ob } \nu$ that is if ν is T_2 -separable.

18.6. Expressing limits as implications

When you studied limits in the school, you was told that $\lim_{x \rightarrow \alpha} f(x) = \beta$ when $x \rightarrow \alpha$ implies $f(x) \rightarrow \beta$. Now let us formalize this.

PROPOSITION 1483. The following are pairwise equivalent for funcoids μ, ν, f of suitable (“compatible”) sources and destinations:

- 1°. $f|_{\langle \mu \rangle^* @ \{\alpha\}} \xrightarrow{\nu} \beta$;
- 2°. $\forall x \in \mathcal{F}(\text{Ob } \mu) : (x \xrightarrow{\mu} \alpha \Rightarrow \langle f \rangle x \xrightarrow{\nu} \beta)$;
- 3°. $\forall x \in \text{atoms}^{\mathcal{F}(\text{Ob } \mu)} : (x \xrightarrow{\mu} \alpha \Rightarrow \langle f \rangle x \xrightarrow{\nu} \beta)$.

PROOF.

$1^\circ \Leftrightarrow 2^\circ$.

$$\begin{aligned} \forall x \in \mathcal{F}(\text{Ob } \mu) : \left(x \xrightarrow{\mu} \alpha \Rightarrow \langle f \rangle x \xrightarrow{\nu} \beta \right) &\Leftrightarrow \\ \forall x \in \mathcal{F}(\text{Ob } \mu) : \left(x \sqsubseteq \langle \mu \rangle \alpha \Rightarrow \right. & \\ \left. \langle f \rangle x \sqsubseteq \langle \nu \rangle \beta \right) &\Leftrightarrow \langle f \rangle \langle \mu \rangle \alpha \sqsubseteq \langle \nu \rangle \beta \Leftrightarrow f|_{\langle \mu \rangle^* \{ \alpha \}} \xrightarrow{\nu} \beta. \end{aligned}$$

$2^\circ \Rightarrow 3^\circ$. Obvious.

$3^\circ \Rightarrow 2^\circ$. Let 3° hold. Then for $x \in \mathcal{F}(\text{Ob } \mu)$ we have $x \xrightarrow{\mu} \alpha \Leftrightarrow x \sqsubseteq \langle \mu \rangle \alpha \Leftrightarrow \forall x' \in \text{atoms } x : x' \sqsubseteq \langle \mu \rangle \alpha \Leftrightarrow \forall x' \in \text{atoms } x : x' \xrightarrow{\mu} \alpha \Rightarrow \forall x' \in \text{atoms } x : \langle f \rangle x' \xrightarrow{\nu} \beta \Leftrightarrow \forall x' \in \text{atoms } x : \langle f \rangle x' \sqsubseteq \langle \nu \rangle \beta \Leftrightarrow \bigsqcup_{x' \in \text{atoms } x} \langle f \rangle x' \sqsubseteq \langle \nu \rangle \beta \Leftrightarrow \langle f \rangle x \sqsubseteq \langle \nu \rangle \beta \Leftrightarrow \langle f \rangle x \xrightarrow{\nu} \beta$.

□

LEMMA 1484. If f is an entirely defined monovalued funcoid and x is an ultrafilter, y is a filter, then $\langle f \rangle x \sqsubseteq y \Leftrightarrow x \sqsubseteq \langle f^{-1} \rangle y$.

PROOF. $\langle f \rangle x$ is an ultrafilter. $\langle f \rangle x \sqsubseteq y \Leftrightarrow \langle f \rangle x \not\sqsubseteq y \Leftrightarrow x \not\sqsubseteq \langle f^{-1} \rangle y \Leftrightarrow x \sqsubseteq \langle f^{-1} \rangle y$. □

PROPOSITION 1485. The following are pairwise equivalent for funcoids μ, ν, f, g of suitable (“compatible”) sources and destinations provided that g is entirely defined and monovalued:

- 1°. $(f \circ g^{-1})|_{\langle \mu \rangle^* \{ \alpha \}} \xrightarrow{\nu} \beta$;
- 2°. $\forall x \in \mathcal{F}(\text{Ob } \mu) : \left(\langle g \rangle x \xrightarrow{\mu} \alpha \Rightarrow \langle f \rangle x \xrightarrow{\nu} \beta \right)$;
- 3°. $\forall x \in \text{atoms}^{\mathcal{F}(\text{Ob } \mu)} : \left(\langle g \rangle x \xrightarrow{\mu} \alpha \Rightarrow \langle f \rangle x \xrightarrow{\nu} \beta \right)$.

PROOF.

$1^\circ \Leftrightarrow 3^\circ$. Equivalently transforming: $(f \circ g^{-1})|_{\langle \mu \rangle^* \{ \alpha \}} \xrightarrow{\nu} \beta$; $\langle f \rangle \langle g^{-1} \rangle^* \langle \mu \rangle^* \{ \alpha \} \sqsubseteq \langle \nu \rangle^* \{ \beta \}$; for every $x \in \text{atoms}^{\mathcal{F}(\text{Ob } \mu)}$ we have $x \sqsubseteq \langle g^{-1} \rangle \langle \mu \rangle^* \{ \alpha \} \Rightarrow \langle f \rangle x \sqsubseteq \langle \nu \rangle^* \{ \beta \}$; what by the lemma is equivalent to $\langle g \rangle x \sqsubseteq \langle \mu \rangle^* \{ \alpha \} \Rightarrow \langle f \rangle x \sqsubseteq \langle \nu \rangle^* \{ \beta \}$ that is $\langle g \rangle x \xrightarrow{\mu} \alpha \Rightarrow \langle f \rangle x \xrightarrow{\nu} \beta$.

$3^\circ \Leftrightarrow 2^\circ$. Let $x \in \mathcal{F}(\text{Ob } \mu)$ and 3° holds. Let $\langle g \rangle x \xrightarrow{\mu} \alpha$. Then $\forall x' \in \text{atoms } x : \langle g \rangle x' \xrightarrow{\mu} \alpha$ and thus $\langle f \rangle x' \xrightarrow{\nu} \beta$ that is $\langle f \rangle x' \sqsubseteq \langle \nu \rangle \beta$. $\langle f \rangle x = \bigsqcup_{x' \in \text{atoms } x} \langle f \rangle x' \sqsubseteq \langle \nu \rangle \beta$ that is $\langle f \rangle x \xrightarrow{\nu} \beta$.

□

PROBLEM 1486. Can the theorem be strenhtened for: a. non-monovalued; b. not entirely defined g ? (The problem seems easy but I have not checked it.)

Unfixed categories

FiXme: This is a draft not thoroughly checked for errors.

Unfixed categories like my other ideas is a great idea. However, previously I thought it is also great for studying funcoids and reroids, because unfixed funcoids is a generalization of funcoids, etc.

Unfixed funcoids are not a so important generalization as I imagined, because there is a simpler and yet more general generalization of funcoids: Every Hom-set of small funcoids can be embedded into $\text{FCD}(\bigcup \mathcal{U}, \bigcup \mathcal{U})$ where \mathcal{U} is the Grothendieck universe. Thus in principle it would be enough to study the semigroup $\text{FCD}(\bigcup \mathcal{U}, \bigcup \mathcal{U})$ rather than all categories of funcoids.

In this chapter I show how to embed one Hom-set into another Hom-set, so this chapter is indeed important. But the topic after which this chapter was titled, “Unfixed categories” is not so much important for our book.

19.1. Axiomatics for unfixed morphisms

DEFINITION 1487. *Category with restricted identities* is defined axiomatically: *Restricted identity* $\text{id}_X^{\mathcal{C}(A,B)}$ and *projection* $A \mapsto [A]$ are described by the axioms:

- 1°. \mathcal{C} is a category with the set of objects \mathfrak{J} ;
- 2°. every Hom-set $\mathcal{C}(A, B)$ is a lattice;
- 3°. \mathfrak{J} and \mathfrak{A} are lattices;
- 4°. $A \rightarrow [A]$ is a lattice embedding from \mathfrak{J} to \mathfrak{A} ;
- 5°. $\text{id}_X^{\mathcal{C}(A,B)} \in \text{Hom}_{\mathcal{C}}(A, B)$ whenever $\mathfrak{A} \ni X \sqsubseteq [A] \sqcap [B]$;
- 6°. $\text{id}_{[A]}^{\mathcal{C}(A,A)} = 1_A^{\mathcal{C}}$;
- 7°. $\text{id}_Y^{\mathcal{C}(B,C)} \circ \text{id}_X^{\mathcal{C}(A,B)} = \text{id}_{X \sqcap Y}^{\mathcal{C}(A,C)}$ whenever $\mathfrak{A} \ni X \sqsubseteq [A] \sqcap [B]$ and $\mathfrak{A} \ni Y \sqsubseteq [B] \sqcap [C]$;
- 8°. $\forall A \in \mathfrak{A} \exists B \in \mathfrak{J} : A \sqsubseteq [B]$.

For a *partially ordered category with restricted identities* introduce additional axiom $X \sqsubseteq Y \Rightarrow \text{id}_X^{\mathcal{C}(A,B)} \sqsubseteq \text{id}_Y^{\mathcal{C}(A,B)}$.

For *dagger categories with restricted identities* introduce additional axiom $(\text{id}_X^{\mathcal{C}(A,B)})^\dagger = \text{id}_X^{\mathcal{C}(B,A)}$.

DEFINITION 1488. I call a category with restricted identities *injective* when the axiom $X \neq Y \Rightarrow \text{id}_X^{\mathcal{C}(A,B)} \neq \text{id}_Y^{\mathcal{C}(A,B)}$ whenever $X, Y \sqsubseteq [A] \sqcap [B]$ holds.

DEFINITION 1489. Define $\mathcal{E}_C^{A,B} = \text{id}_{[A] \sqcap [B]}^{\mathcal{C}(A,B)}$.

PROPOSITION 1490.

- 1°. If $[A] \sqsubseteq [B]$ then $\mathcal{E}_C^{A,B}$ is a monomorphism.
- 2°. If $[A] \supseteq [B]$ then $\mathcal{E}_C^{A,B}$ is an epimorphism.

PROOF. We'll prove only the first as the second is dual.

Let $\mathcal{E}_C^{A,B} \circ f = \mathcal{E}_C^{A,B} \circ g$. Then $\mathcal{E}_C^{B,A} \circ \mathcal{E}_C^{A,B} \circ f = \mathcal{E}_C^{B,A} \circ \mathcal{E}_C^{A,B} \circ g$; $1^A \circ f = 1^A \circ g$; $f = g$. \square

PROPOSITION 1491. $\mathcal{E}_C^{B,C} \circ \mathcal{E}_C^{A,B} = \mathcal{E}_C^{A,C}$ if $B \supseteq A \cap C$ (for every sets A, B, C).

PROOF. $\mathcal{E}_C^{B,C} \circ \mathcal{E}_C^{A,B} = \mathcal{E}_C^{A,C}$ is equivalent to:

$$\text{id}_{B \cap C}^{C(B,C)} \circ \text{id}_{A \cap B}^{C(A,B)} = \text{id}_{A \cap C}^{C(A,C)} \text{ what is obviously true. } \square$$

DEFINITION 1492. $\text{id}_X^{C(A)} = \text{id}_{[X]}^{C(A,A)}$.

19.2. Rectangular embedding-restriction

DEFINITION 1493. $\iota_{B_0, B_1} f = \mathcal{E}_C^{\text{Dst } f, B_1} \circ f \circ \mathcal{E}_C^{B_0, \text{Src } f}$ for $f \in \text{Hom}_C(A_0, A_1)$.

For brevity $\iota_B f = \iota_{B, B} f$.

OBVIOUS 1494. $\iota_{B_0, B_1} f \sqsubseteq f$.

PROPOSITION 1495. $\iota_{\text{Src } f, \text{Dst } f} f = f$.

PROOF.

$$\iota_{\text{Src } f, \text{Dst } f} f =$$

$$\mathcal{E}_C^{\text{Dst } f, \text{Dst } f} \circ f \circ \mathcal{E}_C^{\text{Src } f, \text{Src } f} =$$

$$1_C^{\text{Dst } f} \circ f \circ 1_C^{\text{Src } f} = f. \quad \square$$

PROPOSITION 1496. The function $\iota_{B_0, B_1} |_{f \in \text{Hom}_C(A_0, A_1)}$ is injective, provided that $A_0 \sqsubseteq B_0$ and $A_1 \sqsubseteq B_1$.

PROOF. Because $\mathcal{E}_C^{A_1, B_1}$ is a monomorphism and $\mathcal{E}_C^{A_0, B_0}$ is an epimorphism. \square

COROLLARY 1497. The function $\iota_{B_0, B_1} |_{f \in \text{Hom}_C(A_0, A_1)}$ is order embedding if $A_0 \sqsubseteq B_0 \wedge A_1 \sqsubseteq B_1$ for ordered categories with restricted identities.

19.3. Image and domain

Let define that $\mathcal{S}\mathcal{A} = \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A}: X \subseteq K} \right\}$ holds not only for filters but for any set \mathcal{A} of sets.

OBVIOUS 1498. $\mathcal{S}\mathcal{A} \supseteq \mathcal{A}$.

DEFINITION 1499.

$$1^\circ. \text{IM } f = \left\{ \frac{Y \in \mathfrak{Z}}{\mathcal{E}_C^{Y, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, Y} \circ f = f} \right\};$$

$$2^\circ. \text{DOM } f = \left\{ \frac{X \in \mathfrak{Z}}{f \circ \mathcal{E}_C^{\text{Src } f, X} \circ \mathcal{E}_C^{X, \text{Src } f} = f} \right\}.$$

OBVIOUS 1500.

$$1^\circ. \text{IM } f = \left\{ \frac{Y \in \mathfrak{Z}}{\text{id}_{[Y] \cap [\text{Dst } f]}^{C(\text{Dst } f, \text{Dst } f)} \circ f = f} \right\} = \left\{ \frac{Y \in \mathfrak{Z}}{\text{id}_{Y \cap \text{Dst } f}^{C(\text{Dst } f)} \circ f = f} \right\};$$

$$2^\circ. \text{DOM } f = \left\{ \frac{X \in \mathfrak{Z}}{f \circ \text{id}_{[X] \cap [\text{Src } f]}^{C(\text{Src } f, \text{Src } f)} = f} \right\} = \left\{ \frac{X \in \mathfrak{Z}}{f \circ \text{id}_{X \cap \text{Src } f}^{C(\text{Src } f, \text{Src } f)} = f} \right\}.$$

DEFINITION 1501.

$$1^\circ. \text{Im } f = \left\{ \frac{Y \in \text{IM } f}{Y \sqsubseteq \text{Dst } f} \right\};$$

$$2^\circ. \text{Dom } f = \left\{ \frac{X \in \text{DOM } f}{X \sqsubseteq \text{Src } f} \right\}.$$

PROPOSITION 1502.

$$1^\circ. \text{IM } f = \mathcal{S} \text{Im } f;$$

- 2°. $\text{DOM } f = \mathcal{S} \text{ Dom } f$;
 3°. $\text{Im } f = \langle \text{Dst } f \cap \rangle^* \text{IM } f$;
 4°. $\text{Dom } f = \langle \text{Dst } f \cap \rangle^* \text{DOM } f$.

PROOF. $\text{IM } f = \left\{ \frac{Y \in \mathfrak{Z}}{\text{id}_{[Y] \cap [\text{Dst } f]}^{\mathcal{C}(\text{Dst } f, \text{Dst } f)} \circ f = f} \right\}$.

Suppose $Y \in \text{IM } f$. Then take $Y' = Y \cap \text{Dst } f$. We have $Y \supseteq Y'$ and $Y' \in \text{Im } f$. So $Y \in \mathcal{S} \text{ Im } f$. If $Y \in \mathcal{S} \text{ Im } f$ then $Y \in \text{IM } f$ obviously. So $\text{IM } f = \mathcal{S} \text{ Im } f$.

$\langle \text{Dst } f \cap \rangle^* \text{IM } f \subseteq \text{Im } f$ is obvious. If $\text{Im } f \subseteq \langle \text{Dst } f \cap \rangle^* \text{IM } f$ is also obvious.

The rest follows from symmetry. \square

CONJECTURE 1503. $\text{Im } f$ may be not a filter for an injective category with restricted morphisms.

PROPOSITION 1504. $\text{Dom } f = \left\{ \frac{X \in \mathfrak{Z}}{X \sqsubseteq \text{Src } f, f \text{oid}_{[X]}^{\mathcal{C}(\text{Dst } f)} = f} \right\}$.

PROOF. $\text{Dom } f = \langle \text{Dst } f \cap \rangle^* \left\{ \frac{X \in \mathfrak{Z}}{f \text{oid}_{[X] \cap [\text{Src } f]}^{\mathcal{C}(\text{Src } f, \text{Src } f)} = f} \right\} = \left\{ \frac{X \in \mathfrak{Z}}{X \sqsubseteq \text{Src } f, f \text{oid}_{[X]}^{\mathcal{C}(\text{Dst } f)} = f} \right\}$. \square

PROPOSITION 1505. $\text{Dst } f \in \text{Im } f$; $\text{Src } f \in \text{Dom } f$ for every morphism f of a category with restricted identities.

PROOF. Prove $\text{Dst } f \in \text{Im } f$ (the other is similar): We need to prove that $\mathcal{E}_C^{\text{Dst } f, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, \text{Dst } f} \circ f = f$ what follows from $\mathcal{E}_C^{\text{Dst } f, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, \text{Dst } f} = 1^{\text{Dst } f}$. \square

PROPOSITION 1506. $\text{IM } f$, $\text{Im } f$, $\text{DOM } f$, $\text{Dom } f$ are upper sets.

PROOF. For $\text{Im } f$, $\text{Dom } f$ it follows from the previous proposition.

For $\text{IM } f$, $\text{DOM } f$ it follows from the thesis for $\text{Im } f$, $\text{Dom } f$. \square

DEFINITION 1507.

- 1°. An ordered category with restricted identities is *with ordered image* iff $f \sqsubseteq g \Rightarrow \text{IM } f \subseteq \text{IM } g$.
 2°. An ordered category with restricted identities is *with ordered domain* iff $f \sqsubseteq g \Rightarrow \text{DOM } f \subseteq \text{DOM } g$.
 3°. An ordered category with restricted identities is *with ordered domain and image* iff it is both with ordered domain and with ordered image.

OBVIOUS 1508.

- 1°. An ordered category with restricted identities is with ordered image iff $f \sqsubseteq g \Rightarrow \text{Im } f \subseteq \text{Im } g$.
 2°. An ordered category with restricted identities is with ordered domain iff $f \sqsubseteq g \Rightarrow \text{Dom } f \subseteq \text{Dom } g$.
 3°. An ordered category with restricted identities is with ordered domain and image iff it is both with ordered domain and with ordered image.

OBVIOUS 1509.

- 1°. For an ordered category \mathcal{C} with restricted identities to be with ordered image it's enough that $\text{id}_{[X]}^{\mathcal{C}(\text{Dst } f, \text{Dst } f)} \circ f = f \wedge g \sqsubseteq f \Rightarrow \text{id}_{[X]}^{\mathcal{C}(\text{Dst } f, \text{Dst } f)} \circ g = g$ for every parallel morphisms f and g and $\mathfrak{Z} \ni X \sqsubseteq \text{Dst } f$.
 2°. For an ordered category \mathcal{C} with restricted identities to be with ordered domain it's enough that $f \text{oid}_{[X]}^{\mathcal{C}(\text{Src } f, \text{Src } f)} = f \wedge g \sqsubseteq f \Rightarrow g \text{oid}_{[X]}^{\mathcal{C}(\text{Src } f, \text{Src } f)} = g$ for every parallel morphisms f and g and $\mathfrak{Z} \ni X \sqsubseteq \text{Src } f$.

CONJECTURE 1510. There exists a category with restricted identities which is not with ordered image.

OBVIOUS 1511. For an ordered category with restricted identities with ordered domain and image we have $\iota_{\text{Src } f, \text{Dst } f} \iota_{A, B} f = f \wedge g \sqsubseteq f \Rightarrow \iota_{\text{Src } f, \text{Dst } f} \iota_{A, B} g = g$ for parallel morphisms f and g .

DEFINITION 1512.

- 1°. $\underline{\text{im}} f = \min \text{Im } f$;
- 2°. $\underline{\text{dom}} f = \min \text{Dom } f$.

NOTE 1513. It seems that $\underline{\text{im}}$ and $\underline{\text{dom}}$ are defined not for every category with restricted identities.

PROPOSITION 1514.

- 1°. $\underline{\text{im}} f = \min \text{IM } f$;
- 2°. $\underline{\text{dom}} f = \min \text{DOM } f$.

PROOF. It follows from $\text{IM } f = \mathcal{S} \text{Im } f$ (and likewise for $\underline{\text{dom}} f$). \square

THEOREM 1515. $\text{DOM}(g \circ f) \supseteq \text{DOM } f$, $\text{IM}(g \circ f) \supseteq \text{IM } g$, $\text{Dom}(g \circ f) \supseteq \text{Dom } f$, $\text{Im}(g \circ f) \supseteq \text{Im } g$.

PROOF. $\mathcal{E}_C^{Y, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, Y} \circ g \circ f = g \circ f \Leftarrow \mathcal{E}_C^{Y, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, Y} \circ g = g$ and it implies $\text{IM}(g \circ f) \supseteq \text{IM } g$. The rest follows easily. \square

COROLLARY 1516. $\underline{\text{dom}}(g \circ f) \sqsubseteq \underline{\text{dom}} f$, $\underline{\text{im}}(g \circ f) \sqsubseteq \underline{\text{im}} g$ whenever $\underline{\text{dom}}/\underline{\text{im}}$ are defined.

19.4. Equivalent morphisms

PROPOSITION 1517. $\iota_{A, B} \iota_{X, Y} f = \iota_{A, B} f$ for every sets A, B, X, Y whenever $\text{DOM } f$ and $\text{IM } f$ are filters and $X \in \text{DOM } f$, $Y \in \text{IM } f$.

PROOF.

$$\begin{aligned} \iota_{A, B} f &= \mathcal{E}_C^{\text{Dst } f, B} \circ f \circ \mathcal{E}_C^{A, \text{Src } f} = \\ &\quad (\text{by definition of IM } f \text{ and DOM } f) = \\ &\mathcal{E}_C^{\text{Dst } f, B} \circ \mathcal{E}_C^{Y, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, Y} \circ f \circ \mathcal{E}_C^{X, \text{Src } f} \circ \mathcal{E}_C^{\text{Src } f, X} \circ \mathcal{E}_C^{A, \text{Src } f} = \\ &\mathcal{E}_C^{Y, B} \circ \mathcal{E}_C^{\text{Dst } f, Y} \circ f \circ \mathcal{E}_C^{X, \text{Src } f} \circ \mathcal{E}_C^{A, X} = \\ &\hspace{15em} \iota_{A, B} \iota_{X, Y} f \end{aligned}$$

because

$$\begin{aligned} \mathcal{E}_C^{\text{Dst } f, B} \circ \mathcal{E}_C^{Y, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, Y} &= \\ \text{id}_{Y \cap \text{Dst } f \cap B}^{\mathcal{C}(\text{Dst } f, B)} &= \text{id}_{Y \cap B}^{\mathcal{C}(Y, B)} \circ \text{id}_{Y \cap \text{Dst } f}^{\mathcal{C}(\text{Dst } f, Y)} = \\ &\mathcal{E}_C^{Y, B} \circ \mathcal{E}_C^{\text{Dst } f, Y} \end{aligned}$$

and thus $\mathcal{E}_C^{\text{Dst } f, B} \circ \mathcal{E}_C^{Y, \text{Dst } f} \circ \mathcal{E}_C^{\text{Dst } f, Y} = \mathcal{E}_C^{Y, B} \circ \mathcal{E}_C^{\text{Dst } f, Y}$ and similarly for $\mathcal{E}_C^{X, \text{Src } f} \circ \mathcal{E}_C^{\text{Src } f, X} \circ \mathcal{E}_C^{A, \text{Src } f}$. \square

DEFINITION 1518. I call two morphisms $f \in \mathcal{C}(A_0, B_0)$ and $g \in \mathcal{C}(A_1, B_1)$ of a category with restricted morphisms *equivalent* (and denote $f \sim g$) when

$$\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g.$$

PROPOSITION 1519. $f \sim g$ iff $\iota_{A, B} f = \iota_{A, B} g$ for some $A \in \text{DOM } f \cap \text{DOM } g$, $B \in \text{IM } f \cap \text{IM } g$.

PROOF. Both

$$\iota_{A,B}f = \iota_{A,B}g \Rightarrow \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g$$

and

$$\iota_{A,B}f = \iota_{A,B}g \Leftarrow \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g$$

follow from proposition 1517. \square

THEOREM 1520. Let $f : A_0 \rightarrow B_0$ and $g : A_1 \rightarrow B_1$ (for a partially ordered category with restricted identities). The following are pairwise equivalent:

- 1°. $f \sim g$;
- 2°. $\iota_{A_1, B_1}f = g$ and $\iota_{A_0, B_0}g = f$;
- 3°. $\iota_{A_1, B_1}f \sqsupseteq g$ and $\iota_{A_0, B_0}g \sqsupseteq f$.

PROOF.

1° \Rightarrow 2°. $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g$; $\iota_{A_1, B_1}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_1, B_1}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g$; $\iota_{A_1, B_1}f = \iota_{A_1, B_1}g$; $\iota_{A_1, B_1}f = g \cdot \iota_{A_0, B_0}g = f$ is similar.

3° \Rightarrow 1°. Let $\iota_{A_1, B_1}f \sqsupseteq g$ and $\iota_{A_0, B_0}g \sqsupseteq f$.

$$\begin{aligned} & \iota_{A_1, B_1}\iota_{A_0, B_0}g \sqsupseteq g; \\ & \mathcal{E}^{B_0, B_1} \circ \mathcal{E}^{B_1, B_0} \circ g \circ \mathcal{E}^{A_0, A_1} \circ \mathcal{E}^{A_1, A_0} \sqsupseteq g; \\ & \text{id}_{[B_0] \sqcap [B_1]}^{C(B_1, B_1)} \circ g \circ \text{id}_{[A_0] \sqcap [A_1]}^{C(A_1, A_1)} \sqsupseteq g; \text{id}_{[B_0] \sqcap [B_1]}^{C(B_1, B_1)} \circ g \sqsupseteq g; \text{id}_{[B_0] \sqcap [B_1]}^{C(B_1, B_1)} \circ g = g; \\ & \text{id}_{[B_0] \sqcap [B_1]}^{C(B_0 \sqcap B_1, B_1)} \circ \text{id}_{[B_0] \sqcap [B_1]}^{C(B_1, B_0 \sqcap B_1)} \circ g = g; \mathcal{E}^{B_0 \sqcap B_1, B_1} \circ \mathcal{E}^{B_1, B_0 \sqcap B_1} \circ g = g. \end{aligned}$$

Thus $B_0 \sqcap B_1 \in \text{Im } g$. Similarly $A_0 \sqcap A_1 \in \text{Dom } g$.

$$\text{So } \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}\iota_{A_0, B_0}g = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g.$$

2° \Rightarrow 3°. Obvious. \square

PROPOSITION 1521. Above defined equivalence of morphisms (for a small category) is an equivalence relation.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $f \sim g$ and $g \sim h$ for $f : A_0 \rightarrow B_0$, $g : A_1 \rightarrow B_1$, $h : A_2 \rightarrow B_2$. Then $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g$ and $\iota_{A_1 \sqcup A_2, B_1 \sqcup B_2}g = \iota_{A_1 \sqcup A_2, B_1 \sqcup B_2}h$.

Thus

$$\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f = \iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}g$$

and

$$\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}\iota_{A_1 \sqcup A_2, B_1 \sqcup B_2}g = \iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}\iota_{A_1 \sqcup A_2, B_1 \sqcup B_2}h$$

that is (proposition 1517)

$$\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}f = \iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}g$$

and

$$\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}g = \iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}h.$$

Combining, $\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}f = \iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}h$ and thus

$$\iota_{A_0 \sqcup A_2, B_0 \sqcup B_2}\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}f = \iota_{A_0 \sqcup A_2, B_0 \sqcup B_2}\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}h;$$

(again proposition 1517) $\iota_{A_0 \sqcup A_2, B_0 \sqcup B_2}f = \iota_{A_0 \sqcup A_2, B_0 \sqcup B_2}h$ that is $f \sim h$. \square

PROPOSITION 1522. $[f] = \left\{ \frac{\iota_{A,B}f}{A \in \text{DOM } f, B \in \text{IM } f} \right\}$.

PROOF. If $A \in \text{DOM } f$, $B \in \text{IM } f$ then

$$\iota_{A \sqcup \text{Src } f, B \sqcup \text{Dst } f} \iota_{A, B} f = \iota_{A \sqcup \text{Src } f, B \sqcup \text{Dst } f} f.$$

Thus $\iota_{A, B} f \sim f$ that is $\iota_{A, B} f \in [f]$.

Let now $g \in [f]$ that is $f \sim g$;

$$\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} f = \iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} g.$$

Take $A = \text{Src } g$, $B = \text{Dst } g$. We have

$$\begin{aligned} \iota_{A, B} \iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} f &= \iota_{A, B} \iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} g; \\ \iota_{A, B} f &= \iota_{A, B} g = g. \end{aligned}$$

□

PROPOSITION 1523.

$$\begin{aligned} 1^\circ. \text{IM } f &= \left\{ \frac{Y \in \mathfrak{Z}}{\mathcal{E}_C^{\text{Dst } f, Y} \circ f \sim f} \right\}; \\ 2^\circ. \text{DOM } f &= \left\{ \frac{X \in \mathfrak{Z}}{f \circ \mathcal{E}_C^{X, \text{Src } f} \sim f} \right\}. \end{aligned}$$

PROOF.

$$\begin{aligned} \mathcal{E}_C^{\text{Dst } f, Y} \circ f \sim f &\Leftrightarrow \iota_{\text{Src } f, Y \sqcup \text{Dst } f} (\mathcal{E}_C^{\text{Dst } f, Y} \circ f) = \iota_{\text{Src } f, Y \sqcup \text{Dst } f} f \Leftrightarrow \\ &\mathcal{E}^{Y, Y \sqcup \text{Dst } f} \circ \mathcal{E}^{\text{Dst } f, Y} \circ f \circ \mathcal{E}^{\text{Src } f, \text{Src } f} = \mathcal{E}^{\text{Dst } f, Y \sqcup \text{Dst } f} \circ f \circ \mathcal{E}^{\text{Src } f, \text{Src } f} \Leftrightarrow \\ &\mathcal{E}^{Y, Y \sqcup \text{Dst } f} \circ \mathcal{E}^{\text{Dst } f, Y} \circ f = \mathcal{E}^{\text{Dst } f, Y \sqcup \text{Dst } f} \circ f \Leftrightarrow (\text{proposition 1490}) \\ \Leftrightarrow \mathcal{E}^{Y \sqcup \text{Dst } f, \text{Dst } f} \circ \mathcal{E}^{Y, Y \sqcup \text{Dst } f} \circ \mathcal{E}^{\text{Dst } f, Y} \circ f &= \mathcal{E}^{Y \sqcup \text{Dst } f, \text{Dst } f} \circ \mathcal{E}^{\text{Dst } f, Y \sqcup \text{Dst } f} \circ f \Leftrightarrow \\ &\mathcal{E}^{Y, \text{Dst } f} \circ \mathcal{E}^{\text{Dst } f, Y} \circ f = f. \end{aligned}$$

From this our thesis follows obviously. □

PROPOSITION 1524. $\iota_{A_1, B_1} \iota_{A_0, B_0} f \sqsubseteq \iota_{A_1, B_1} f$.

PROOF.

$$\begin{aligned} \iota_{A_1, B_1} \iota_{A_0, B_0} f &= \\ &\mathcal{E}^{B_0, B_1} \circ \mathcal{E}^{\text{Dst } f, B_0} \circ f \circ \mathcal{E}^{A_0, \text{Src } f} \circ \mathcal{E}^{A_1, A_0} = \\ &\text{id}_{[B_0] \sqcap [B_1]}^{\mathcal{C}(B_0, B_1)} \circ \text{id}_{[\text{Dst } f] \sqcap [B_0]}^{\mathcal{C}(\text{Dst } f, B_0)} \circ f \circ \text{id}_{[A_0] \sqcap [\text{Src } f]}^{\mathcal{C}(A_0, \text{Src } f)} \circ \text{id}_{[A_1] \sqcap [A_0]}^{\mathcal{C}(A_1, A_0)} = \\ &\text{id}_{[\text{Dst } f] \sqcap [B_0] \sqcap [B_1]}^{\mathcal{C}(\text{Dst } f, B_1)} \circ f \circ \text{id}_{[A_0] \sqcap [A_1] \sqcap [\text{Src } f]}^{\mathcal{C}(A_1, \text{Src } f)} \sqsubseteq \\ &\text{id}_{[\text{Dst } f] \sqcap [B_1]}^{\mathcal{C}(\text{Dst } f, B_1)} \circ f \circ \text{id}_{[A_1] \sqcap [\text{Src } f]}^{\mathcal{C}(A_1, \text{Src } f)} = \\ &\iota_{A_1, B_1} f. \end{aligned}$$

□

19.5. Binary product

DEFINITION 1525. The category *with binary product morphism* is a category with restricted identities and additional axioms

- 1°. $\text{id}_Y^{\mathcal{C}(B, B)} \circ f \circ \text{id}_X^{\mathcal{C}(A, A)} = f \sqcap (X \times_{A, B} Y)$ (holding for every $A, B \in \mathfrak{Z}$, $\mathfrak{A} \ni X \sqsubseteq [A]$, $\mathfrak{A} \ni Y \sqsubseteq [B]$, $X \times_{A, B} Y \in \mathcal{C}(A, B)$ and morphism $f \in \mathcal{C}(A, B)$);
- 2°. $\iota_{A_1, B_1} (X \times_{A_0, B_0} Y) = X \times_{A_1, B_1} Y$ whenever $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$.

PROPOSITION 1526. The second axiom is equivalent to the following axiom:

- 1°. $f \sim X \times_{A_0, B_0} Y \Leftrightarrow f = X \times_{A_1, B_1} Y$ whenever $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$, $f : A_1 \rightarrow B_1$.

PROOF.

\Leftarrow . Obvious.

\Rightarrow . $f \sim X \times_{A_0, B_0} Y \Leftarrow f = X \times_{A_1, B_1} Y$ because $\iota_{A_1, B_1}(X \times_{A_0, B_0} Y) = X \times_{A_1, B_1} Y$ and $\iota_{A_0, B_0}(X \times_{A_1, B_1} Y) = X \times_{A_0, B_0} Y$.

Let's prove $f \sim X \times_{A_0, B_0} Y \Rightarrow f = X \times_{A_1, B_1} Y$. Really, if $f \sim X \times_{A_0, B_0} Y$ then $f = \iota_{A_1, B_1} f \sim \iota_{A_1, B_1}(X \times_{A_0, B_0} Y) = X \times_{A_1, B_1} Y$ and thus $f = X \times_{A_1, B_1} Y$. \square

PROPOSITION 1527. $[A] \times_{A, B} [B]$ is the greatest morphism $\top^{C(A, B)} : A \rightarrow B$.

PROOF. It's enough to prove $f \sqcap ([A] \times_{A, B} [B]) = f$ for every $f : A \rightarrow B$. Really, $f \sqcap ([A] \times_{A, B} [B]) = \text{id}_B^{C(B, B)} \circ f \circ \text{id}_A^{C(A, A)} = 1^B \circ f \circ 1^A = f$. \square

PROPOSITION 1528. For every category with binary product morphism

$$X \times_{A, B} Y = \text{id}_Y^{C(B, B)} \circ \top^{C(A, B)} \circ \text{id}_X^{C(A, A)}$$

PROOF. $X \times_{A, B} Y \sqsupseteq \text{id}_Y^{C(B, B)} \circ \top^{C(A, B)} \circ \text{id}_X^{C(A, A)}$ because $\text{id}_Y^{C(B, B)} \circ \top^{C(A, B)} \circ \text{id}_X^{C(A, A)} = \top^{C(A, B)} \sqcap (X \times_{A, B} Y)$.

$$\begin{aligned} \text{id}_Y^{C(B, B)} \circ \top^{C(A, B)} \circ \text{id}_X^{C(A, A)} &\sqsupseteq \\ &\text{id}_Y^{C(B, B)} \circ (X \times_{A, B} Y) \circ \text{id}_X^{C(A, A)} = \\ &(X \times_{A, B} Y) \sqcap (X \times_{A, B} Y) = X \times_{A, B} Y. \end{aligned}$$

\square

PROPOSITION 1529. $\iota_{A, B}(f \sqcap g) = \iota_{A, B} f \sqcap \iota_{A, B} g$ for every parallel morphisms f and g and objects A and B , whenever all $\mathcal{E}^{X, Y}$ are metamonovalued and metainjective.

PROOF.

$$\begin{aligned} \iota_{A, B}(f \sqcap g) &= \\ &\mathcal{E}^{\text{Dst } f, B} \circ (f \sqcap g) \circ \mathcal{E}^{A, \text{Src } f} = \\ &(\mathcal{E}^{\text{Dst } f, B} \circ f \circ \mathcal{E}^{A, \text{Src } f}) \sqcap (\mathcal{E}^{\text{Dst } f, B} \circ g \circ \mathcal{E}^{A, \text{Src } f}) = \\ &\iota_{A, B} f \sqcap \iota_{A, B} g. \end{aligned}$$

\square

PROPOSITION 1530. $(X_0 \times_{A, B} Y_0) \sqcap (X_1 \times_{A, B} Y_1) = (X_0 \sqcap X_1) \times_{A, B} (Y_0 \sqcap Y_1)$.

PROOF. $(X_0 \times_{A, B} Y_0) \sqcap (X_1 \times_{A, B} Y_1) = \text{id}_{Y_1}^{C(B, B)} \circ (X_0 \times_{A, B} Y_0) \circ \text{id}_{X_1}^{C(A, A)} = \text{id}_{Y_1}^{C(B, B)} \circ \text{id}_{Y_0}^{C(B, B)} \circ \top^{C(A, B)} \circ \text{id}_{X_1}^{C(A, A)} \circ \text{id}_{X_0}^{C(A, A)} = \text{id}_{Y_0 \sqcap Y_1}^{C(B, B)} \circ \top^{C(A, B)} \circ \text{id}_{X_0 \sqcap X_1}^{C(A, A)} = (X_0 \sqcap X_1) \times_{A, B} (Y_0 \sqcap Y_1)$. \square

PROPOSITION 1531. For a category with binary product morphism $\text{Im } f$, $\text{Dom } f$, $\text{IM } f$, and $\text{DOM } f$ are filters.

PROOF. That they are upper sets was proved above.

To prove that $\text{Im } f$ is a filter it remains to show $A, B \in \text{Im } f \Leftrightarrow A \sqcap B \in \text{Im } f$. Really,

$$A, B \in \text{Im } f \Leftrightarrow \top \times A \sqsupseteq f \wedge \top \times B \sqsupseteq f \Rightarrow \top \times (A \sqcap B) \sqsupseteq f \Leftrightarrow A \sqcap B \in \text{Im } f.$$

$\text{Dom } f$ is similar.

The thesis for $\text{IM } f$, $\text{DOM } f$ follows from above proved for $\text{Im } f$, $\text{Dom } f$. \square

NOTE 1532. For example for below defined category of functors (with binary product morphism), these filters are filters on filters on sets not filters of sets and thus are not the same as im and dom .

19.6. Operations on the set of unfixed morphisms

19.6.1. Semigroup of unfixed morphisms.

PROPOSITION 1533. Let $f : A_0 \rightarrow A_1$ and $g : A_1 \rightarrow A_2$ and $A_1 \sqsubseteq B_1$. Then $\iota_{B_0, B_2}(g \circ f) = \iota_{B_1, B_2}g \circ \iota_{B_0, B_1}f$.

PROOF.

$$\begin{aligned} \iota_{B_0, B_2}(g \circ f) &= \\ & \mathcal{E}_C^{A_2, B_2} \circ g \circ f \circ \mathcal{E}_C^{B_0, A_0} = \\ & \mathcal{E}_C^{A_2, B_2} \circ g \circ 1^{A_1} \circ f \circ \mathcal{E}_C^{B_0, A_0} = \\ & \mathcal{E}_C^{A_2, B_2} \circ g \circ \text{id}_{A_1}^{C(\text{Dst } f, \text{Src } g)} \circ f \circ \mathcal{E}_C^{B_0, A_0} = \\ & \mathcal{E}_C^{A_2, B_2} \circ g \circ \mathcal{E}^{B_1, A_1} \circ \mathcal{E}^{A_1, B_1} \circ f \circ \mathcal{E}_C^{B_0, A_0} = \\ & \iota_{B_1, B_2}g \circ \iota_{B_0, B_1}f. \end{aligned}$$

□

DEFINITION 1534. We will turn the category \mathcal{C} into a semigroup \mathcal{C}/\sim (*the semigroup of unfixed morphisms*) by taking the partition regarding the relation \sim and the formula for the composition $[g] \circ [f] = [g \circ f]$ whenever f and g are composable morphisms.

We need to prove that $[g] \circ [f]$ does not depend on choice of f and g (provided that f and g are composable). We also need to prove that $[g] \circ [f]$ is always defined for every morphisms (not necessarily composable) f and g . That the resulting structure is a semigroup (that is, \circ is associative) is then obvious.

PROOF. That $[g] \circ [f]$ is defined in at least one way for every morphisms f and g is simple to prove. Just consider the morphisms $f' = \iota_{\text{Src } f, \text{Dst } f} \sqcup \text{Src } g \circ f \sim f$ and $g' = \iota_{\text{Dst } f \sqcup \text{Src } g, \text{Dst } g} \circ g \sim g$. Then we can take $[g] \circ [f] = [g' \circ f']$.

It remains to prove that $[g] \circ [f]$ does not depend on choice of f and g . Really, take arbitrary composable pairs of morphisms $(f_0 : A_0 \rightarrow B_0, g_0 : B_0 \rightarrow C_0)$ and $(f_1 : A_1 \rightarrow B_1, g_1 : B_1 \rightarrow C_1)$ such that $f_0 \sim f_1$ and $g_0 \sim g_1$. It remains to prove that $g_0 \circ f_0 \sim g_1 \circ f_1$. We have

$$\begin{aligned} \iota_{B_0 \sqcup B_1, C_0 \sqcup C_1}g_0 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f_0 &= (\text{proposition 1533}) = \\ & \mathcal{E}_C^{C_0, C_0 \sqcup C_1} \circ g_0 \circ f_0 \circ \mathcal{E}_C^{A_0 \sqcup A_1, B_0} = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_0 \circ f_0). \end{aligned}$$

Similarly

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1}g_1 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f_1 = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_1 \circ f_1).$$

But

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1}g_0 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f_1 = \iota_{B_0 \sqcup B_1, C_0 \sqcup C_1}g_1 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1}f_0$$

thus having $\iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_0 \circ f_0) = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_1 \circ f_1)$ and so $g_0 \circ f_0 \sim g_1 \circ f_1$. □

19.6.2. Restricted identities.

DEFINITION 1535. *Restricted identity* for unfixed morphisms is defined as: $\text{id}_X = [\text{id}_X^{C(A,B)}]$ for an $X \sqsubseteq [A] \sqcap [B]$.

We need to prove that it does not depend on the choice of A and B .

PROOF. Let $\mathfrak{A} \ni X \sqsubseteq [A_0] \sqcap [B_0]$ and $\mathfrak{A} \ni X \sqsubseteq [A_1] \sqcap [B_1]$ for $A_0, B_0, A_1, B_1 \in \mathfrak{A}$.

3. We need to prove $\text{id}_X^{C(A_0, B_0)} \sim \text{id}_X^{C(A_1, B_1)}$.

Really,

$$\begin{aligned} \iota_{A_1, B_1} \text{id}_X^{C(A_0, B_0)} &= \\ & \mathcal{E}^{B_0, B_1} \circ \text{id}_X^{C(A_0, B_0)} \circ \mathcal{E}^{A_1, A_0} = \\ & \text{id}_{[B_0] \sqcap [B_1]}^{C(B_0, B_1)} \circ \text{id}_X^{C(A_0, B_0)} \circ \text{id}_{[A_0] \sqcap [A_1]}^{C(A_1, A_0)} = \\ & \text{id}_{[A_0] \sqcap [A_1] \sqcap [B_0] \sqcap [B_1] \sqcap X}^{C(A_1, B_1)} = \\ & \text{id}_X^{C(A_1, B_1)}. \end{aligned}$$

Similarly $\iota_{A_0, B_0} \text{id}_X^{C(A_1, B_1)} = \text{id}_X^{C(A_0, B_0)}$.

So $\text{id}_X^{C(A_0, B_0)} \sim \text{id}_X^{C(A_1, B_1)}$. \square

PROPOSITION 1536. $\text{id}_Y \circ \text{id}_X = \text{id}_{X \sqcap Y}$ for every $X, Y \in \mathfrak{A}$.

PROOF. Take arbitrary $\text{id}_X^{C(A, B_0)} \in \text{id}_X$ and $\text{id}_Y^{C(B_1, C)} \in \text{id}_Y$.

Obviously, $\text{id}_X^{C(A, B_0 \sqcup B_1)} \in \text{id}_X$ and $\text{id}_Y^{C(B_0 \sqcup B_1, C)} \in \text{id}_Y$. Thus $\text{id}_Y \circ \text{id}_X = [\text{id}_Y^{C(B_0 \sqcup B_1, C)}] \circ [\text{id}_X^{C(A, B_0 \sqcup B_1)}] = [\text{id}_{X \sqcap Y}^{C(A, C)}] = \text{id}_{X \sqcap Y}$. \square

19.6.3. Poset of unfixed morphisms.

LEMMA 1537. $f \sqsubseteq g \Rightarrow \iota_{A, B} f \sqsubseteq \iota_{A, B} g$ for every morphisms f and g such that $\text{Src } f = \text{Src } g$ and $\text{Dst } f = \text{Dst } g$.

PROOF.

$$\begin{aligned} \iota_{A, B} f \sqsubseteq \iota_{A, B} g &\Leftrightarrow \\ & \mathcal{E}^{\text{Dst } f, B} \circ f \circ \mathcal{E}^{A, \text{Src } f} \sqsubseteq \mathcal{E}^{\text{Dst } g, B} \circ g \circ \mathcal{E}^{A, \text{Src } g} \Leftrightarrow \\ & \text{id}_{[B] \sqcap [\text{Dst } f]}^{C(\text{Dst } f, B)} \circ f \circ \text{id}_{[A] \sqcap [\text{Src } f]}^{C(A, \text{Src } f)} \sqsubseteq \text{id}_{[B] \sqcap [\text{Dst } g]}^{C(\text{Dst } g, B)} \circ g \circ \text{id}_{[A] \sqcap [\text{Src } g]}^{C(A, \text{Src } g)} \Leftrightarrow \\ & f \sqsubseteq g \end{aligned}$$

because $\text{id}_{[B] \sqcap [\text{Dst } f]}^{C(\text{Dst } f, B)} = \text{id}_{[B] \sqcap [\text{Dst } g]}^{C(\text{Dst } g, B)}$ and $\text{id}_{[A] \sqcap [\text{Src } f]}^{C(A, \text{Src } f)} = \text{id}_{[A] \sqcap [\text{Src } g]}^{C(A, \text{Src } g)}$. \square

COROLLARY 1538.

- 1°. $f_0 \sqsubseteq g_0 \wedge f_0 \sim f_1 \wedge g_0 \sim g_1 \Rightarrow f_1 \sqsubseteq g_1$ whenever $\text{Src } f_0 = \text{Src } g_0$ and $\text{Dst } f_0 = \text{Dst } g_0$ and $\text{Src } f_1 = \text{Src } g_1$ and $\text{Dst } f_1 = \text{Dst } g_1$.
- 2°. $f_0 \sqsubseteq g_0 \Leftrightarrow f_1 \sqsubseteq g_1$ whenever $\text{Src } f_0 = \text{Src } g_0$ and $\text{Dst } f_0 = \text{Dst } g_0$ and $\text{Src } f_1 = \text{Src } g_1$ and $\text{Dst } f_1 = \text{Dst } g_1$ and $f_0 \sim f_1 \wedge g_0 \sim g_1$.

PROOF.

1°. Because $f_1 = \iota_{\text{Src } f_1, \text{Dst } f_1} f_0$ and $g_1 = \iota_{\text{Src } g_1, \text{Dst } g_1} g_0$.

2°. A consequence of the previous. \square

The above corollary warrants validity of the following definition:

DEFINITION 1539. The order on the set of unfixed morphisms is defined by the formula $[f] \sqsubseteq [g] \Leftrightarrow f \sqsubseteq g$ whenever $\text{Src } f = \text{Src } g \wedge \text{Dst } f = \text{Dst } g$.

It is really an order:

PROOF.

Reflexivity. Obvious.

Transitivity. Obvious.

Antisymmetry. Let $[f] \sqsubseteq [g]$ and $[g] \sqsubseteq [f]$ and $\text{Src } f = \text{Src } g \wedge \text{Dst } f = \text{Dst } g$. Then $f \sqsubseteq g$ and $g \sqsubseteq f$ and thus $f = g$ so having $[f] = [g]$.

□

OBVIOUS 1540. $f \mapsto [f]$ is an order embedding from the set $\mathcal{C}(A, B)$ to unfixed morphisms, for every objects A, B .

PROPOSITION 1541. If S is a set of parallel morphisms of a partially ordered category with an equivalence relation respecting the order, then

- 1°. $\prod_{X \in S} [X]$ exists and $\prod_{X \in S} [X] = [\prod S]$;
- 2°. $\bigsqcup_{X \in S} [X]$ exists and $\bigsqcup_{X \in S} [X] = [\bigsqcup S]$.

PROOF.

1°. $[\prod S] \sqsubseteq [X]$ for every $X \in S$ because $\prod S \sqsubseteq X$.

Let now $L \sqsubseteq [X]$ for every $X \in S$ for an equivalence class L . Then $L \sqsubseteq [\prod S]$ because $l \sqsubseteq \prod S$ for $l \in L$ because $l \sqsubseteq X$ for every $X \in S$.

Thus $[\prod S]$ is the greatest lower bound of $\left\{ \frac{[X]}{X \in S} \right\}$.

2°. By duality.

□

PROPOSITION 1542.

- 1°. If every Hom-set is a join-semilattice, then the poset of unfixed morphism is a join-semilattice.
- 2°. If every Hom-set is a join-semilattice, then the poset of unfixed morphism is a meet-semilattice.

PROOF. Let f and g be arbitrary morphisms.

$$\begin{aligned} [f] \sqcup [g] &= [\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} f] \sqcup [\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} g] = \\ & \text{(obvious 1540)} = [\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} f \sqcup \iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} g] \end{aligned}$$

and

$$\begin{aligned} [f] \sqcap [g] &= [\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} f] \sqcap [\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} g] = \\ & \text{(obvious 1540)} = [\iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} f \sqcap \iota_{\text{Src } f \sqcup \text{Src } g, \text{Dst } f \sqcup \text{Dst } g} g]. \end{aligned}$$

□

COROLLARY 1543. If every Hom-set is a lattice, then the poset of unfixed morphisms is a lattice.

THEOREM 1544. Meet of nonempty set of unfixed morphisms exists provided that the orders of Hom-sets are posets, every nonempty subset of which has a meet, and our category is with ordered domain and image and that morphisms \mathcal{E} are metamonovaled and metainjective.

PROOF. Let S be a nonempty set of unfixed morphisms. Take an arbitrary unfixed morphism $f \in S$. Take an arbitrary $F \in f$. Let $A = \text{Src } F$ and $B = \text{Dst } F$.

$$\begin{aligned} \sqcap S &= \sqcap \langle f \sqcap \rangle^* S = \sqcap \langle [F] \sqcap \rangle^* S = \sqcap \left\{ \frac{[F] \sqcap [G]}{g \in S, G \in g} \right\} = \\ &= \sqcap \left\{ \frac{[\iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F \sqcap \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} G]}{g \in S, G \in g} \right\}. \end{aligned}$$

We will prove $\iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F \sqcap \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} G \sim F \sqcap \iota_{A, B} G$.

$\iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F \sqcap \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} G \sqsubseteq \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F$ and $\iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} \iota_{A, B} \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F = \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F$, thus by being with ordered domain and image

$$\begin{aligned} \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F \sqcap \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} G &= \\ \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} \iota_{A, B} (\iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F \sqcap \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} G) &= \\ \text{(by being metamonovaled and metainjective)} &= \\ \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} (\iota_{A, B} \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} F \sqcap \iota_{A, B} \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} G) &= \\ \iota_{A \sqcup \text{Src } G, B \sqcup \text{Dst } G} (\iota_{A, B} F \sqcap \iota_{A, B} G) \sim \iota_{A, B} F \sqcap \iota_{A, B} G &= F \sqcap \iota_{A, B} G. \end{aligned}$$

Due the proved equivalence we have $\sqcap S = \sqcap \left\{ \frac{[F \sqcap \iota_{A, B} G]}{g \in S, G \in g} \right\}$. Now we can apply proposition 1541: $\sqcap S = \left[\sqcap \left\{ \frac{F \sqcap \iota_{A, B} G}{g \in S, G \in g} \right\} \right]$. We have provided an explicit formula for $\sqcap S$. \square

The poset of unfixed morphisms may be not a complete lattice even if every Hom-set is a complete lattice. We will show this below for funcoids.

19.6.4. Domain and image of unfixed morphisms.

PROPOSITION 1545. $\text{IM } f = \left\{ \frac{Y \in \mathfrak{Z}}{\text{id}_Y \circ [f] = [f]} \right\}$; $\text{DOM } f = \left\{ \frac{X \in \mathfrak{Z}}{[f] \circ \text{id}_X = [f]} \right\}$.

PROOF. We will prove only the first, as the second is similar.

$$\begin{aligned} \text{id}_Y \circ [f] = [f] &\Leftrightarrow \\ \text{id}_Y^{\mathcal{C}(\text{Y} \sqcup \text{Dst } f, \text{Y} \sqcup \text{Dst } f)} \circ \mathcal{E}^{\text{Dst } f, \text{Y} \sqcup \text{Dst } f} \circ f &= \mathcal{E}^{\text{Dst } f, \text{Y} \sqcup \text{Dst } f} \circ f \Leftrightarrow \\ \text{id}_{[Y] \sqcap [\text{Dst } f]}^{\mathcal{C}(\text{Dst } f, \text{Y} \sqcup \text{Dst } f)} \circ f &= \mathcal{E}^{\text{Dst } f, \text{Y} \sqcup \text{Dst } f} \circ f \Leftrightarrow \\ \mathcal{E}^{\text{Y} \sqcup \text{Dst } f, \text{Dst } f} \circ \text{id}_{[Y] \sqcap [\text{Dst } f]}^{\mathcal{C}(\text{Dst } f, \text{Y} \sqcup \text{Dst } f)} \circ f &= f \Leftrightarrow \\ \text{id}_{[Y] \sqcap [\text{Dst } f]}^{\mathcal{C}(\text{Dst } f, \text{Dst } f)} \circ f &= f \text{Y} \in \text{IM } f. \end{aligned}$$

\square

The above proposition allows to define:

DEFINITION 1546. $\text{DOM } f = \text{DOM } F$ and $\text{IM } f = \text{IM } F$ for $F \in f$.

DEFINITION 1547. $\text{dom } f = \min \text{DOM } f$ and $\text{im } f = \min \text{IM } f$ for an unfixed morphism f .

NOTE 1548. $\text{dom } f$ and $\text{im } f$ are not always defined.

19.6.5. Rectangular restriction.

PROPOSITION 1549. $\iota_{A, B} f = \iota_{A, B} g$ if $f \sim g$.

PROOF. Let $f \sim g$. Then $g = \iota_{\text{Src } g, \text{Dst } g} f$. So $\iota_{A, B} g = \iota_{A, B} \iota_{\text{Src } g, \text{Dst } g} f \sqsubseteq$ (proposition 1524) $\sqsubseteq \iota_{A, B} f$. Similarly, $\iota_{A, B} f \sqsubseteq \iota_{A, B} g$. So $\iota_{A, B} f = \iota_{A, B} g$. \square

The above proposition allows to define:

DEFINITION 1550. $\iota_{A,B}F = \iota_{A,B}f$ for an unfixed morphism F and arbitrary $f \in F$.

DEFINITION 1551. $F \square_{A,B} = [\iota_{A,B}F]$ for every unfixed morphism F .

PROPOSITION 1552. $F \square_{A,B} = \text{id}_B \circ F \circ \text{id}_A$ for every unfixed morphism F and objects A and B .

PROOF. Take $f \in F$. $F \square_{A,B} = [\iota_{A,B}F] = [\iota_{A,B}f] = [\mathcal{E}^{\text{Dst } f, B} \circ f \circ \mathcal{E}^{A, \text{Src } f}] = [\text{id}_B^{\mathcal{C}(\text{Dst } f, B)} \circ f \circ \text{id}_A^{\mathcal{C}(A, \text{Src } f)}] = [\text{id}_B^{\mathcal{C}(\text{Dst } f, B)} \circ \text{id}_B^{\mathcal{C}(\text{Dst } f, \text{Dst } f)} \circ f \circ \text{id}_A^{\mathcal{C}(\text{Src } f, \text{Src } f)} \circ \text{id}_A^{\mathcal{C}(A, \text{Src } f)}] = [\text{id}_B^{\mathcal{C}(\text{Dst } f, B)} \circ f \circ \text{id}_A^{\mathcal{C}(A, \text{Src } f)}] = [\text{id}_B^{\mathcal{C}(\text{Dst } f, B)}] \circ [f] \circ [\text{id}_A^{\mathcal{C}(A, \text{Src } f)}] = \text{id}_B \circ F \circ \text{id}_A. \quad \square$

PROPOSITION 1553. $f \square_{A_0, B_0} \square_{A_1, B_1} = f \square_{A_0 \cap A_1, A_1 \cap B_1}$.

PROOF. From the previous $f \square_{A_0, B_0} \square_{A_1, B_1} = \text{id}_{B_1} \circ \text{id}_{B_0} \circ f \circ \text{id}_{A_0} \circ \text{id}_{A_1} = \text{id}_{B_0 \cap B_1} \circ f \circ \text{id}_{A_0 \cap A_1} = f \square_{A_0 \cap A_1, A_1 \cap B_1}$. \square

DEFINITION 1554. $f|_X = f \circ \text{id}_X$ for every unfixed morphism f and $X \in \mathfrak{A}$.

OBVIOUS 1555. $(f|_X)|_Y = f_{X \cap Y}$.

19.6.6. Algebraic properties of the lattice of unfixed morphisms. The following proposition allows to easily prove algebraic properties (cf. distributivity) of the poset of unfixed morphisms:

THEOREM 1556. The following are mutually inverse bijections:

- 1°. Let A and B be objects. $f \mapsto [f]$ and $F \mapsto \iota_{A,B}F$ are mutually inverse order isomorphisms between $\left\{ \frac{f \in \text{unfixed morphisms}}{A \in \text{DOM } f, B \in \text{IM } f} \right\}$ and $\mathcal{C}(A, B)$. If $A = B$ they are also semigroup isomorphisms.
- 2°. Let T be an unfixed morphism. $f \mapsto [f]$ and $F \mapsto \iota_{\text{Src } t, \text{Dst } t}F$ are mutually inverse order isomorphisms between the lattice DT and Dt whenever $t \in T$.

PROOF. We will prove that these functions are mutually inverse bijections. That they are order-preserving is obvious.

1°. $\iota_{A,B}F \in \mathcal{C}(A, B)$ is obvious.

We need to prove that $[f] \in \left\{ \frac{f \in \text{unfixed morphisms}}{A \in \text{DOM } f, B \in \text{IM } f} \right\}$. For this it's enough to prove $A \in \text{DOM}[f] \wedge B \in \text{IM}[f]$ what is the same as $A \in \text{DOM } f \wedge B \in \text{IM } f$ what follows from proposition 1505.

Because $f \mapsto [f]$ is an injection, it is enough¹ to prove that $\iota_{A,B}[f] = f$. Really, $\iota_{A,B}[f] = \iota_{A,B}f = f$.

That they are semigroup isomorphisms follows from the already proved formula $[g \circ f] = [g] \circ [f]$.

2°. Because of the previous, it is enough to prove that $[f] \in DT \Leftrightarrow f \in Dt$. Really, it is equivalent to $[f] \sqsubseteq T \Leftrightarrow f \sqsubseteq t$ what is obvious. \square

PROPOSITION 1557. If every Hom-set is a distributive lattice, then the poset of unfixed morphisms is a distributive lattice.

PROOF. It follows from the above isomorphism. \square

PROPOSITION 1558. If every Hom-set is a co-brouwerian lattice, then the poset of unfixed morphisms is a co-brouwerian lattice.

¹<https://math.stackexchange.com/a/3007051/4876>

PROOF. It follows from the above isomorphism and the definition of pseudodifference. \square

PROPOSITION 1559. If every Hom-set is a lattice with quasidifference, then the poset of unfixed morphisms is a lattice with quasidifference.

PROOF. It follows from the above isomorphism and the definition of quasidifference. \square

PROPOSITION 1560.

1°. If every Hom-set is an atomic lattice, then the poset of unfixed morphisms is an atomic lattice.

2°. If every Hom-set is an atomistic lattice, then the poset of unfixed morphisms is an atomistic lattice.

PROOF. Follows from the above isomorphism. \square

19.6.7. Binary product morphism.

DEFINITION 1561. For a category \mathcal{C} with binary product morphism and $X, Y \in \mathfrak{A}$ define $X \times Y = [X \times_{A,B} Y]$ where $A \in \mathfrak{Z}$, $[A] \sqsupseteq X$, $B \in \mathfrak{Z}$, $[B] \sqsupseteq Y$. (Such A and B exist by an axiom of categories with restricted identities.)

We need to prove validity of this definition:

PROOF. Let $A_0 \in \mathfrak{Z}$, $[A_0] \sqsupseteq X$, $B_0 \in \mathfrak{Z}$, $[B_0] \sqsupseteq Y$, $A_1 \in \mathfrak{Z}$, $[A_1] \sqsupseteq X$, $B_1 \in \mathfrak{Z}$, $[B_1] \sqsupseteq Y$. We need to prove $X \times_{A_0, B_0} Y \sim X \times_{A_1, B_1} Y$, but it trivially follows from an axiom in the definition of category with binary product morphism. \square

PROPOSITION 1562. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$ for every $X_0, X_1, Y_0, Y_1 \in \mathfrak{A}$.

PROOF. Take $A_0 \in \mathfrak{Z}$, $[A_0] \sqsupseteq X_0$, $B_0 \in \mathfrak{Z}$, $[B_0] \sqsupseteq Y_0$, $A_1 \in \mathfrak{Z}$, $[A_1] \sqsupseteq X_1$, $B_1 \in \mathfrak{Z}$, $[B_1] \sqsupseteq Y_1$.

Then

$$\begin{aligned} (X_0 \times Y_0) \sqcap (X_1 \times Y_1) &= \\ &= [X_0 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_0] \sqcap [X_1 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_1] = \\ &= [(X_0 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_0) \sqcap (X_1 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_1)] = \\ &= [(X_0 \sqcap X_1) \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} (Y_0 \sqcap Y_1)] = \\ &= (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1). \end{aligned}$$

\square

PROPOSITION 1563. $f \sqcap_{A,B} = f \sqcap (A \times B)$.

PROOF. Take $F \in f$. Let $F' = \iota_{A \sqcup \text{Src } F, B \sqcup \text{Dst } F} F$. We have $F' \in f$.

$$\begin{aligned} f \sqcap_{A,B} &= [\iota_{A,B} F'] = \\ &= [\mathcal{E}^{B \sqcup \text{Dst } F, B} \circ F' \circ \mathcal{E}^{A, A \sqcup \text{Src } F}] = \\ &= [\text{id}_{[B]}^{\mathcal{C}(B \sqcup \text{Dst } F, B)} \circ F' \circ \text{id}_{[A]}^{\mathcal{C}(A, A \sqcup \text{Src } F)}] = \\ &= [\text{id}_{[B]}^{\mathcal{C}(B \sqcup \text{Dst } F, B)}] \circ [F'] \circ [\text{id}_{[A]}^{\mathcal{C}(A, A \sqcup \text{Src } F)}] = \\ &= [\text{id}_{[B]}^{\mathcal{C}(B \sqcup \text{Dst } F, B \sqcup \text{Dst } F)}] \circ [F'] \circ [\text{id}_{[A]}^{\mathcal{C}(A \sqcup \text{Src } F, A \sqcup \text{Src } F)}] = \\ &= [\text{id}_{[B]}^{\mathcal{C}(B \sqcup \text{Dst } F, B \sqcup \text{Dst } F)} \circ F' \circ \text{id}_{[A]}^{\mathcal{C}(A \sqcup \text{Src } F, A \sqcup \text{Src } F)}] = \\ &= [F' \sqcap (A \times_{A \sqcup \text{Src } F, B \sqcup \text{Dst } F} B)] = \\ &= [F'] \sqcap [A \times_{A \sqcup \text{Src } F, B \sqcup \text{Dst } F} B] = f \sqcap (A \times B). \end{aligned}$$

□

19.7. Examples of categories with restricted identities

19.7.1. Category Rel. Category **Rel** of relations between small sets can be considered as a category with restricted identities with $\mathfrak{J} = \mathfrak{A}$ being the set of all small sets, projection being the identity function and restricted identity being the identity relation between the given sets.

Moreover it is a category with binary product morphism with usual Cartesian product.

Proofs of this are trivial.

19.7.2. Category FCD. Category **FCD** can be considered as a category with restricted identities with \mathfrak{J} being the set of all small sets, \mathfrak{A} is the set of unfixed filters, projection being the projection function for the equivalence classes of filters, restricted identity being defined by the formulas

$$\begin{aligned} \langle \text{id}_{\mathcal{F}^{\text{FCD}(A,B)}} \rangle \mathcal{X} &= ([\mathcal{X}] \sqcap \mathcal{F}) \div B; \\ \langle (\text{id}_{\mathcal{F}^{\text{FCD}(A,B)}})^{-1} \rangle \mathcal{Y} &= ([\mathcal{Y}] \sqcap \mathcal{F}) \div A \end{aligned}$$

(whenever $\mathcal{F} \sqsubseteq [A] \sqcap [B]$).

We need to prove that this really defines a functor.

PROOF.

$$\begin{aligned} \mathcal{Y} \neq \langle \text{id}_{\mathcal{F}^{\text{FCD}(A,B)}} \rangle \mathcal{X} &\Leftrightarrow \\ \mathcal{Y} \neq ([\mathcal{X}] \sqcap \mathcal{F}) \div B &\Leftrightarrow \mathcal{Y} \neq (\mathcal{X} \div B) \sqcap (\mathcal{F} \div B) \Leftrightarrow \\ &[\mathcal{Y}] \neq [\mathcal{X}] \sqcap \mathcal{F}. \end{aligned}$$

Similarly $\langle (\text{id}_{\mathcal{F}^{\text{FCD}(A,B)}})^{-1} \rangle \mathcal{Y} \Leftrightarrow [\mathcal{X}] \neq [\mathcal{Y}] \sqcap \mathcal{F}$.

Thus $\mathcal{Y} \neq \langle \text{id}_{\mathcal{F}^{\text{FCD}(A,B)}} \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \neq \langle (\text{id}_{\mathcal{F}^{\text{FCD}(A,B)}})^{-1} \rangle \mathcal{Y}$. □

We need to prove that the restricted identities conform to the axioms:

PROOF. The first five **axioms** are obvious. Let's prove the remaining ones:

$$\begin{aligned} \text{id}_{[A]}^{\text{FCD}(A,A)} &= 1_A^{\text{FCD}} \text{ because } \langle \text{id}_{[A]}^{\text{FCD}(A,A)} \rangle \mathcal{X} = ([\mathcal{X}] \sqcap [A]) \div A = [\mathcal{X}] \div A = \mathcal{X}. \\ \text{id}_Y^{\text{FCD}(B,C)} \circ \text{id}_X^{\text{FCD}(A,B)} &= \text{id}_{X \sqcap Y}^{\text{FCD}(A,C)} \text{ because } \langle \text{id}_Y^{\text{FCD}(B,C)} \circ \text{id}_X^{\text{FCD}(A,B)} \rangle \mathcal{X} = \\ \langle \text{id}_Y^{\text{FCD}(B,C)} \rangle \langle \text{id}_X^{\text{FCD}(A,B)} \rangle \mathcal{X} &= \langle \text{id}_Y^{\text{FCD}(B,C)} \rangle (([\mathcal{X}] \sqcap X) \div B) = ((([\mathcal{X}] \sqcap X) \div B) \sqcap Y) \div \\ C &= ((([\mathcal{X}] \sqcap X \sqcap Y) \div B) \div C = (\text{because } [\mathcal{X}] \sqcap X \sqcap Y \sqsubseteq [B]) = ([\mathcal{X}] \sqcap X \sqcap Y) \div C = \\ \langle \text{id}_{X \sqcap Y}^{\text{FCD}(A,C)} \rangle \mathcal{X}. \\ \forall A \in \mathfrak{A} \exists B \in \mathfrak{J} : A \sqsubseteq [B] &\text{ is obvious. } \quad \square \end{aligned}$$

PROPOSITION 1564. $\mathcal{E}_{\text{FCD}}^{A,B} = (A, B, \lambda \mathcal{X} \in \mathfrak{F}(A) : \mathcal{X} \div B, \lambda \mathcal{Y} \in \mathfrak{F}(B) : \mathcal{Y} \div A)$ for objects $A \subseteq B$ of **FCD**.

PROOF. Take $\mathcal{F} = [A] \sqcap [B]$. Then $\mathcal{F} \sqsupseteq [\mathcal{X}]$ and $\mathcal{F} \sqsupseteq [\mathcal{Y}]$, thus $[\mathcal{X}] \sqcap \mathcal{F} = [\mathcal{X}]$ and $[\mathcal{Y}] \sqcap \mathcal{F} = [\mathcal{Y}]$. So, it follows from the above. □

PROPOSITION 1565. $\text{id}_X^{\text{FCD}(A,A)} = \text{id}_{X \div A}^{\text{FCD}}$ whenever $A \in \mathfrak{J}$ and $\mathfrak{A} \ni X \sqsubseteq [A]$.

PROOF. $\langle \text{id}_X^{\text{FCD}(A,A)} \rangle \mathcal{X} = ([\mathcal{X}] \sqcap X) \div A = ([\mathcal{X}] \div A) \sqcap (X \div A) = \mathcal{X} \sqcap (X \div A) = \langle \text{id}_{X \div A}^{\text{FCD}} \rangle \mathcal{X}$ (used bijections for unfixed filters) for every $\mathcal{X} \in \mathfrak{F}(A)$. □

DEFINITION 1566. Category FCD can be considered as a category with binary product morphism with the binary product defined as: $\mathcal{X} \times_{A,B} \mathcal{Y} = (\mathcal{X} \div A) \times^{\text{FCD}} (\mathcal{Y} \div B)$ for every unfixed filters \mathcal{X} and \mathcal{Y} .

It is really a binary product morphism:

PROOF. Need to prove the axioms:

$$1^\circ. f \sqcap (X \times_{A,B} Y) = f \sqcap ((X \div A) \times^{\text{FCD}} (Y \div B)) = \text{id}_{Y \div B}^{\text{FCD}} \circ f \circ \text{id}_{X \div A}^{\text{FCD}} = \text{id}_Y^{\text{FCD}(B,B)} \circ f \circ \text{id}_X^{\text{FCD}(A,A)}.$$

$$2^\circ. \text{ Let unfixed filters } X \sqsubseteq [A_0] \sqcap [A_1] \text{ and } Y \sqsubseteq [B_0] \sqcap [B_1]. \text{ Then for } \mathcal{X} \in \mathcal{F}(A_1) \text{ we have } \langle \iota_{A_1, B_1}(X \times_{A_0, B_0} Y) \rangle \mathcal{X} = \langle \mathcal{E}^{\text{FCD}(B_0, B_1)} \rangle (X \times_{A_0, B_0} Y) \langle \mathcal{E}^{\text{FCD}(A_1, A_0)} \rangle \mathcal{X} = (\langle X \times_{A_0, B_0} Y \rangle (\mathcal{X} \div A_0)) \div B_1 = (\langle (X \div A_0) \times^{\text{FCD}} (Y \div B_0) \rangle (\mathcal{X} \div A_0)) \div B_1.$$

On the other hand, $\langle X \times_{A_1, B_1} Y \rangle \mathcal{X} = \langle (X \div A_1) \times^{\text{FCD}} (Y \div B_1) \rangle \mathcal{X}$

If $[\mathcal{X}] \simeq X$ then (use isomorphisms) $\mathcal{X} \simeq X \div A_1$ and $\mathcal{X} \div A_0 \simeq X \div A_0$. So $\langle \iota_{A_1, B_1}(X \times_{A_0, B_0} Y) \rangle \mathcal{X} = \perp$ and $\langle X \times_{A_1, B_1} Y \rangle \mathcal{X} = \perp$.

If $[\mathcal{X}] \not\simeq X$ then (use isomorphisms) $\mathcal{X} \not\simeq X \div A_1$ and $\mathcal{X} \div A_0 \not\simeq X \div A_0$. So $\langle \iota_{A_1, B_1}(X \times_{A_0, B_0} Y) \rangle \mathcal{X} = (Y \div B_0) \div B_1 = Y \div B_1$ and $\langle X \times_{A_1, B_1} Y \rangle \mathcal{X} = Y \div B_1$.

So in all cases, $\langle \iota_{A_1, B_1}(X \times_{A_0, B_0} Y) \rangle \mathcal{X} = \langle X \times_{A_1, B_1} Y \rangle \mathcal{X}$.

□

LEMMA 1567. $\mathcal{X} \div A = (\mathcal{X} \sqcap [A]) \div A$ for every unfixed filter \mathcal{X} and small set A .

PROOF. $(\mathcal{X} \sqcap [A]) \div A = (\mathcal{X} \div A) \sqcap ([A] \div A) = (\mathcal{X} \div A) \sqcap \top^{\mathcal{F}(A)} = \mathcal{X} \div A$. □

COROLLARY 1568. There is a pointfree funcoid p such that $\langle p \rangle \mathcal{X} = \mathcal{X} \div A$.

PROOF. Let q be the order embedding (see the diagram) from unfixed filters \mathcal{F} such that $A \in \mathcal{F}$ to filters on A .

$$\text{Then } \langle \mathcal{X} \div A \rangle \mathcal{X} = \langle (\mathcal{X} \sqcap [A]) \div A \rangle \mathcal{X} = \langle q \circ \text{id}_{[A]}^{\text{pFCD}(\text{unfixed filters})} \rangle \mathcal{X}. \quad \square$$

Let f be a funcoid. Define pointfree funcoid $\mathcal{S}f$ between unfixed filters as:

DEFINITION 1569. For every unfixed filters \mathcal{X} and \mathcal{Y}

$$\langle \mathcal{S}f \rangle \mathcal{X} = [\langle f \rangle (\mathcal{X} \div \text{Src } f)]; \quad \langle (\mathcal{S}f)^{-1} \rangle \mathcal{Y} = [\langle f^{-1} \rangle (\mathcal{Y} \div \text{Dst } f)].$$

It is really a pointfree funcoid:

PROOF. For an unfixed filter \mathcal{Y} we have

$$\mathcal{Y} \not\simeq \langle \mathcal{S}f \rangle \mathcal{X} \Leftrightarrow$$

$$\mathcal{Y} \not\simeq [\langle f \rangle (\mathcal{X} \div \text{Src } f)] \Leftrightarrow$$

$$\mathcal{Y} \div \text{Dst } f \not\simeq \langle f \rangle (\mathcal{X} \div \text{Src } f) \Leftrightarrow$$

$$\mathcal{X} \div \text{Src } f \not\simeq \langle f^{-1} \rangle (\mathcal{Y} \div \text{Dst } f) \Leftrightarrow$$

$$\mathcal{X} \not\simeq [\langle f^{-1} \rangle (\mathcal{Y} \div \text{Dst } f)] \Leftrightarrow$$

$$\mathcal{X} \not\simeq \langle (\mathcal{S}f)^{-1} \rangle \mathcal{Y}.$$

□

DEFINITION 1570. $\mathcal{S}F = \mathcal{S}f$ for an unfixed funcoid F and $f \in F$.

We need to prove validity of the above definition:

PROOF. Let $f, g \in F$, let $f : A_0 \rightarrow B_0$, $g : A_1 \rightarrow B_1$. Need to prove $\mathcal{S}f = \mathcal{S}g$. We have

$$\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g.$$

$$\begin{aligned}
\langle \mathcal{S} \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f \rangle \mathcal{X} &= \\
&= [\langle \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f \rangle (\mathcal{X} \div (A_0 \sqcup A_1))] = \\
&= \left[\left\langle \mathcal{E}^{\text{FCD}(B_0, B_0 \sqcup B_1)} \right\rangle \langle f \rangle \left\langle \mathcal{E}^{\text{FCD}(A_0 \sqcup A_1, A_0)} \right\rangle (\mathcal{X} \div (A_0 \sqcup A_1)) \right] = \\
&= [\langle f \rangle ((\mathcal{X} \div (A_0 \sqcup A_1)) \div A_0) \div B_0] = \\
&= [\langle f \rangle (\mathcal{X} \div A_0)] = \\
&= \langle \mathcal{S} f \rangle \mathcal{X}.
\end{aligned}$$

Similarly $\langle \mathcal{S} \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g \rangle \mathcal{X} = \langle \mathcal{S} g \rangle \mathcal{X}$.
So $\langle \mathcal{S} f \rangle \mathcal{X} = \langle \mathcal{S} g \rangle \mathcal{X}$. \square

DEFINITION 1571. So, we can define $\langle f \rangle \mathcal{X} = \langle \mathcal{S} f \rangle \mathcal{X}$ for every unfixed funcoid f and an unfixed filter \mathcal{X} .

PROPOSITION 1572.

- 1°. \mathcal{S} from a Hom-set $\text{FCD}(A, B)$ is an order embedding.
- 2°. \mathcal{S} from the category FCD is a prefunctor.
- 3°. \mathcal{S} from unfixed funcoids is an order embedding and a prefunctor (= semi-group homomorphism).

PROOF.

1°. $(\langle \mathcal{S} f \rangle \mathcal{X}) \div \text{Dst } f = \langle f \rangle \mathcal{X}$. Thus for different f we have different $\mathcal{X} \mapsto \langle \mathcal{S} f \rangle \mathcal{X}$. So it is an injection. That it is a monotone function is obvious.

2°. $\langle \mathcal{S} g \circ \mathcal{S} f \rangle \mathcal{X} = \langle \mathcal{S} g \rangle \langle \mathcal{S} f \rangle \mathcal{X} = \langle \mathcal{S} g \rangle [\langle f \rangle (\mathcal{X} \div \text{Src } f)] = [\langle g \rangle (\langle f \rangle (\mathcal{X} \div \text{Src } f) \div \text{Src } g)] = [\langle g \rangle (\langle f \rangle (\mathcal{X} \div \text{Src } f) \div \text{Src } g)] = [\langle g \rangle \langle f \rangle (\mathcal{X} \div \text{Src } f)] = [\langle g \circ f \rangle (\mathcal{X} \div \text{Src } f)] = \langle \mathcal{S} (g \circ f) \rangle \mathcal{X}$ for every composable funcoids f and g and an unfixed filter \mathcal{X} . Thus $\mathcal{S} g \circ \mathcal{S} f = \mathcal{S} (g \circ f)$.

3°. To prove that it is an order embedding, it is enough to show that $f \approx g$ implies $\mathcal{S} f \neq \mathcal{S} g$ (monotonicity is obvious). Let $f \approx g$ that is $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f \neq \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$. Then there exist filter $\mathcal{X} \in \mathfrak{F}(A_0 \sqcup A_1)$ such that $\langle \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f \rangle \mathcal{X} \neq \langle \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g \rangle \mathcal{X}$.

Consequently, $\langle \mathcal{S} f \rangle \mathcal{X} = \langle \mathcal{S} \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f \rangle \mathcal{X} \neq \langle \mathcal{S} \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g \rangle \mathcal{X} = \langle \mathcal{S} g \rangle \mathcal{X}$.

It remains to prove that $\mathcal{S} G \circ \mathcal{S} F = \mathcal{S} (G \circ F)$ but it is equivalent to $\mathcal{S} g \circ \mathcal{S} f = \mathcal{S} (g \circ f)$ for arbitrarily taken $f \in F$ and $g \in G$, what is already proved above. \square

LEMMA 1573. For every meet-semilattice $a \not\leq b$ and $c \sqsupseteq b$ implies $a \sqcap c \not\leq b$.

PROOF. Suppose $a \not\leq b$. Then there is a non-least x such that $x \sqsubseteq a, b$. Thus $x \sqsubseteq c$, so $x \sqsubseteq a \sqcap c$. We have $a \sqcap c \not\leq b$. \square

PROPOSITION 1574. $\mathcal{S}(X \times Y) = X \times^{\text{pFCD}(\mathfrak{F}(U))} Y$ for every unfixed filters X and Y .

PROOF. $\mathcal{S}(X \times Y) = \mathcal{S}(X \times_{A, B} Y)$ for arbitrary filters A , and B such that $X \sqsubseteq [A]$ and $Y \sqsubseteq [B]$. So for every unfixed filter \mathcal{X} we have

$$\begin{aligned}
\langle \mathcal{S}(X \times Y) \rangle \mathcal{X} &= \langle \mathcal{S}(X \times_{A, B} Y) \rangle \mathcal{X} = \\
&= [\langle X \times_{A, B} Y \rangle (\mathcal{X} \div A)] = [\langle (X \div A) \times^{\text{FCD}} (Y \div B) \rangle (\mathcal{X} \div A)].
\end{aligned}$$

Thus if $\mathcal{P} \not\leq X$ then (by the lemma) $\mathcal{P} \sqcap A \not\leq X$; $\mathcal{P} \div A \not\leq X \div A$; $\langle \mathcal{S}(X \times Y) \rangle \mathcal{X} = [Y \div B] = Y$.

if $\mathcal{P} \asymp X$ then $\mathcal{P} \sqcap A \asymp X$; $\mathcal{P} \div A \asymp X \div A$; $\langle \mathcal{S}(X \times Y) \rangle \mathcal{X} = [\perp] = \perp$.

So $\mathcal{S}(X \times Y) = X \times^{\text{pFCD}(\mathfrak{F}(\mathcal{U}))} Y$. \square

PROPOSITION 1575. $\mathcal{S} \text{id}_X = \text{id}_X^{\text{pFCD}(\mathfrak{F}(\mathcal{U}))}$ for every unfixed filter X .

PROOF. For every unfixed filter \mathcal{X} we for arbitrary filters A and B such that $X \sqsubseteq [A] \sqcap [B]$ have

$$\begin{aligned} \langle \mathcal{S} \text{id}_X \rangle \mathcal{X} &= \langle \mathcal{S} [\text{id}_X^{\text{FCD}(A,B)}] \rangle \mathcal{X} = \langle \mathcal{S} \text{id}_X^{\text{FCD}(A,B)} \rangle \mathcal{X} = \\ &= \left[\langle \text{id}_X^{\text{FCD}(A,B)} \rangle (\mathcal{X} \div A) \right] = [([\mathcal{X} \div A] \sqcap X) \div B] = \\ &= [(\mathcal{X} \sqcap X) \div B] = \mathcal{X} \sqcap X. \end{aligned}$$

Thus $\mathcal{S} \text{id}_X = \text{id}_X^{\text{pFCD}(\mathfrak{F}(\mathcal{U}))}$. \square

19.7.3. Category RLD.

DEFINITION 1576. $f \div D = (A, B, (\text{GR } f) \div D)$ for every reloid f and a binary relation D .

Category RLD can be considered as a category with restricted identities with \mathfrak{J} being the set of all small sets, \mathfrak{A} is the set of unfixed filters, projection being the projection function for the equivalence classes of filters, restricted identity being defined by the formula

$$\text{id}_{\mathcal{F}}^{\text{RLD}(A,B)} = \text{id}_{\mathcal{F} \div (A \sqcap B)}^{\text{RLD}} \div (A \times B).$$

We need to prove that the restricted identities conform to the axioms:

PROOF. The first five **axioms** are obvious. Let's prove the remaining ones:

$$\text{id}_{[A]}^{\text{RLD}(A,A)} = \text{id}_{[A] \div A}^{\text{RLD}} \div (A \times A) = \text{id}_A^{\text{RLD}} \div (A \times A) = 1_A^{\text{RLD}}.$$

$$\begin{aligned} \text{id}_Y^{C(B,C)} \circ \text{id}_X^{C(A,B)} &= \prod_{x \in \text{up } X, y \in \text{up } Y} (\text{id}_y^{C(B,C)} \circ \text{id}_x^{C(A,B)}) = \\ &= \prod_{x \in \text{up } X, y \in \text{up } Y} \text{id}_{x \sqcap y}^{C(A,B)} = \text{id}_{X \sqcap Y}^{C(A,B)}. \end{aligned}$$

$\forall A \in \mathfrak{A} \exists B \in \mathfrak{J} : A \sqsubseteq [B]$ is obvious. \square

OBVIOUS 1577. $\mathcal{E}_{\text{RLD}}^{A,B} = \uparrow^{\text{RLD}(A,B)} \text{id}_{A \sqcap B}$.

PROPOSITION 1578. RLD with $\mathcal{X} \times_{A,B} \mathcal{Y} = (\mathcal{X} \div A) \times^{\text{RLD}} (\mathcal{X} \div B)$ for every unfixed filters \mathcal{X} and \mathcal{Y} is a category with binary product morphism.

PROOF. $\text{id}_Y^{C(B,B)} \circ f \circ \text{id}_X^{C(A,A)} = f \sqcap (\mathcal{X} \times_{A,B} \mathcal{Y})$ because

$$\begin{aligned} \text{id}_Y^{C(B,B)} \circ f \circ \text{id}_X^{C(A,A)} &= \\ &= (\text{id}_{Y \div B}^{\text{RLD}} \div (B \times B)) \circ f \circ (\text{id}_{X \div A}^{\text{RLD}} \div (A \times A)) = \\ &= \text{id}_{Y \div B}^{\text{RLD}} \circ f \circ \text{id}_{X \div A}^{\text{RLD}} = \\ &= f \sqcap ((\mathcal{X} \div A) \times^{\text{RLD}} (\mathcal{Y} \div B)) = \\ &= f \sqcap (\mathcal{X} \times_{A,B} \mathcal{Y}). \end{aligned}$$

$$\begin{aligned}
\iota_{A_1, B_1}(\mathcal{X} \times_{A_0, B_0} \mathcal{Y}) &= \\
& \mathcal{E}^{B_0, B_1} \circ (\mathcal{X} \times_{A_0, B_0} \mathcal{Y}) \circ \mathcal{E}^{A_1, A_0} = \\
& \uparrow^{\text{RLD}(B_0, B_1)} \text{id}_{B_0 \cap B_1} ((\mathcal{X} \div B_0) \times (\mathcal{Y} \div A_0)) \circ \uparrow^{\text{RLD}(A_1, A_0)} \text{id}_{A_0 \cap A_1} = \\
& ((\mathcal{X} \div B_0) \div B_1) \times ((\mathcal{Y} \div A_0) \div A_1) = (\mathcal{X} \div B_1) \times (\mathcal{Y} \div A_1) = \\
& \mathcal{X} \times_{A_1, B_1} \mathcal{Y}.
\end{aligned}$$

□

PROOF.

$$\begin{aligned}
\iota_{A, B} f &= \\
& \mathcal{E}_{\text{RLD}}^{\text{Dst } f, B} \circ f \circ \mathcal{E}_{\text{RLD}}^{A, \text{Src } f} = \\
& \prod_{F \in \text{up } f}^{\text{RLD}} (\uparrow^{\text{Rel}(\text{Dst } f, B)} \text{id}_{\text{Dst } f \cap B} \circ F \circ \uparrow^{\text{Rel}(A, \text{Src } f)} \text{id}_{A \cap \text{Src } f}) = \\
& \prod_{F \in \text{up } f}^{\text{RLD}} (\uparrow^{\text{Rel}(A, B)} (\text{id}_{\text{Dst } f \cap B} \circ \text{GR } F \circ \text{id}_{\text{Dst } f \cap B})) = \\
& \prod_{F \in \text{up } f}^{\text{RLD}} \uparrow^{\text{Rel}(A, B)} (F \cap (A \times B)) = \\
& f \div (A \times B).
\end{aligned}$$

□

PROPOSITION 1579. $\text{id}_X^{\text{RLD}(A, A)} = \text{id}_{X \div A}^{\text{RLD}}$ whenever $A \in \mathfrak{J}$ and $\mathfrak{A} \ni X \sqsubseteq [A]$.PROOF. $\text{id}_X^{\text{RLD}(A, A)} = \text{id}_{X \div (A \cap A)}^{\text{RLD}} \div (A \times A) = \text{id}_{X \div A}^{\text{RLD}}$. □DEFINITION 1580. Category RLD can be considered as a category with binary product morphism with the binary product defined as: $\mathcal{X} \times_{A, B} \mathcal{Y} = (\mathcal{X} \div A) \times^{\text{RLD}} (\mathcal{Y} \div B)$ for every unfixed filters \mathcal{X} and \mathcal{Y} .

It is really a binary product morphism:

PROOF. Need to prove the axioms:

$$1^\circ. f \sqcap (X \times_{A, B} Y) = f \sqcap ((X \div A) \times^{\text{RLD}} (Y \div B)) = \text{id}_{Y \div B}^{\text{RLD}} \circ f \circ \text{id}_{X \div A}^{\text{RLD}} = \text{id}_Y^{\text{RLD}(B, B)} \circ f \circ \text{id}_X^{\text{RLD}(A, A)}.$$

2°. Let unfixed filters $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$. Then we have $\iota_{A_1, B_1}(X \times_{A_0, B_0} Y) = \mathcal{E}^{C(B_0, B_1)} \circ (X \times_{A_0, B_0} Y) \circ \mathcal{E}^{C(A_1, A_0)} = \uparrow^{\text{RLD}(B_0, B_1)} \text{id}_{B_0 \cap B_1} \circ ((X \div A_0) \times^{\text{RLD}} (Y \div B_0)) \circ \uparrow^{\text{RLD}(A_1, A_0)} \text{id}_{A_0 \cap A_1}$.

$$\text{But } (X \div A_0) \times^{\text{RLD}} (Y \div B_0) = \prod_{x \in \text{up}(X \div A_0), y \in \text{up}(Y \div B_0)}^{\text{RLD}} (x \times y) =$$

(by the bijection) $= \prod_{x \in \text{up } X, y \in \text{up } Y}^{\text{RLD}} ((x \div A_0) \times (y \div B_0)).$

$$\text{Thus by definition of reloidal product } \iota_{A_1, B_1}(X \times_{A_0, B_0} Y) = \prod_{x \in \text{up } X, y \in \text{up } Y}^{\text{RLD}(A_1, B_1)} (\text{id}_{B_0 \cap B_1} \circ ((x \div A_0) \times (y \div B_0)) \circ \text{id}_{A_0 \cap A_1}) = \prod_{x \in \text{up } X, y \in \text{up } Y}^{\text{RLD}(A_1, B_1)} ((x \div A_0) \times (y \div B_0)) = \prod_{x \in \text{up}(X \div A_0), y \in \text{up}(Y \div B_0)}^{\text{RLD}(A_1, B_1)} (x \times y) = (X \div A_1) \times^{\text{RLD}} (Y \div B_1) = X \times_{A_1, B_1} Y.$$

□

DEFINITION 1581. Reloid $\mathcal{S}f \in \text{End}_{\text{RLD}}(\text{small sets})$ is defined by the formula $\text{GR } \mathcal{S}f = \mathcal{S} \text{GR } f$ for every reloid f .

DEFINITION 1582. Reloid $\mathcal{S}f \in \text{End}_{\text{RLD}}(\text{small sets})$ if defined by the formula $\mathcal{S}f = \mathcal{S}F$ for arbitrary $F \in f$ for every unfixed reloid f .

That the result does not depend on the choice of F obviously follows from the corresponding result for filters.

PROPOSITION 1583.

- 1°. \mathcal{S} from a Hom-set $\text{RLD}(A, B)$ to $\text{End}_{\text{RLD}}(\text{small sets})$ is an order embedding.
- 2°. \mathcal{S} from the category RLD to $\text{End}_{\text{RLD}}(\text{small sets})$ is a prefunctor.
- 3°. \mathcal{S} from unfixed reloids is an order embedding and a prefunctor (= semi-group homomorphism).

PROOF.

1°. That it's monotone is obvious. That it is an injection follows from \mathcal{S} for filters being an injection.

2°. Let f and g be composable reloids.

If $H \in \text{up } \mathcal{S}(g \circ f)$ then $H \supseteq H' \in \text{up}(g \circ f)$, $H' \supseteq G \circ F$ for some $H', F \in \text{up } f$ and $G \in \text{up } g$. Consequently $F \in \text{GR } \mathcal{S}f$, $G \in \text{GR } \mathcal{S}g$. So $G \circ F \in \text{up}(\mathcal{S}g \circ \mathcal{S}f)$ and thus $\mathcal{S}(g \circ f) \supseteq \mathcal{S}g \circ \mathcal{S}f$.

Whenever $H \in \text{up}(\mathcal{S}g \circ \mathcal{S}f)$, we have $H \supseteq G \circ F$ where $F \in \text{up } \mathcal{S}f$, $G \in \text{up } \mathcal{S}g$. Thus $F \supseteq F' \in \text{up } f$, $G \supseteq G' \in \text{up } g$; $H \supseteq G' \circ F' \in \text{up}(g \circ f)$ for some F', G' and so $H \in \text{up}(\mathcal{S}(g \circ f))$. So $\mathcal{S}g \circ \mathcal{S}f \supseteq \mathcal{S}(g \circ f)$.

So $\mathcal{S}(g \circ f) = \mathcal{S}g \circ \mathcal{S}f$.

3°. That it is a prefunctor easily follows from the above.

Suppose f, g are unfixed reloids and $\mathcal{S}f = \mathcal{S}g$. Let $F \in f$, $G \in g$ and thus $\mathcal{S}F = \mathcal{S}G$. It is enough to prove that $F \sim G$.

Really, $\mathcal{S}F = \mathcal{S}G \Rightarrow \mathcal{S} \text{GR } F = \mathcal{S} \text{GR } G \Rightarrow \text{GR } F \sim \text{GR } G \Rightarrow \text{GR } G = (\text{GR } F) \div (\text{dom } G \times \text{im } G) \Leftrightarrow G = F \div (\text{dom } G \times \text{im } G) = \iota_{\text{dom } G, \text{im } G} F$. Similarly $F = \iota_{\text{dom } F, \text{im } F} G$. So $F \sim G$. □

I yet failed to generalize propositions 1574 and 1575. The generalization may require first research pointfree reloids.

19.8. More results on restricted identities

In the next three propositions assume $A \in \mathfrak{J}$, $\mathfrak{A} \ni X \sqsubseteq A$.

PROPOSITION 1584. $\text{id}_X^{\text{Rel}(A)} = \text{id}_{[X]}^{\text{Rel}(A, A)}$.

PROOF. $\text{id}_{[X]}^{\text{Rel}(A, A)} = \text{id}_X^{\text{Rel}(A, A)} = \text{id}_X^{\text{Rel}(A)}$. □

PROPOSITION 1585. $\text{id}_X^{\text{FCD}(A)} = \text{id}_{[X]}^{\text{FCD}(A, A)}$.

PROOF. $\langle \text{id}_{[X]}^{\text{FCD}(A, A)} \rangle \mathcal{X} = ([\mathcal{X}] \sqcap [X]) \div A = [\mathcal{X} \sqcap X] \div A = \mathcal{X} \sqcap X = \langle \text{id}_X^{\text{FCD}(A)} \rangle \mathcal{X}$ for $\mathfrak{A} \ni \mathcal{X} \sqsubseteq A$. □

PROPOSITION 1586. $\text{id}_X^{\text{RLD}(A)} = \text{id}_{[X]}^{\text{RLD}(A, A)}$.

PROOF. $\text{id}_{[X]}^{\text{RLD}(A, A)} = \text{id}_{[X] \div (A \cap A)}^{\text{RLD}} \div (A \times A) = \text{id}_X^{\text{RLD}} \div (A \times A) = \text{id}_X^{\text{RLD}(A)}$. □

PROPOSITION 1587. $\left\{ \frac{(A \div A, A \sqcap A)}{A \in \mathfrak{F}(U)} \right\}$ is a function and moreover is an order isomorphism for a set $A \subseteq U$.

PROOF. $\mathcal{A} \div A$ and $\mathcal{A} \sqcap A$ are determined by each other by the following formulas:

$$\mathcal{A} \div A = (\mathcal{A} \sqcap A) \div A \quad \text{and} \quad \mathcal{A} \sqcap A = (\mathcal{A} \div A) \div \text{Base}(\mathcal{A}).$$

Prove the formulas: $(\mathcal{A} \sqcap A) \div A = \prod \left\{ \frac{\uparrow^A(X \cap A)}{X \in \mathcal{A} \sqcap A} \right\} = \prod \left\{ \frac{\uparrow^A(X \cap A)}{X \in \mathcal{A}} \right\} = \mathcal{A} \div A$.

$$(\mathcal{A} \div A) \div \text{Base}(\mathcal{A}) =$$

$$\begin{aligned} & \prod \left\{ \frac{\uparrow^A(X \cap A)}{X \in \mathcal{A}} \right\} \div \text{Base}(\mathcal{A}) = \\ & \prod \left\{ \frac{\uparrow^{\text{Base}(\mathcal{A})}(Y \cap \text{Base}(\mathcal{A}))}{Y \in \prod \left\{ \frac{\uparrow^A(X \cap A)}{X \in \mathcal{A}} \right\}} \right\} = \\ & \text{(by properties of filter bases)} = \\ & \prod \left\{ \frac{\uparrow^{\text{Base}(\mathcal{A})}(X \cap A \cap \text{Base}(\mathcal{A}))}{X \in} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{Base}(\mathcal{A})}(X \cap A)}{X \in \mathcal{A}} \right\} = \end{aligned}$$

$$\mathcal{A} \sqcap A.$$

That this defines a bijection, follows from $\mathcal{A} \div A \sim \mathcal{A} \sqcap A$ what easily follows from the above formulas. \square

PROPOSITION 1588. $\left\{ \frac{(\iota_{X,Y} f, \text{id}_Y^{\text{Rel}(\text{Dst } f)} \circ f \circ \text{id}_X^{\text{Rel}(\text{Src } f)})}{f \in \text{Rel}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \text{Src } f$, $Y \subseteq \text{Dst } f$.

PROOF. $\iota_{X,Y} f = (X, Y, \text{GR } f \cap (X \times Y))$; $\text{id}_Y^{\text{Rel}} \circ f \circ \text{id}_X^{\text{Rel}} = (\text{Src } f, \text{Dst } f, \text{GR } f \cap (X \times Y))$. The isomorphism (both order and semigroup) is evident. \square

PROPOSITION 1589. $\left\{ \frac{(\iota_{X,Y} f, \text{id}_Y^{\text{FCD}(\text{Dst } f)} \circ f \circ \text{id}_X^{\text{FCD}(\text{Src } f)})}{f \in \text{FCD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \text{Src } f$, $Y \subseteq \text{Dst } f$.

PROOF. From symmetry it follows that it's enough to prove that $\left\{ \frac{(\mathcal{E}^Y \circ f, \text{id}_Y^{\text{FCD}} \circ f)}{f \in \text{FCD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for a set $Y \subseteq \text{Dst } f$.

Really, $\left\{ \frac{((\mathcal{E}^Y)_x, (\text{id}_Y^{\text{FCD}})_x)}{x \in \text{Dst } f} \right\} = \left\{ \frac{(x \div Y, x \sqcap Y)}{x \in \text{Dst } f} \right\}$ is an order isomorphism by proved above. This implies that $\left\{ \frac{(\mathcal{E}^Y \circ f, \text{id}_Y^{\text{FCD}} \circ f)}{f \in \text{FCD}(A,B)} \right\}$ is an isomorphism (both order and semigroup). \square

PROPOSITION 1590. $\left\{ \frac{(\iota_{X,Y} f, \text{id}_Y^{\text{RLD}(\text{Dst } f)} \circ f \circ \text{id}_X^{\text{RLD}(\text{Src } f)})}{f \in \text{RLD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \text{Src } f$, $Y \subseteq \text{Dst } f$.

PROOF. $\iota_{X,Y} f = (X, Y, (\text{up } f) \div (X \times Y))$; $\text{id}_Y^{\text{RLD}} \circ f \circ \text{id}_X^{\text{RLD}} = (\text{Src } f, \text{Dst } f, (\text{up } f) \sqcap (X \times Y))$. They are order isomorphic by proved above.

$$\iota_{Y,Z} g \circ \iota_{X,Y} f =$$

$$\begin{aligned} & \mathcal{E}^{\text{Dst } g, Z} \circ g \circ \mathcal{E}^{Y, \text{Src } g} \circ \mathcal{E}^{\text{Dst } f, Y} \circ f \circ \mathcal{E}^{X, \text{Src } f} = \\ & \mathcal{E}^{\text{Dst } g, Z} \circ g \circ \text{id}_Y^{\text{RLD}} \circ \text{id}_Y^{\text{RLD}} \circ f \circ \mathcal{E}^{X, \text{Src } f} \end{aligned}$$

because $\mathcal{E}^{Y, \text{Src } g} \circ \mathcal{E}^{\text{Dst } f, Y} = \text{id}_Y^{\text{Rel}} = \text{id}_Y^{\text{Rel}} \circ \text{id}_Y^{\text{Rel}}$. Thus by proved above

$$\left\{ \frac{(\iota_{Y, Z} g \circ \iota_{X, Y} f, \text{id}_Z^{\text{RLD}} \circ g \circ \text{id}_Y^{\text{RLD}} \circ \text{id}_Y^{\text{RLD}} \circ f \circ \text{id}_X^{\text{RLD}})}{f \in \text{RLD}(A, B)} \right\}$$

is a bijection. \square

Can three last propositions be generalized into one?

PROPOSITION 1591. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for unfixed morphisms whenever the same formula holds for (composable) morphisms.

PROOF. $f \circ (g \sqcup h) = [\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} (f \circ (g \sqcup h))]$ because $\text{dom}(f \circ (g \sqcup h)) \sqsubseteq \text{Src } g \sqcup \text{Src } h$ and $\text{im}(f \circ (g \sqcup h)) \sqsubseteq \text{Dst } f$.

So

$$\begin{aligned} f \circ (g \sqcup h) &= \\ & [\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} f \circ \iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} (g \sqcup h)] = \\ & [\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} f \circ (\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} g \sqcup \iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} h)] = \\ & [\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} f \circ \iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} g \sqcup \iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} f \circ \iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} h] = \\ & [\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} (f \circ g) \sqcup \iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} (f \circ h)] = \\ & [\iota_{\text{Src } g \sqcup \text{Src } h, \text{Dst } f} (f \circ g \sqcup f \circ h)] = \\ & f \circ g \sqcup f \circ h \end{aligned}$$

because $\text{dom}(f \circ g \sqcup f \circ h) \sqsubseteq \text{Src } g \sqcup \text{Src } h$ and $\text{im}(f \circ g \sqcup f \circ h) \sqsubseteq \text{Dst } f$. \square

Part 3

Pointfree funcoids and reloids

Pointfree functors

This chapter is based on [29].

This is a routine chapter. There is almost nothing creative here. I just generalize theorems about functors to the maximum extent for *pointfree functors* (defined below) preserving the proof idea. The main idea behind this chapter is to find weakest theorem conditions enough for the same theorem statement as for above theorems for functors.

For those who know pointfree topology: Pointfree topology notions of frames and locales is a non-trivial generalization of topological spaces. Pointfree functors are different: I just replace the set of filters on a set with an arbitrary poset, this readily gives the definition of *pointfree functor*, almost no need of creativity here.

Pointfree functors are used in the below definitions of products of functors.

20.1. Definition

DEFINITION 1592. *Pointfree functor* is a quadruple $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ where \mathfrak{A} and \mathfrak{B} are posets, $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta \in \mathfrak{A}^{\mathfrak{B}}$ such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (y \not\prec \alpha x \Leftrightarrow x \not\prec \beta y).$$

DEFINITION 1593. The *source* $\text{Src}(\mathfrak{A}, \mathfrak{B}, \alpha, \beta) = \mathfrak{A}$ and *destination* $\text{Dst}(\mathfrak{A}, \mathfrak{B}, \alpha, \beta) = \mathfrak{B}$ for every pointfree functor $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$.

To every functor (A, B, α, β) corresponds pointfree functor $(\mathcal{P}A, \mathcal{P}B, \alpha, \beta)$. Thus pointfree functors are a generalization of functors.

DEFINITION 1594. I will denote $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ the set of pointfree functors from \mathfrak{A} to \mathfrak{B} (that is with source \mathfrak{A} and destination \mathfrak{B}), for every posets \mathfrak{A} and \mathfrak{B} .

$$\langle (\mathfrak{A}, \mathfrak{B}, \alpha, \beta) \rangle \stackrel{\text{def}}{=} \alpha \text{ for every pointfree functor } (\mathfrak{A}, \mathfrak{B}, \alpha, \beta).$$

DEFINITION 1595. $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)^{-1} = (\mathfrak{B}, \mathfrak{A}, \beta, \alpha)$ for every pointfree functor $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$.

PROPOSITION 1596. If f is a pointfree functor then f^{-1} is also a pointfree functor.

PROOF. It follows from symmetry in the definition of pointfree functor. \square

OBVIOUS 1597. $(f^{-1})^{-1} = f$ for every pointfree functor f .

DEFINITION 1598. The relation $[f] \in \mathcal{P}(\text{Src } f \times \text{Dst } f)$ is defined by the formula (for every pointfree functor f and $x \in \text{Src } f, y \in \text{Dst } f$)

$$x [f] y \stackrel{\text{def}}{=} y \not\prec \langle f \rangle x.$$

OBVIOUS 1599. $x [f] y \Leftrightarrow y \not\prec \langle f \rangle x \Leftrightarrow x \not\prec \langle f^{-1} \rangle y$ for every pointfree functor f and $x \in \text{Src } f, y \in \text{Dst } f$.

OBVIOUS 1600. $[f^{-1}] = [f]^{-1}$ for every pointfree functor f .

THEOREM 1601. Let \mathfrak{A} and \mathfrak{B} be posets. Then:

- 1°. If \mathfrak{A} is separable, for given value of $\langle f \rangle$ there exists no more than one $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.
- 2°. If \mathfrak{A} and \mathfrak{B} are separable, for given value of $[f]$ there exists no more than one $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROOF. Let $f, g \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

1°. Let $\langle f \rangle = \langle g \rangle$. Then for every $x \in \mathfrak{A}, y \in \mathfrak{B}$ we have

$$x \not\prec \langle f^{-1} \rangle y \Leftrightarrow y \not\prec \langle f \rangle x \Leftrightarrow y \not\prec \langle g \rangle x \Leftrightarrow x \not\prec \langle g^{-1} \rangle y$$

and thus by separability of \mathfrak{A} we have $\langle f^{-1} \rangle y = \langle g^{-1} \rangle y$ that is $\langle f^{-1} \rangle = \langle g^{-1} \rangle$ and so $f = g$.

2°. Let $[f] = [g]$. Then for every $x \in \mathfrak{A}, y \in \mathfrak{B}$ we have

$$x \not\prec \langle f^{-1} \rangle y \Leftrightarrow x [f] y \Leftrightarrow x [g] y \Leftrightarrow x \not\prec \langle g^{-1} \rangle y$$

and thus by separability of \mathfrak{A} we have $\langle f^{-1} \rangle y = \langle g^{-1} \rangle y$ that is $\langle f^{-1} \rangle = \langle g^{-1} \rangle$. Similarly we have $\langle f \rangle = \langle g \rangle$. Thus $f = g$. □

PROPOSITION 1602. If $\text{Src } f$ and $\text{Dst } f$ have least elements, then $\langle f \rangle \perp^{\text{Src } f} = \perp^{\text{Dst } f}$ for every pointfree funcoid f .

PROOF. $y \not\prec \langle f \rangle \perp^{\text{Src } f} \Leftrightarrow \perp^{\text{Src } f} \not\prec \langle f^{-1} \rangle y \Leftrightarrow 0$ for every $y \in \text{Dst } f$. Thus $\langle f \rangle \perp^{\text{Src } f} \asymp \langle f \rangle \perp^{\text{Src } f}$. So $\langle f \rangle \perp^{\text{Src } f} = \perp^{\text{Dst } f}$. □

PROPOSITION 1603. If $\text{Dst } f$ is strongly separable then $\langle f \rangle$ is a monotone function (for a pointfree funcoid f).

PROOF.

$$\begin{aligned} a \sqsubseteq b &\Rightarrow \\ \forall x \in \text{Dst } f : (a \not\prec \langle f^{-1} \rangle x \Rightarrow b \not\prec \langle f^{-1} \rangle x) &\Rightarrow \\ \forall x \in \text{Dst } f : (x \not\prec \langle f \rangle a \Rightarrow x \not\prec \langle f \rangle b) &\Leftrightarrow \\ \star \langle f \rangle a \sqsubseteq \star \langle f \rangle b &\Rightarrow \\ \langle f \rangle a \sqsubseteq \langle f \rangle b. & \end{aligned}$$

□

THEOREM 1604. Let f be a pointfree funcoid from a starrish join-semilattice $\text{Src } f$ to a separable starrish join-semilattice $\text{Dst } f$. Then $\langle f \rangle(i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j$ for every $i, j \in \text{Src } f$.

PROOF.

$$\begin{aligned} \star \langle f \rangle(i \sqcup j) &= \\ \left\{ \frac{y \in \text{Dst } f}{y \not\prec \langle f \rangle(i \sqcup j)} \right\} &= \\ \left\{ \frac{y \in \text{Dst } f}{i \sqcup j \not\prec \langle f^{-1} \rangle y} \right\} &= \\ \left\{ \frac{y \in \text{Dst } f}{i \not\prec \langle f^{-1} \rangle y \vee j \not\prec \langle f^{-1} \rangle y} \right\} &= \\ \left\{ \frac{y \in \text{Dst } f}{y \not\prec \langle f \rangle i \vee y \not\prec \langle f \rangle j} \right\} &= \\ \left\{ \frac{y \in \text{Dst } f}{y \not\prec \langle f \rangle i \sqcup \langle f \rangle j} \right\} &= \\ \star \langle f \rangle i \sqcup \star \langle f \rangle j. & \end{aligned}$$

Thus $\langle f \rangle(i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j$ by separability. \square

PROPOSITION 1605. Let f be a pointfree functor. Then:

- 1°. $k [f] i \sqcup j \Leftrightarrow k [f] i \vee k [f] j$ for every $i, j \in \text{Dst } f$, $k \in \text{Src } f$ if $\text{Dst } f$ is a starrish join-semilattice.
- 2°. $i \sqcup j [f] k \Leftrightarrow i [f] k \vee j [f] k$ for every $i, j \in \text{Src } f$, $k \in \text{Dst } f$ if $\text{Src } f$ is a starrish join-semilattice.

PROOF.

- 1°. $k [f] i \sqcup j \Leftrightarrow i \sqcup j \not\prec \langle f \rangle k \Leftrightarrow i \not\prec \langle f \rangle k \vee j \not\prec \langle f \rangle k \Leftrightarrow k [f] i \vee k [f] j$.
- 2°. Similar.

\square

20.2. Composition of pointfree functors

DEFINITION 1606. *Composition* of pointfree functors is defined by the formula

$$(\mathfrak{B}, \mathfrak{C}, \alpha_2, \beta_2) \circ (\mathfrak{A}, \mathfrak{B}, \alpha_1, \beta_1) = (\mathfrak{A}, \mathfrak{C}, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2).$$

DEFINITION 1607. I will call functors f and g *composable* when $\text{Dst } f = \text{Src } g$.

PROPOSITION 1608. If f, g are composable pointfree functors then $g \circ f$ is pointfree functor.

PROOF. Let $f = (\mathfrak{A}, \mathfrak{B}, \alpha_1, \beta_1)$, $g = (\mathfrak{B}, \mathfrak{C}, \alpha_2, \beta_2)$. For every $x, y \in \mathfrak{A}$ we have

$$y \not\prec (\alpha_2 \circ \alpha_1)x \Leftrightarrow y \not\prec \alpha_2 \alpha_1 x \Leftrightarrow \alpha_1 x \not\prec \beta_2 y \Leftrightarrow x \not\prec \beta_1 \beta_2 y \Leftrightarrow x \not\prec (\beta_1 \circ \beta_2)y.$$

So $(\mathfrak{A}, \mathfrak{C}, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$ is a pointfree functor. \square

OBVIOUS 1609. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable pointfree functors f and g .

THEOREM 1610. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable pointfree functors f and g .

PROOF.

$$\begin{aligned} \langle (g \circ f)^{-1} \rangle &= \langle f^{-1} \rangle \circ \langle g^{-1} \rangle = \langle f^{-1} \circ g^{-1} \rangle; \\ \langle ((g \circ f)^{-1})^{-1} \rangle &= \langle g \circ f \rangle = \langle (f^{-1} \circ g^{-1})^{-1} \rangle. \end{aligned}$$

\square

PROPOSITION 1611. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable pointfree functors f, g, h .

PROOF. $\langle (h \circ g) \circ f \rangle = \langle h \circ g \rangle \circ \langle f \rangle = \langle h \rangle \circ \langle g \rangle \circ \langle f \rangle = \langle h \rangle \circ \langle g \circ f \rangle = \langle h \circ (g \circ f) \rangle$;

$$\begin{aligned} \langle ((h \circ g) \circ f)^{-1} \rangle &= \langle f^{-1} \circ (h \circ g)^{-1} \rangle = \langle f^{-1} \circ g^{-1} \circ h^{-1} \rangle = \\ &= \langle (g \circ f)^{-1} \circ h^{-1} \rangle = \langle (h \circ (g \circ f))^{-1} \rangle. \end{aligned}$$

\square

EXERCISE 1612. Generalize section 7.4 for pointfree functors.

20.3. Pointfree funcoid as continuation

PROPOSITION 1613. Let f be a pointfree funcoid. Then for every $x \in \text{Src } f$, $y \in \text{Dst } f$ we have

- 1°. If $(\text{Src } f, \mathfrak{Z})$ is a filtrator with separable core then $x [f] y \Leftrightarrow \forall X \in \text{up}^3 x : X [f] y$.
- 2°. If $(\text{Dst } f, \mathfrak{Z})$ is a filtrator with separable core then $x [f] y \Leftrightarrow \forall Y \in \text{up}^3 y : x [f] Y$.

PROOF. We will prove only the second because the first is similar.

$$x [f] y \Leftrightarrow y \not\prec^{\text{Dst } f} \langle f \rangle x \Leftrightarrow \forall Y \in \text{up}^3 y : Y \not\prec \langle f \rangle x \Leftrightarrow \forall Y \in \text{up}^3 y : x [f] Y.$$

□

COROLLARY 1614. Let f be a pointfree funcoid and $(\text{Src } f, \mathfrak{Z}_0)$, $(\text{Dst } f, \mathfrak{Z}_1)$ be filtrators with separable core. Then

$$x [f] y \Leftrightarrow \forall X \in \text{up}^{\mathfrak{Z}_0} x, Y \in \text{up}^{\mathfrak{Z}_1} y : X [f] Y.$$

PROOF. Apply the proposition twice. □

THEOREM 1615. Let f be a pointfree funcoid. Let $(\text{Src } f, \mathfrak{Z}_0)$ be a binarily meet-closed filtrator with separable core which is a meet-semilattice and $\forall x \in \text{Src } f : \text{up}^{\mathfrak{Z}_0} x \neq \emptyset$ and $(\text{Dst } f, \mathfrak{Z}_1)$ be a primary filtrator over a boolean lattice.

$$\langle f \rangle x = \prod^{\text{Dst } f} \langle \langle f \rangle \rangle^* \text{up}^{\mathfrak{Z}_0} x.$$

PROOF. By the previous proposition for every $y \in \text{Dst } f$:

$$y \not\prec^{\text{Dst } f} \langle f \rangle x \Leftrightarrow x [f] y \Leftrightarrow \forall X \in \text{up}^{\mathfrak{Z}_0} x : X [f] y \Leftrightarrow \forall X \in \text{up}^{\mathfrak{Z}_0} x : y \not\prec^{\text{Dst } f} \langle f \rangle X.$$

Let's denote $W = \left\{ \frac{y \cap^{\text{Dst } f} \langle f \rangle X}{X \in \text{up}^{\mathfrak{Z}_0} x} \right\}$. We will prove that W is a generalized filter base over \mathfrak{Z}_1 . To prove this enough to show that $V = \left\{ \frac{\langle f \rangle X}{X \in \text{up}^{\mathfrak{Z}_0} x} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f \rangle A$, $\mathcal{Q} = \langle f \rangle B$ where $A, B \in \text{up}^{\mathfrak{Z}_0} x$; $A \cap^{\mathfrak{Z}_0} B \in \text{up}^{\mathfrak{Z}_0} x$ (used the fact that it is a binarily meet-closed and theorem 535) and $\mathcal{R} \sqsubseteq \mathcal{P} \cap^{\text{Dst } f} \mathcal{Q}$ for $\mathcal{R} = \langle f \rangle (A \cap^{\mathfrak{Z}_0} B) \in V$ because $\text{Dst } f$ is strongly separable by proposition 579. So V is a generalized filter base and thus W is a generalized filter base.

$\perp^{\text{Dst } f} \notin W \Leftrightarrow \perp^{\text{Dst } f} \notin \prod^{\text{Dst } f} W$ by theorem 572. That is

$$\forall X \in \text{up}^{\mathfrak{Z}_0} x : y \cap^{\text{Dst } f} \langle f \rangle X \neq \perp^{\text{Dst } f} \Leftrightarrow y \cap^{\text{Dst } f} \prod^{\text{Dst } f} \langle \langle f \rangle \rangle^* \text{up}^{\mathfrak{Z}_0} x \neq \perp^{\text{Dst } f}.$$

Comparing with the above,

$$y \cap^{\text{Dst } f} \langle f \rangle x \neq \perp^{\text{Dst } f} \Leftrightarrow y \cap^{\text{Dst } f} \prod^{\text{Dst } f} \langle \langle f \rangle \rangle^* \text{up}^{\mathfrak{Z}_0} x \neq \perp^{\text{Dst } f}.$$

So $\langle f \rangle x = \prod^{\text{Dst } f} \langle \langle f \rangle \rangle^* \text{up}^{\mathfrak{Z}_0} x$ because $\text{Dst } f$ is separable (proposition 579 and the fact that \mathfrak{Z}_1 is a boolean lattice). □

THEOREM 1616. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices.

- 1°. A function $\alpha \in \mathfrak{B}^{\mathfrak{Z}_0}$ conforming to the formulas (for every $I, J \in \mathfrak{Z}_0$)

$$\alpha \perp^{\mathfrak{Z}_0} = \perp^{\mathfrak{B}}, \quad \alpha(I \sqcup J) = \alpha I \sqcup \alpha J$$

can be continued to the function $\langle f \rangle$ for a unique $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\langle f \rangle \mathcal{X} = \prod^{\mathfrak{B}} \langle \alpha \rangle^* \text{up}^{\mathfrak{B}_0} \mathcal{X} \quad (20)$$

for every $\mathcal{X} \in \mathfrak{A}$.

2°. A relation $\delta \in \mathcal{P}(\mathfrak{Z}_0 \times \mathfrak{Z}_1)$ conforming to the formulas (for every $I, J, K \in \mathfrak{Z}_0$ and $I', J', K' \in \mathfrak{Z}_1$)

$$\begin{aligned} \neg(\perp^{\mathfrak{Z}_0} \delta I'), \quad I \sqcup J \delta K' &\Leftrightarrow I \delta K' \vee J \delta K', \\ \neg(I \delta \perp^{\mathfrak{Z}_1}), \quad K \delta I' \sqcup J' &\Leftrightarrow K \delta I' \vee K \delta J' \end{aligned} \quad (21)$$

can be continued to the relation $[f]$ for a unique $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up}^{\mathfrak{Z}_0} \mathcal{X}, Y \in \text{up}^{\mathfrak{Z}_1} \mathcal{Y} : X \delta Y \quad (22)$$

for every $\mathcal{X} \in \mathfrak{A}, \mathcal{Y} \in \mathfrak{B}$.

PROOF. Existence of no more than one such pointfree functors and formulas (20) and (22) follow from two previous theorems.

2°. $\left\{ \frac{Y \in \mathfrak{Z}_1}{X \delta Y} \right\}$ is obviously a free star for every $X \in \mathfrak{Z}_0$. By properties of filters on boolean lattices, there exist a unique filter αX such that $\partial(\alpha X) = \left\{ \frac{Y \in \mathfrak{Z}_1}{X \delta Y} \right\}$ for every $X \in \mathfrak{Z}_0$. Thus $\alpha \in \mathfrak{B}^{\mathfrak{Z}_0}$. Similarly it can be defined $\beta \in \mathfrak{A}^{\mathfrak{Z}_1}$ by the formula $\partial(\beta Y) = \left\{ \frac{X \in \mathfrak{Z}_0}{X \delta Y} \right\}$. Let's continue the functions α and β to $\alpha' \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta' \in \mathfrak{A}^{\mathfrak{B}}$ by the formulas

$$\alpha' \mathcal{X} = \prod^{\mathfrak{B}} \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X} \quad \text{and} \quad \beta' \mathcal{Y} = \prod^{\mathfrak{A}} \langle \beta \rangle^* \text{up}^{\mathfrak{Z}_1} \mathcal{Y}$$

and δ to $\delta' \in \mathcal{P}(\mathfrak{A} \times \mathfrak{B})$ by the formula

$$\mathcal{X} \delta' \mathcal{Y} \Leftrightarrow \forall X \in \text{up}^{\mathfrak{Z}_0} \mathcal{X}, Y \in \text{up}^{\mathfrak{Z}_1} \mathcal{Y} : X \delta Y.$$

$\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp^{\mathfrak{B}} \Leftrightarrow \mathcal{Y} \sqcap \prod^{\mathfrak{B}} \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X} \neq \perp^{\mathfrak{B}} \Leftrightarrow \prod^{\mathfrak{B}} \langle \mathcal{Y} \sqcap \rangle^* \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X} \neq \perp^{\mathfrak{B}}$. Let's prove that

$$W = \langle \mathcal{Y} \sqcap \rangle^* \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X}$ is a generalized filter base.

If $\mathcal{A}, \mathcal{B} \in \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X}$ then exist $X_1, X_2 \in \text{up}^{\mathfrak{Z}_0} \mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{B} = \alpha X_2$. Then $\alpha(X_1 \sqcap^{\mathfrak{Z}_0} X_2) \in \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X}$. So $\langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\prod^{\mathfrak{B}} \langle \mathcal{Y} \sqcap \rangle^* \langle \alpha \rangle^* \text{up}^{\mathfrak{Z}_0} \mathcal{X} \neq \perp^{\mathfrak{B}}$ is equivalent to

$$\forall X \in \text{up}^{\mathfrak{Z}_0} \mathcal{X} : \mathcal{Y} \sqcap \alpha X \neq \perp^{\mathfrak{B}},$$

what is equivalent to

$$\begin{aligned} \forall X \in \text{up}^{\mathfrak{Z}_0} \mathcal{X}, Y \in \text{up}^{\mathfrak{Z}_1} \mathcal{Y} : Y \sqcap^{\mathfrak{B}} \alpha X \neq \perp^{\mathfrak{B}} &\Leftrightarrow \\ \forall X \in \text{up}^{\mathfrak{Z}_0} \mathcal{X}, Y \in \text{up}^{\mathfrak{Z}_1} \mathcal{Y} : Y \in \partial(\alpha X) &\Leftrightarrow \\ \forall X \in \text{up}^{\mathfrak{Z}_0} \mathcal{X}, Y \in \text{up}^{\mathfrak{Z}_1} \mathcal{Y} : X \delta Y. & \end{aligned}$$

Combining the equivalencies we get $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp^{\mathfrak{B}} \Leftrightarrow \mathcal{X} \delta' \mathcal{Y}$. Analogously $\mathcal{X} \sqcap \beta' \mathcal{Y} \neq \perp^{\mathfrak{A}} \Leftrightarrow \mathcal{X} \delta' \mathcal{Y}$. So $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp^{\mathfrak{B}} \Leftrightarrow \mathcal{X} \sqcap \beta' \mathcal{Y} \neq \perp^{\mathfrak{A}}$, that is $(\mathfrak{A}, \mathfrak{B}, \alpha', \beta')$ is a pointfree functor. From the formula $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \perp^{\mathfrak{B}} \Leftrightarrow \mathcal{X} \delta' \mathcal{Y}$ it follows that $[(\mathfrak{A}, \mathfrak{B}, \alpha', \beta')]$ is a continuation of δ .

1°. Let define the relation $\delta \in \mathcal{P}(\mathfrak{Z}_0 \times \mathfrak{Z}_1)$ by the formula $X \delta Y \Leftrightarrow Y \sqcap^{\mathfrak{B}} \alpha X \neq \perp^{\mathfrak{B}}$.

That $\neg(\perp^{\mathfrak{Z}_0} \delta I')$ and $\neg(I \delta \perp^{\mathfrak{Z}_1})$ is obvious. We have

$$\begin{aligned}
& K \delta I' \sqcup^{\mathfrak{Z}_1} J' \Leftrightarrow \\
& (I' \sqcup^{\mathfrak{Z}_1} J') \sqcap^{\mathfrak{B}} \alpha K \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& (I' \sqcup^{\mathfrak{B}} J') \sqcap \alpha K \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& (I' \sqcap^{\mathfrak{B}} \alpha K) \sqcup (J' \sqcap^{\mathfrak{B}} \alpha K) \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& I' \sqcap^{\mathfrak{B}} \alpha K \neq \perp^{\mathfrak{B}} \vee J' \sqcap^{\mathfrak{B}} \alpha K \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& K \delta I' \vee K \delta J'
\end{aligned}$$

and

$$\begin{aligned}
& I \sqcup^{\mathfrak{Z}_0} J \delta K' \Leftrightarrow \\
& K' \sqcap^{\mathfrak{B}} \alpha(I \sqcup^{\mathfrak{Z}_0} J) \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& K' \sqcap^{\mathfrak{B}} (\alpha I \sqcup \alpha J) \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& (K' \sqcap^{\mathfrak{B}} \alpha I) \sqcup (K' \sqcap^{\mathfrak{B}} \alpha J) \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& K' \sqcap^{\mathfrak{B}} \alpha I \neq \perp^{\mathfrak{B}} \vee K' \sqcap^{\mathfrak{B}} \alpha J \neq \perp^{\mathfrak{B}} \Leftrightarrow \\
& I \delta K' \vee J \delta K'.
\end{aligned}$$

That is the formulas (21) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

$\forall X \in \mathfrak{Z}_0, Y \in \mathfrak{Z}_1 : (Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \perp^{\mathfrak{B}} \Leftrightarrow X [f] Y \Leftrightarrow Y \sqcap^{\mathfrak{B}} \alpha X \neq \perp^{\mathfrak{B}})$, consequently $\forall X \in \mathfrak{Z}_0 : \alpha X = \langle f \rangle X$ because our filtrator is with separable core. So $\langle f \rangle$ is a continuation of α .

□

THEOREM 1617. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If $\alpha \in \mathfrak{B}^{\mathfrak{Z}_0}, \beta \in \mathfrak{A}^{\mathfrak{Z}_1}$ are functions such that $Y \neq \alpha X \Leftrightarrow X \neq \beta Y$ for every $X \in \mathfrak{Z}_0, Y \in \mathfrak{Z}_1$, then there exists exactly one pointfree funcoid $f : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\langle f \rangle|_{\mathfrak{Z}_0} = \alpha, \langle f^{-1} \rangle|_{\mathfrak{Z}_1} = \beta$.

PROOF. Prove $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$. Really, $Y \neq \alpha(I \sqcup J) \Leftrightarrow I \sqcup J \neq \beta Y \Leftrightarrow I \neq \beta Y \vee J \neq \beta Y \Leftrightarrow Y \neq \alpha I \vee Y \neq \alpha J \Leftrightarrow Y \neq \alpha I \sqcup \alpha J$. So $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$ by star-separability. Similarly $\beta(I \sqcup J) = \beta I \sqcup \beta J$.

Thus by the theorem above there exists a pointfree funcoid f such that $\langle f \rangle|_{\mathfrak{Z}_0} = \alpha, \langle f^{-1} \rangle|_{\mathfrak{Z}_1} = \beta$.

That this pointfree funcoid is unique, follows from the above. □

PROPOSITION 1618. Let $(\text{Src } f, \mathfrak{Z}_0)$ be a primary filtrator over a bounded distributive lattice and $(\text{Dst } f, \mathfrak{Z}_1)$ be a primary filtrator over boolean lattice. If S is a generalized filter base on $\text{Src } f$ then $\langle f \rangle \sqcap^{\text{Src } f} S = \sqcap^{\text{Dst } f} \langle \langle f \rangle \rangle^* S$ for every pointfree funcoid f .

PROOF. First the meets $\sqcap^{\text{Src } f} S$ and $\sqcap^{\text{Dst } f} \langle \langle f \rangle \rangle^* S$ exist by corollary 518.

$(\text{Src } f, \mathfrak{Z}_0)$ is a binarily meet-closed filtrator by corollary 536 and with separable core by theorem 537; thus we can apply theorem 1615 (up $x \neq \emptyset$ is obvious).

$\langle f \rangle \sqcap^{\text{Src } f} S \subseteq \langle f \rangle X$ for every $X \in S$ because $\text{Dst } f$ is strongly separable by proposition 579 and thus $\langle f \rangle \sqcap^{\text{Src } f} S \subseteq \sqcap^{\text{Dst } f} \langle \langle f \rangle \rangle^* S$.

Taking into account properties of generalized filter bases:

$$\begin{aligned}
& \langle f \rangle \bigsqcap^{\text{Src } f} S = \\
& \bigsqcap^{\text{Dst } f} \langle \langle f \rangle \rangle^* \text{up} \bigsqcap S = \\
& \bigsqcap^{\text{Dst } f} \langle \langle f \rangle \rangle^* \left\{ \frac{X}{\exists \mathcal{P} \in S : X \in \text{up } \mathcal{P}} \right\} = \\
& \bigsqcap^{\text{Dst } f} \left\{ \frac{\langle f \rangle^* X}{\exists \mathcal{P} \in S : X \in \text{up } \mathcal{P}} \right\} \sqsupseteq \text{(because Dst } f \text{ is a strongly separable poset)} \\
& \bigsqcap^{\text{Dst } f} \left\{ \frac{\langle f \rangle \mathcal{P}}{\mathcal{P} \in S} \right\} = \\
& \bigsqcap^{\text{Dst } f} \langle \langle f \rangle \rangle^* S.
\end{aligned}$$

□

PROPOSITION 1619. $\mathcal{X} [f] \bigsqcap S \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X} [f] \mathcal{Y}$ if f is a pointfree functor, $\text{Dst } f$ is a meet-semilattice with least element and S is a generalized filter base on $\text{Dst } f$.

PROOF.

$$\begin{aligned}
\mathcal{X} [f] \bigsqcap S & \Leftrightarrow \bigsqcap S \cap \langle f \rangle \mathcal{X} \neq \perp \Leftrightarrow \bigsqcap \langle \langle f \rangle \mathcal{X} \cap \rangle^* S \neq \perp \Leftrightarrow \\
& \text{(by properties of generalized filter bases)} \Leftrightarrow \\
& \exists \mathcal{Y} \in \langle \langle f \rangle \mathcal{X} \cap \rangle^* S : \mathcal{Y} \neq \perp \Leftrightarrow \exists \mathcal{Y} \in S : \langle f \rangle \mathcal{X} \cap \mathcal{Y} \neq \perp \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X} [f] \mathcal{Y}.
\end{aligned}$$

□

THEOREM 1620. A function $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$, where $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ are primary filtrators over boolean lattices, preserves finite joins (including nullary joins) and filtered meets iff there exists a pointfree functor f such that $\langle f \rangle = \varphi$.

PROOF. Backward implication follows from above.

Let $\psi = \varphi|_{\mathfrak{Z}_0}$. Then ψ preserves bottom element and binary joins. Thus there exists a functor f such that $\langle f \rangle^* = \psi$.

It remains to prove that $\langle f \rangle = \varphi$.

Really, $\langle f \rangle \mathcal{X} = \bigsqcap \langle \langle f \rangle \rangle^* \text{up } \mathcal{X} = \bigsqcap \langle \psi \rangle^* \text{up } \mathcal{X} = \bigsqcap \langle \varphi \rangle^* \text{up } \mathcal{X} = \varphi \bigsqcap \text{up } \mathcal{X} = \varphi \mathcal{X}$ for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$. □

COROLLARY 1621. Pointfree functors f from a lattice \mathfrak{A} of filters on a boolean lattice to a lattice \mathfrak{B} of filters on a boolean lattice bijectively correspond by the formula $\langle f \rangle = \varphi$ to functions $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ preserving finite joins and filtered meets.

THEOREM 1622. The set of pointfree functors between sets of filters on boolean lattices is a co-frame.

PROOF. Theorems 1616 and 533. □

20.4. The order of pointfree functors

DEFINITION 1623. The order of pointfree functors $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is defined by the formula:

$$f \sqsubseteq g \Leftrightarrow \forall x \in \mathfrak{A} : \langle f \rangle x \sqsubseteq \langle g \rangle x \wedge \forall y \in \mathfrak{B} : \langle f^{-1} \rangle y \sqsubseteq \langle g^{-1} \rangle y.$$

PROPOSITION 1624. It is really a partial order on the set $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROOF.

Reflexivity. Obvious.

Transitivity. It follows from transitivity of the order relations on \mathfrak{A} and \mathfrak{B} .

Antisymmetry. It follows from antisymmetry of the order relations on \mathfrak{A} and \mathfrak{B} . \square

REMARK 1625. It is enough to define order of pointfree funcoids on every set $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ where \mathfrak{A} and \mathfrak{B} are posets. We do not need to compare pointfree funcoids with different sources or destinations.

OBVIOUS 1626. $f \sqsubseteq g \Rightarrow [f] \sqsubseteq [g]$ for every $f, g \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ for every posets \mathfrak{A} and \mathfrak{B} .

THEOREM 1627. If \mathfrak{A} and \mathfrak{B} are separable posets then $f \sqsubseteq g \Leftrightarrow [f] \sqsubseteq [g]$.

PROOF. From the theorem 1601. \square

PROPOSITION 1628. If \mathfrak{A} and \mathfrak{B} have least elements, then $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ has least element.

PROOF. It is $(\perp_{\mathfrak{A}}, \perp_{\mathfrak{B}}, \perp_{\mathfrak{A}} \times \{\perp_{\mathfrak{B}}\}, \mathfrak{B} \times \{\perp_{\mathfrak{A}}\})$. \square

THEOREM 1629. If \mathfrak{A} and \mathfrak{B} are bounded posets, then $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is bounded.

PROOF. That $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ has least element was proved above. I will demonstrate that $(\perp_{\mathfrak{A}}, \perp_{\mathfrak{B}}, \alpha, \beta)$ is the greatest element of $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ for

$$\alpha X = \begin{cases} \perp_{\mathfrak{B}} & \text{if } X = \perp_{\mathfrak{A}} \\ \top_{\mathfrak{B}} & \text{if } X \neq \perp_{\mathfrak{A}} \end{cases}; \quad \beta Y = \begin{cases} \perp_{\mathfrak{A}} & \text{if } Y = \perp_{\mathfrak{B}} \\ \top_{\mathfrak{A}} & \text{if } Y \neq \perp_{\mathfrak{B}} \end{cases}.$$

First prove $Y \not\leq \alpha X \Leftrightarrow X \not\leq \beta Y$.

If $\top_{\mathfrak{B}} = \perp_{\mathfrak{B}}$ then $Y \not\leq \alpha X \Leftrightarrow Y \not\leq \perp_{\mathfrak{B}} \Leftrightarrow 0 \Leftrightarrow X \not\leq \perp_{\mathfrak{A}} \Leftrightarrow X \not\leq \beta \perp_{\mathfrak{A}}$ (proposition 1602). The case $\top_{\mathfrak{A}} = \perp_{\mathfrak{A}}$ is similar. So we can assume $\top_{\mathfrak{A}} \neq \perp_{\mathfrak{A}}$ and $\top_{\mathfrak{B}} \neq \perp_{\mathfrak{B}}$.

Consider all variants:

$X = \perp_{\mathfrak{A}}$ and $Y = \perp_{\mathfrak{B}}$. $Y \not\leq \alpha X \Leftrightarrow 0 \Leftrightarrow X \not\leq \beta Y$.

$X \neq \perp_{\mathfrak{A}}$ and $Y \neq \perp_{\mathfrak{B}}$. $\alpha X = \top_{\mathfrak{B}}$ and $\beta Y = \top_{\mathfrak{A}}$; $Y \not\leq \alpha X \Leftrightarrow Y \not\leq \top_{\mathfrak{B}} \Leftrightarrow 1 \Leftrightarrow X \not\leq \top_{\mathfrak{A}} \Leftrightarrow X \not\leq \beta Y$ (used that $\top_{\mathfrak{A}} \neq \perp_{\mathfrak{A}}$ and $\top_{\mathfrak{B}} \neq \perp_{\mathfrak{B}}$).

$X = \perp_{\mathfrak{A}}$ and $Y \neq \perp_{\mathfrak{B}}$. $\alpha X = \perp_{\mathfrak{B}}$ (proposition 1602) and $\beta Y = \top_{\mathfrak{A}}$; $Y \not\leq \alpha X \Leftrightarrow Y \not\leq \perp_{\mathfrak{B}} \Leftrightarrow 0 \Leftrightarrow \perp_{\mathfrak{A}} \not\leq \beta Y \Leftrightarrow X \not\leq \beta Y$.

$X \neq \perp_{\mathfrak{A}}$ and $Y \neq \perp_{\mathfrak{B}}$. Similar.

It's easy to show that both α and β are the greatest possible components of a pointfree funcoid taking into account proposition 1602. \square

THEOREM 1630. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then for $R \in \mathcal{P}\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $X \in \mathfrak{Z}_0, Y \in \mathfrak{Z}_1$ we have:

- 1°. $X \llbracket R \rrbracket Y \Leftrightarrow \exists f \in R : X \llbracket f \rrbracket Y$;
- 2°. $\langle \llbracket R \rrbracket X \rangle = \llbracket_{f \in R} \langle f \rangle X$.

PROOF.

2°. $\alpha X \stackrel{\text{def}}{=} \bigsqcup_{f \in R} \langle f \rangle X$ (by corollary 518 all joins on \mathfrak{B} exist). We have $\alpha \perp^{\mathfrak{A}} = \perp^{\mathfrak{B}}$;

$$\begin{aligned} \alpha(I \sqcup^{\mathfrak{A}_0} J) &= \\ \bigsqcup \left\{ \frac{\langle f \rangle (I \sqcup^{\mathfrak{A}_0} J)}{f \in R} \right\} &= \\ \bigsqcup \left\{ \frac{\langle f \rangle (I \sqcup^{\mathfrak{A}} J)}{f \in R} \right\} &= \\ \bigsqcup \left\{ \frac{\langle f \rangle I \sqcup^{\mathfrak{B}} \langle f \rangle J}{f \in R} \right\} &= \\ \bigsqcup \left\{ \frac{\langle f \rangle I}{f \in R} \right\} \sqcup^{\mathfrak{B}} \bigsqcup \left\{ \frac{\langle f \rangle J}{f \in R} \right\} &= \\ \alpha I \sqcup^{\mathfrak{B}} \alpha J & \end{aligned}$$

(used theorem 1604). By theorem 1616 the function α can be continued to $\langle h \rangle$ for an $h \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$. Obviously

$$\forall f \in R : h \sqsupseteq f. \quad (23)$$

And h is the least element of $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ for which the condition (23) holds. So $h = \bigsqcup R$.

1°.

$$\begin{aligned} X \left[\bigsqcup R \right] Y &\Leftrightarrow \\ Y \sqcap^{\mathfrak{B}} \left\langle \bigsqcup R \right\rangle X \neq \perp^{\mathfrak{B}} &\Leftrightarrow \\ Y \sqcap^{\mathfrak{B}} \bigsqcup \left\{ \frac{\langle f \rangle X}{f \in R} \right\} \neq \perp^{\mathfrak{B}} &\Leftrightarrow \\ \exists f \in R : Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \perp^{\mathfrak{B}} &\Leftrightarrow \\ \exists f \in R : X [f] Y & \end{aligned}$$

(used theorem 610). □

COROLLARY 1631. If $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_1)$ are primary filtrators over boolean lattices then $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a complete lattice.

PROOF. Apply [27]. □

THEOREM 1632. Let \mathfrak{A} and \mathfrak{B} be starrish join-semilattices. Then for $f, g \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$:

- 1°. $\langle f \sqcup g \rangle x = \langle f \rangle x \sqcup \langle g \rangle x$ for every $x \in \mathfrak{A}$;
- 2°. $[f \sqcup g] = [f] \cup [g]$.

PROOF. □

1°. Let $\alpha \mathcal{X} \stackrel{\text{def}}{=} \langle f \rangle x \sqcup \langle g \rangle x$; $\beta \mathcal{Y} \stackrel{\text{def}}{=} \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Then

$$\begin{aligned} y \not\prec^{\mathfrak{B}} \alpha x &\Leftrightarrow \\ y \not\prec \langle f \rangle x \vee y \not\prec \langle g \rangle x &\Leftrightarrow \\ x \not\prec \langle f^{-1} \rangle y \vee x \not\prec \langle g^{-1} \rangle y &\Leftrightarrow \\ x \not\prec \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y &\Leftrightarrow \\ x \not\prec \beta y. & \end{aligned}$$

So $h = (\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ is a pointfree funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some $p \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\langle p \rangle x \sqsupseteq \langle f \rangle x \sqcup \langle g \rangle x = \langle h \rangle x$ and $\langle p^{-1} \rangle y \sqsupseteq \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y = \langle h^{-1} \rangle y$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.

2°.

$$\begin{aligned} x [f \sqcup g] y &\Leftrightarrow \\ y \not\prec \langle f \sqcup g \rangle x &\Leftrightarrow \\ y \not\prec \langle f \rangle x \sqcup \langle g \rangle x &\Leftrightarrow \\ y \not\prec \langle f \rangle x \vee y \not\prec \langle g \rangle x &\Leftrightarrow \\ x [f] y \vee x [g] y & \end{aligned}$$

for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$.

20.5. Domain and range of a pointfree funcoid

DEFINITION 1633. Let \mathfrak{A} be a poset. The *identity pointfree funcoid* $1_{\mathfrak{A}}^{\mathbf{pFCD}} = (\mathfrak{A}, \mathfrak{A}, \text{id}_{\mathfrak{A}}, \text{id}_{\mathfrak{A}})$.

It is trivial that identity funcoid is really a pointfree funcoid.

Let now \mathfrak{A} be a meet-semilattice.

DEFINITION 1634. Let $a \in \mathfrak{A}$. The *restricted identity pointfree funcoid* $\text{id}_a^{\mathbf{pFCD}(\mathfrak{A})} = (\mathfrak{A}, \mathfrak{A}, a \sqcap^{\mathfrak{A}}, a \sqcap^{\mathfrak{A}})$.

PROPOSITION 1635. The restricted pointfree funcoid is a pointfree funcoid.

PROOF. We need to prove that $(a \sqcap^{\mathfrak{A}} x) \not\prec^{\mathfrak{A}} y \Leftrightarrow (a \sqcap^{\mathfrak{A}} y) \not\prec^{\mathfrak{A}} x$ what is obvious. \square

OBVIOUS 1636. $(\text{id}_a^{\mathbf{pFCD}(\mathfrak{A})})^{-1} = \text{id}_a^{\mathbf{pFCD}(\mathfrak{A})}$.

OBVIOUS 1637. $x \left[\text{id}_a^{\mathbf{pFCD}(\mathfrak{A})} \right] y \Leftrightarrow a \not\prec^{\mathfrak{A}} x \sqcap^{\mathfrak{A}} y$ for every $x, y \in \mathfrak{A}$.

DEFINITION 1638. I will define *restricting* of a pointfree funcoid f to an element $a \in \text{Src } f$ by the formula $f|_a \stackrel{\text{def}}{=} f \circ \text{id}_a^{\mathbf{pFCD}(\text{Src } f)}$.

DEFINITION 1639. Let f be a pointfree funcoid whose source is a set with greatest element. *Image* of f will be defined by the formula $\text{im } f = \langle f \rangle \top$.

PROPOSITION 1640. $\text{im } f \sqsupseteq \langle f \rangle x$ for every $x \in \text{Src } f$ whenever $\text{Dst } f$ is a strongly separable poset with greatest element.

PROOF. $\langle f \rangle \top$ is greater than every $\langle f \rangle x$ (where $x \in \text{Src } f$) by proposition 1603. \square

DEFINITION 1641. *Domain* of a pointfree funcoid f is defined by the formula $\text{dom } f = \text{im } f^{-1}$.

PROPOSITION 1642. $\langle f \rangle \text{dom } f = \text{im } f$ if f is a pointfree funcoid and $\text{Src } f$ is a strongly separable poset with greatest element and $\text{Dst } f$ is a separable poset with greatest element.

PROOF. For every $y \in \text{Dst } f$

$$y \not\prec \langle f \rangle \text{dom } f \Leftrightarrow \text{dom } f \not\prec \langle f^{-1} \rangle y \Leftrightarrow \langle f^{-1} \rangle \top \not\prec \langle f^{-1} \rangle y \Leftrightarrow$$

(by strong separability of $\text{Src } f$)

$$\langle f^{-1} \rangle y \text{ is not least} \Leftrightarrow \top \not\prec \langle f^{-1} \rangle y \Leftrightarrow y \not\prec \langle f \rangle \top \Leftrightarrow y \not\prec \text{im } f.$$

So $\langle f \rangle \text{dom } f = \text{im } f$ by separability of $\text{Dst } f$. \square

PROPOSITION 1643. $\langle f \rangle x = \langle f \rangle (x \sqcap \text{dom } f)$ for every $x \in \text{Src } f$ for a pointfree funcoid f whose source is a bounded separable meet-semilattice and destination is a bounded separable poset.

PROOF. $\text{Src } f$ is strongly separable by theorem 225. For every $y \in \text{Dst } f$ we have

$$\begin{aligned} y \not\leq \langle f \rangle (x \sqcap \text{dom } f) &\Leftrightarrow x \sqcap \text{dom } f \sqcap \langle f^{-1} \rangle y \neq \perp^{\text{Src } f} \Leftrightarrow \\ &x \sqcap \text{im } f^{-1} \sqcap \langle f^{-1} \rangle y \neq \perp^{\text{Src } f} \Leftrightarrow \\ &\text{(by strong separability of Src } f) \\ &x \sqcap \langle f^{-1} \rangle y \neq \perp^{\text{Src } f} \Leftrightarrow y \not\leq \langle f \rangle x. \end{aligned}$$

Thus $\langle f \rangle x = \langle f \rangle (x \sqcap \text{dom } f)$ by separability of $\text{Dst } f$. \square

PROPOSITION 1644. $x \not\leq \text{dom } f \Leftrightarrow (\langle f \rangle x \text{ is not least})$ for every pointfree funcoid f and $x \in \text{Src } f$ if $\text{Dst } f$ has greatest element \top .

PROOF. $x \not\leq \text{dom } f \Leftrightarrow x \not\leq \langle f^{-1} \rangle \top^{\text{Dst } f} \Leftrightarrow \top^{\text{Dst } f} \not\leq \langle f \rangle x \Leftrightarrow (\langle f \rangle x \text{ is not least})$. \square

PROPOSITION 1645. $\text{dom } f = \bigsqcup \left\{ \frac{a \in \text{atoms}^{\text{Src } f}}{\langle f \rangle a \neq \perp^{\text{Dst } f}} \right\}$ for every pointfree funcoid f whose destination is a bounded strongly separable poset and source is an atomistic poset.

PROOF. For every $a \in \text{atoms}^{\text{Src } f}$ we have

$$a \not\leq \text{dom } f \Leftrightarrow a \not\leq \langle f^{-1} \rangle \top^{\text{Dst } f} \Leftrightarrow \top^{\text{Dst } f} \not\leq \langle f \rangle a \Leftrightarrow \langle f \rangle a \neq \perp^{\text{Dst } f}.$$

So $\text{dom } f = \bigsqcup \left\{ \frac{a \in \text{atoms}^{\text{Src } f}}{a \not\leq \text{dom } f} \right\} = \bigsqcup \left\{ \frac{a \in \text{atoms}^{\text{Src } f}}{\langle f \rangle a \neq \perp^{\text{Dst } f}} \right\}$. \square

PROPOSITION 1646. $\text{dom}(f|_a) = a \sqcap \text{dom } f$ for every pointfree funcoid f and $a \in \text{Src } f$ where $\text{Src } f$ is a meet-semilattice and $\text{Dst } f$ has greatest element.

PROOF.

$$\begin{aligned} \text{dom}(f|_a) &= \text{im}(\text{id}_a^{\text{PFCd}(\text{Src } f)} \circ f^{-1}) = \\ &\langle \text{id}_a^{\text{PFCd}(\text{Src } f)} \rangle \langle f^{-1} \rangle \top^{\text{Dst } f} = a \sqcap \langle f^{-1} \rangle \top^{\text{Dst } f} = a \sqcap \text{dom } f. \end{aligned}$$

\square

PROPOSITION 1647. For every composable pointfree funcoids f and g

- 1°. If $\text{im } f \sqsupseteq \text{dom } g$ then $\text{im}(g \circ f) = \text{im } g$, provided that the posets $\text{Src } f$, $\text{Dst } f = \text{Src } g$ and $\text{Dst } g$ have greatest elements and $\text{Src } g$ and $\text{Dst } g$ are strongly separable.
- 2°. If $\text{im } f \sqsubseteq \text{dom } g$ then $\text{dom}(g \circ f) = \text{dom } g$, provided that the posets $\text{Dst } g$, $\text{Dst } f = \text{Src } g$ and $\text{Src } f$ have greatest elements and $\text{Dst } f$ and $\text{Src } f$ are strongly separable.

PROOF.

1°. $\text{im}(g \circ f) = \langle g \circ f \rangle \top^{\text{Src } f} = \langle g \rangle \langle f \rangle \top^{\text{Src } f} \sqsubseteq \text{im } g$ by strong separability of $\text{Dst } g$; $\text{im}(g \circ f) = \langle g \circ f \rangle \top^{\text{Src } f} = \langle g \rangle \text{im } f \sqsupseteq \langle g \rangle \text{dom } g = \text{im } g$ by strong separability of $\text{Dst } g$ and proposition 1642.

2°. $\text{dom}(g \circ f) = \text{im}(f^{-1} \circ g^{-1})$ what by the proved is equal to $\text{im } f^{-1}$ that is $\text{dom } f$. \square

20.6. Specifying functors by functions or relations on atomic filters

THEOREM 1648. Let \mathfrak{A} be an atomic poset and $(\mathfrak{B}, \mathfrak{F}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \text{atoms}^{\mathfrak{A}} \mathcal{X}.$$

PROOF. For every $Y \in \mathfrak{F}_1$ we have

$$\begin{aligned} Y \not\prec^{\mathfrak{B}} \langle f \rangle \mathcal{X} &\Leftrightarrow \mathcal{X} \not\prec^{\mathfrak{A}} \langle f^{-1} \rangle Y \Leftrightarrow \\ &\Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{A}} \mathcal{X} : x \not\prec^{\mathfrak{A}} \langle f^{-1} \rangle Y \Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{A}} \mathcal{X} : Y \not\prec^{\mathfrak{B}} \langle f \rangle x. \end{aligned}$$

Thus $\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle^* \langle \langle f \rangle \rangle^* \text{atoms}^{\mathfrak{A}} \mathcal{X} = \partial \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \text{atoms}^{\mathfrak{A}} \mathcal{X}$ (used corollary 569). Consequently $\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \text{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 568. \square

PROPOSITION 1649. Let f be a pointfree functor. Then for every $\mathcal{X} \in \text{Src } f$ and $\mathcal{Y} \in \text{Dst } f$

- 1°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \text{atoms } \mathcal{X} : x [f] \mathcal{Y}$ if $\text{Src } f$ is an atomic poset.
- 2°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists y \in \text{atoms } \mathcal{Y} : \mathcal{X} [f] y$ if $\text{Dst } f$ is an atomic poset.

PROOF. I will prove only the second as the first is similar.

If $\mathcal{X} [f] \mathcal{Y}$, then $\mathcal{Y} \not\prec \langle f \rangle \mathcal{X}$, consequently exists $y \in \text{atoms } \mathcal{Y}$ such that $y \not\prec \langle f \rangle \mathcal{X}$, $\mathcal{X} [f] y$. The reverse is obvious. \square

COROLLARY 1650. If f is a pointfree functor with both source and destination being atomic posets, then for every $\mathcal{X} \in \text{Src } f$ and $\mathcal{Y} \in \text{Dst } f$

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \text{atoms } \mathcal{X}, y \in \text{atoms } \mathcal{Y} : x [f] y.$$

PROOF. Apply the theorem twice. \square

COROLLARY 1651. If \mathfrak{A} is a separable atomic poset and \mathfrak{B} is a separable poset then $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is determined by the values of $\langle f \rangle X$ for $X \in \text{atoms}^{\mathfrak{A}}$.

PROOF.

$$\begin{aligned} y \not\prec \langle f \rangle x &\Leftrightarrow x \not\prec \langle f^{-1} \rangle y \Leftrightarrow \\ &\Leftrightarrow \exists X \in \text{atoms } x : X \not\prec \langle f^{-1} \rangle y \Leftrightarrow \\ &\Leftrightarrow \exists X \in \text{atoms } x : y \not\prec \langle f \rangle X. \end{aligned}$$

Thus by separability of \mathfrak{B} we have $\langle f \rangle$ is determined by $\langle f \rangle X$ for $X \in \text{atoms } x$.

By separability of \mathfrak{A} we infer that f can be restored from $\langle f \rangle$ (theorem 1601). \square

THEOREM 1652. Let $(\mathfrak{A}, \mathfrak{F}_0)$ and $(\mathfrak{B}, \mathfrak{F}_1)$ be primary filtrators over boolean lattices.

- 1°. A function $\alpha \in \mathfrak{B}^{\text{atoms}^{\mathfrak{A}}}$ such that (for every $a \in \text{atoms}^{\mathfrak{A}}$)

$$\alpha a \sqsubseteq \prod \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \text{atoms}^{\mathfrak{A}} \right\rangle^* \text{up}^{\mathfrak{F}_0} a \quad (24)$$

can be continued to the function $\langle f \rangle$ for a unique $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\langle f \rangle \mathcal{X} = \bigsqcup \langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} \mathcal{X} \quad (25)$$

for every $\mathcal{X} \in \mathfrak{A}$.

2°. A relation $\delta \in \mathcal{P}(\text{atoms}^{\mathfrak{A}} \times \text{atoms}^{\mathfrak{B}})$ such that (for every $a \in \text{atoms}^{\mathfrak{A}}$, $b \in \text{atoms}^{\mathfrak{B}}$)

$$\forall X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y \Rightarrow a \delta b \quad (26)$$

can be continued to the relation $[f]$ for a unique $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \text{atoms} \mathcal{X}, y \in \text{atoms} \mathcal{Y} : x \delta y \quad (27)$$

for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.

PROOF. Existence of no more than one such funcoids and formulas (25) and (27) follow from theorem 1648 and corollary 1650 and the fact that our filtrators are separable.

1°. Consider the function $\alpha' \in \mathfrak{B}^{\mathfrak{A}_0}$ defined by the formula (for every $X \in \mathfrak{A}_0$)

$$\alpha' X = \bigsqcup \langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} X.$$

Obviously $\alpha' \perp^{\mathfrak{A}_0} = \perp^{\mathfrak{B}}$. For every $I, J \in \mathfrak{A}_0$

$$\begin{aligned} \alpha'(I \sqcup J) &= \\ \bigsqcup \langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} (I \sqcup J) &= \\ \bigsqcup \langle \alpha \rangle^* (\text{atoms}^{\mathfrak{A}} I \cup \text{atoms}^{\mathfrak{A}} J) &= \\ \bigsqcup (\langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} I \cup \langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} J) &= \\ \bigsqcup \langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} I \sqcup \bigsqcup \langle \alpha \rangle^* \text{atoms}^{\mathfrak{A}} J &= \\ \alpha' I \sqcup \alpha' J. & \end{aligned}$$

Let continue α' till a pointfree funcoid f (by the theorem 1616): $\langle f \rangle \mathcal{X} = \bigsqcup \langle \alpha' \rangle^* \text{up}^{\mathfrak{A}_0} \mathcal{X}$.

Let's prove the reverse of (24):

$$\begin{aligned} \bigsqcap \langle \bigsqcup \langle \alpha \rangle^* \circ \text{atoms}^{\mathfrak{A}} \rangle^* \text{up}^{\mathfrak{A}_0} a &= \\ \bigsqcap \langle \bigsqcup \langle \alpha \rangle^* \rangle^* \langle \text{atoms}^{\mathfrak{A}} \rangle^* \text{up}^{\mathfrak{A}_0} a &\sqsubseteq \\ \bigsqcap \langle \bigsqcup \langle \alpha \rangle^* \rangle^* \{\{a\}\} &= \\ \bigsqcap \{(\bigsqcup \langle \alpha \rangle^*) \{a\}\} &= \\ \bigsqcap \{\bigsqcup \langle \alpha \rangle^* \{a\}\} &= \\ \bigsqcap \{\bigsqcup \{\alpha a\}\} &= \\ \bigsqcap \{\alpha a\} &= \alpha a. \end{aligned}$$

Finally,

$$\alpha a = \bigsqcap \langle \bigsqcup \langle \alpha \rangle^* \circ \text{atoms}^{\mathfrak{A}} \rangle^* \text{up}^{\mathfrak{A}_0} a = \bigsqcap \langle \alpha' \rangle^* \text{up}^{\mathfrak{A}_0} a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α .

2°. Consider the relation $\delta' \in \mathcal{P}(\mathfrak{A}_0 \times \mathfrak{A}_1)$ defined by the formula (for every $X \in \mathfrak{A}_0$, $Y \in \mathfrak{A}_1$)

$$X \delta' Y \Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y.$$

Obviously $\neg(X \delta' \perp^{\mathfrak{A}_1})$ and $\neg(\perp^{\mathfrak{B}_0} \delta' Y)$.

$$\begin{aligned} I \sqcup J \delta' Y &\Leftrightarrow \\ \exists x \in \text{atoms}^{\mathfrak{A}}(I \sqcup J), y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y &\Leftrightarrow \\ \exists x \in \text{atoms}^{\mathfrak{A}} I \cup \text{atoms}^{\mathfrak{A}} J, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y &\Leftrightarrow \\ \exists x \in \text{atoms}^{\mathfrak{A}} I, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y \vee \exists x \in \text{atoms}^{\mathfrak{A}} J, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y &\Leftrightarrow \\ I \delta' Y \vee J \delta' Y; & \end{aligned}$$

similarly $X \delta' I \sqcup J \Leftrightarrow X \delta' I \vee X \delta' J$. Let's continue δ' till a funcoid f (by the theorem 1616):

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \text{up}^{\mathfrak{A}_0} \mathcal{X}, Y \in \text{up}^{\mathfrak{B}_1} \mathcal{Y} : X \delta' Y.$$

The reverse of (26) implication is trivial, so

$$\begin{aligned} \forall X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y &\Leftrightarrow a \delta b; \\ \forall X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y &\Leftrightarrow \\ \forall X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b : X \delta' Y &\Leftrightarrow \\ a [f] b. & \end{aligned}$$

So $a \delta b \Leftrightarrow a [f] b$, that is $[f]$ is a continuation of δ . □

THEOREM 1653. Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_1)$ be primary filtrators over boolean lattices. If $R \in \mathcal{P}\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $x \in \text{atoms}^{\mathfrak{A}}, y \in \text{atoms}^{\mathfrak{B}}$, then

- 1°. $\langle \sqcap R \rangle x = \sqcap_{f \in R} \langle f \rangle x$;
- 2°. $x \langle \sqcap R \rangle y \Leftrightarrow \forall f \in R : x [f] y$.

PROOF.

2°. Let denote $x \delta y \Leftrightarrow \forall f \in R : x [f] y$.

$$\begin{aligned} \forall X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y : x \delta y &\Rightarrow \\ \forall f \in R, X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y : x [f] y &\Rightarrow \\ \forall f \in R, X \in \text{up}^{\mathfrak{A}_0} a, Y \in \text{up}^{\mathfrak{B}_1} b : X [f] Y &\Rightarrow \\ \forall f \in R : a [f] b &\Leftrightarrow \\ a \delta b. & \end{aligned}$$

So by theorem 1652, δ can be continued till $[p]$ for some $p \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

For every $q \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ such that $\forall f \in R : q \sqsubseteq f$ we have $x [q] y \Rightarrow \forall f \in R : x [f] y \Leftrightarrow x \delta y \Leftrightarrow x [p] y$, so $q \sqsubseteq p$. Consequently $p = \sqcap R$.

From this $x \langle \sqcap R \rangle y \Leftrightarrow \forall f \in R : x [f] y$.

1°. From the former

$$\begin{aligned} y \in \text{atoms}^{\mathfrak{B}} \langle \sqcap R \rangle x &\Leftrightarrow y \cap \langle \sqcap R \rangle x \neq \perp^{\mathfrak{B}} \Leftrightarrow \forall f \in R : y \cap \langle f \rangle x \neq \perp^{\mathfrak{B}} \Leftrightarrow \\ y \in \bigcap \langle \text{atoms}^{\mathfrak{B}} \rangle^* \left\{ \frac{\langle f \rangle x}{f \in R} \right\} &\Leftrightarrow y \in \text{atoms} \sqcap \left\{ \frac{\langle f \rangle x}{f \in R} \right\} \end{aligned}$$

for every $y \in \text{atoms}^{\mathfrak{B}}$.

\mathfrak{B} is atomically separable by the corollary 582. Thus $\langle \sqcap R \rangle x = \sqcap_{f \in R} \langle f \rangle x$. □

20.7. More on composition of pointfree funcoids

PROPOSITION 1654. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for every composable pointfree funcoids f and g .

PROOF. For every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$

$$x [g \circ f] y \Leftrightarrow y \not\prec \langle g \circ f \rangle x \Leftrightarrow y \not\prec \langle g \rangle \langle f \rangle x \Leftrightarrow \langle f \rangle x [g] y \Leftrightarrow x ([g] \circ \langle f \rangle) y.$$

Thus $[g \circ f] = [g] \circ \langle f \rangle$.

$$[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} = \langle g^{-1} \rangle^{-1} \circ [f].$$

□

THEOREM 1655. Let f and g be pointfree funcoids and $\mathfrak{A} = \text{Dst } f = \text{Src } g$ be an atomic poset. Then for every $\mathcal{X} \in \text{Src } f$ and $\mathcal{Z} \in \text{Dst } g$

$$\mathcal{X} [g \circ f] \mathcal{Z} \Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{A}} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}).$$

PROOF.

$$\begin{aligned} \exists y \in \text{atoms}^{\mathfrak{A}} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\prec \langle g \rangle y \wedge y \not\prec \langle f \rangle \mathcal{X}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathfrak{A}} : (y \not\prec \langle g^{-1} \rangle \mathcal{Z} \wedge y \not\prec \langle f \rangle \mathcal{X}) &\Leftrightarrow \\ \langle g^{-1} \rangle \mathcal{Z} \not\prec \langle f \rangle \mathcal{X} &\Leftrightarrow \\ \mathcal{X} [g \circ f] \mathcal{Z}. & \end{aligned}$$

□

THEOREM 1656. Let \mathfrak{A} , \mathfrak{B} , \mathfrak{C} be separable starrish join-semilattices and \mathfrak{B} is atomic. Then:

- 1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \in \text{pFCD}(\mathfrak{B}, \mathfrak{C})$.
- 2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $g, h \in \text{pFCD}(\mathfrak{B}, \mathfrak{C})$.

PROOF. I will prove only the first equality because the other is analogous. We can apply theorem 1632.

For every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{C}$

$$\begin{aligned} \mathcal{X} [f \circ (g \sqcup h)] \mathcal{Z} &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathfrak{B}} : (\mathcal{X} [g \sqcup h] y \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathfrak{B}} : ((\mathcal{X} [g] y \vee \mathcal{X} [h] y) \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathfrak{B}} : ((\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee (\mathcal{X} [h] y \wedge y [f] \mathcal{Z})) &\Leftrightarrow \\ \exists y \in \text{atoms}^{\mathfrak{B}} : (\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee \exists y \in \text{atoms}^{\mathfrak{B}} : (\mathcal{X} [h] y \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \mathcal{X} [f \circ g] \mathcal{Z} \vee \mathcal{X} [f \circ h] \mathcal{Z} &\Leftrightarrow \\ \mathcal{X} [f \circ g \sqcup f \circ h] \mathcal{Z}. & \end{aligned}$$

Thus $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ by theorem 1601.

□

THEOREM 1657. Let \mathfrak{A} , \mathfrak{B} , \mathfrak{C} be posets of filters over some boolean lattices, $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$, $g \in \text{pFCD}(\mathfrak{B}, \mathfrak{C})$, $h \in \text{pFCD}(\mathfrak{A}, \mathfrak{C})$. Then

$$g \circ f \not\prec h \Leftrightarrow g \not\prec h \circ f^{-1}.$$

PROOF.

$$\begin{aligned}
& g \circ f \neq h \Leftrightarrow \\
& \exists a \in \text{atoms}^{\mathfrak{A}}, c \in \text{atoms}^{\mathfrak{C}} : a [(g \circ f) \sqcap h] c \Leftrightarrow \\
& \exists a \in \text{atoms}^{\mathfrak{A}}, c \in \text{atoms}^{\mathfrak{C}} : (a [g \circ f] c \wedge a [h] c) \Leftrightarrow \\
& \exists a \in \text{atoms}^{\mathfrak{A}}, b \in \text{atoms}^{\mathfrak{B}}, c \in \text{atoms}^{\mathfrak{C}} : (a [f] b \wedge b [g] c \wedge a [h] c) \Leftrightarrow \\
& \exists b \in \text{atoms}^{\mathfrak{B}}, c \in \text{atoms}^{\mathfrak{C}} : (b [g] c \wedge b [h \circ f^{-1}] c) \Leftrightarrow \\
& \exists b \in \text{atoms}^{\mathfrak{B}}, c \in \text{atoms}^{\mathfrak{C}} : b [g \sqcap (h \circ f^{-1})] c \Leftrightarrow \\
& g \neq h \circ f^{-1}.
\end{aligned}$$

□

20.8. Funcoidal product of elements

DEFINITION 1658. *Funcoidal product* $\mathcal{A} \times^{\text{FCD}} \mathcal{B}$ where $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ and \mathfrak{A} and \mathfrak{B} are posets with least elements is a pointfree funcoid such that for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$

$$\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \neq \mathcal{A}; \\ \perp^{\mathfrak{B}} & \text{if } \mathcal{X} \simeq \mathcal{A}; \end{cases} \quad \text{and} \quad \langle (\mathcal{A} \times^{\text{FCD}} \mathcal{B})^{-1} \rangle \mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \neq \mathcal{B}; \\ \perp^{\mathfrak{A}} & \text{if } \mathcal{Y} \simeq \mathcal{B}. \end{cases}$$

PROPOSITION 1659. $\mathcal{A} \times^{\text{FCD}} \mathcal{B}$ is really a pointfree funcoid and

$$\mathcal{X} [\mathcal{A} \times^{\text{FCD}} \mathcal{B}] \mathcal{Y} \Leftrightarrow \mathcal{X} \neq \mathcal{A} \wedge \mathcal{Y} \neq \mathcal{B}.$$

PROOF. Obvious. □

PROPOSITION 1660. Let \mathfrak{A} and \mathfrak{B} be posets with least elements, $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Rightarrow \text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}.$$

PROOF. If $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ then $\text{dom } f \sqsubseteq \text{dom}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$, $\text{im } f \sqsubseteq \text{im}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \sqsubseteq \mathcal{B}$. □

THEOREM 1661. Let \mathfrak{A} and \mathfrak{B} be strongly separable bounded posets, $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Leftrightarrow \text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}.$$

PROOF. One direction is the proposition above. The other:

If $\text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$ then $\mathcal{X} [f] \mathcal{Y} \Rightarrow \mathcal{Y} \neq \langle f \rangle \mathcal{X} \Rightarrow \mathcal{Y} \neq \text{im } f \Rightarrow \mathcal{Y} \neq \mathcal{B}$ (strong separability used) and similarly $\mathcal{X} [f] \mathcal{Y} \Rightarrow \mathcal{X} \neq \mathcal{A}$.

So $[f] \subseteq [\mathcal{A} \times^{\text{FCD}} \mathcal{B}]$ and thus using separability $f \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$. □

THEOREM 1662. Let \mathfrak{A} , \mathfrak{B} be bounded separable meet-semilattices. For every $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$

$$f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \text{id}_{\mathfrak{B}}^{\text{pFCD}(\mathfrak{B})} \circ f \circ \text{id}_{\mathfrak{A}}^{\text{pFCD}(\mathfrak{A})}.$$

PROOF. $h \stackrel{\text{def}}{=} \text{id}_{\mathfrak{B}}^{\text{pFCD}(\mathfrak{B})} \circ f \circ \text{id}_{\mathfrak{A}}^{\text{pFCD}(\mathfrak{A})}$. For every $\mathcal{X} \in \mathfrak{A}$

$$\langle h \rangle \mathcal{X} = \langle \text{id}_{\mathfrak{B}}^{\text{pFCD}(\mathfrak{B})} \rangle \langle f \rangle \langle \text{id}_{\mathfrak{A}}^{\text{pFCD}(\mathfrak{A})} \rangle \mathcal{X} = \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X})$$

and

$$\langle h^{-1} \rangle \mathcal{X} = \langle \text{id}_{\mathfrak{A}}^{\text{pFCD}(\mathfrak{A})} \rangle \langle f^{-1} \rangle \langle \text{id}_{\mathfrak{B}}^{\text{pFCD}(\mathfrak{B})} \rangle \mathcal{X} = \mathcal{A} \sqcap \langle f^{-1} \rangle (\mathcal{B} \sqcap \mathcal{X}).$$

From this, as easy to show, $h \sqsubseteq f$ and $h \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$. If $g \sqsubseteq f \wedge g \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ for a $g \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\text{dom } g \sqsubseteq \mathcal{A}$. \mathfrak{A} and \mathfrak{B} are strongly separable by theorem 225. Thus by propositions 1643 we have:

$$\begin{aligned} \langle g \rangle \mathcal{X} &= \langle g \rangle (\mathcal{X} \sqcap \text{dom } g) = \langle g \rangle (\mathcal{X} \sqcap \mathcal{A}) = \mathcal{B} \sqcap \langle g \rangle (\mathcal{A} \sqcap \mathcal{X}) \sqsubseteq \\ &\quad \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X}) = \left\langle \text{id}_{\mathcal{B}}^{\text{pFCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle \text{id}_{\mathcal{A}}^{\text{pFCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X}, \end{aligned}$$

and similarly $\langle g^{-1} \rangle \mathcal{Y} \sqsubseteq \langle h^{-1} \rangle \mathcal{Y}$. Thus $g \sqsubseteq h$.

$$\text{So } h = f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}). \quad \square$$

COROLLARY 1663. Let $\mathfrak{A}, \mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$ we have $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\text{FCD}} \top^{\mathfrak{B}})$.

$$\text{PROOF. } f \sqcap (\mathcal{A} \times^{\text{FCD}} \top^{\mathfrak{B}}) = \text{id}_{\top^{\mathfrak{B}}}^{\text{pFCD}(\mathfrak{B})} \circ f \circ \text{id}_{\mathcal{A}}^{\text{pFCD}(\mathfrak{A})} = f \circ \text{id}_{\mathcal{A}}^{\text{pFCD}(\mathfrak{A})} = f|_{\mathcal{A}}. \quad \square$$

COROLLARY 1664. Let $\mathfrak{A}, \mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}, \mathcal{B} \in \mathfrak{B}$ we have

$$f \not\leq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Leftrightarrow \mathcal{A} [f] \mathcal{B}.$$

PROOF. Existence of $f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B})$ follows from the above theorem.

$$\begin{aligned} f \not\leq \mathcal{A} \times^{\text{FCD}} \mathcal{B} &\Leftrightarrow \\ f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) &\neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})} \Leftrightarrow \\ \langle f \sqcap (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \rangle &\top^{\mathfrak{A}} \neq \perp^{\mathfrak{B}} \Leftrightarrow \\ \left\langle \text{id}_{\mathcal{B}}^{\text{pFCD}(\mathfrak{B})} \circ f \circ \text{id}_{\mathcal{A}}^{\text{pFCD}(\mathfrak{A})} \right\rangle &\top^{\mathfrak{A}} \neq \perp^{\mathfrak{B}} \Leftrightarrow \\ \left\langle \text{id}_{\mathcal{B}}^{\text{pFCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle \text{id}_{\mathcal{A}}^{\text{pFCD}(\mathfrak{A})} \right\rangle &\top^{\mathfrak{A}} \neq \perp^{\mathfrak{B}} \Leftrightarrow \\ \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \top^{\mathfrak{A}}) &\neq \perp^{\mathfrak{B}} \Leftrightarrow \\ \mathcal{B} \sqcap \langle f \rangle \mathcal{A} &\neq \perp^{\mathfrak{B}} \Leftrightarrow \\ &\mathcal{A} [f] \mathcal{B}. \end{aligned}$$

□

THEOREM 1665. Let $\mathfrak{A}, \mathfrak{B}$ be bounded separable meet-semilattices. Then the poset $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is separable.

PROOF. Let $f, g \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \neq g$. By the theorem 1601 $[f] \neq [g]$. That is there exist $x, y \in \mathfrak{A}$ such that $x [f] y \Leftrightarrow x [g] y$ that is $f \sqcap (x \times^{\text{FCD}} y) \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})} \Leftrightarrow g \sqcap (x \times^{\text{FCD}} y) \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})}$. Thus $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is separable. □

COROLLARY 1666. Let $\mathfrak{A}, \mathfrak{B}$ be atomic bounded separable meet-semilattices. The poset $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is:

- 1°. separable;
- 2°. strongly separable;
- 3°. atomically separable;
- 4°. conforming to Wallman's disjunction property.

PROOF. By the theorem 233. □

REMARK 1667. For more ways to characterize (atomic) separability of the lattice of pointfree functors see subsections “**Separation subsets and full stars**” and “**Atomically Separable Lattices**”.

COROLLARY 1668. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. The poset $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is an atomistic lattice.

PROOF. By the corollary 1631 $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a complete lattice. We can use theorem 231. \square

THEOREM 1669. Let \mathfrak{A} and \mathfrak{B} be posets of filters over boolean lattices. If $S \in \mathcal{P}(\mathfrak{A} \times \mathfrak{B})$ then

$$\prod_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \prod \text{dom } S \times^{\text{FCD}} \prod \text{im } S.$$

PROOF. If $x \in \text{atoms}^{\mathfrak{A}}$ then by the theorem 1653

$$\left\langle \prod_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \right\rangle x = \prod \left\{ \frac{\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\}.$$

If $x \sqcap \prod \text{dom } S \neq \perp^{\mathfrak{A}}$ then

$$\begin{aligned} \forall (\mathcal{A}, \mathcal{B}) \in S : (x \sqcap \mathcal{A} \neq \perp^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x = \mathcal{B}); \\ \left\{ \frac{\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\} = \text{im } S; \end{aligned}$$

if $x \sqcap \prod \text{dom } S = \perp^{\mathfrak{A}}$ then

$$\begin{aligned} \exists (\mathcal{A}, \mathcal{B}) \in S : (x \sqcap \mathcal{A} = \perp^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x = \perp^{\mathfrak{B}}); \\ \left\{ \frac{\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\} \ni \perp^{\mathfrak{B}}. \end{aligned}$$

So

$$\left\langle \prod_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \right\rangle x = \begin{cases} \prod \text{im } S & \text{if } x \sqcap \prod \text{dom } S \neq \perp^{\mathfrak{A}}; \\ \perp^{\mathfrak{B}} & \text{if } x \sqcap \prod \text{dom } S = \perp^{\mathfrak{A}}. \end{cases}$$

From this by theorem 1652 the statement of our theorem follows. \square

COROLLARY 1670. Let \mathfrak{A} and \mathfrak{B} be posets of filters over boolean lattices. For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{A}$ and $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{B}$

$$(\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\text{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

PROOF. $(\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1) = \prod \{ \mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 \}$ what is by the last theorem equal to $(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\text{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1)$. \square

THEOREM 1671. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If $\mathcal{A} \in \mathfrak{A}$ then $\mathcal{A} \times^{\text{FCD}}$ is a complete homomorphism from the lattice \mathfrak{A} to the lattice $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$, if also $\mathcal{A} \neq \perp^{\mathfrak{A}}$ then it is an order embedding.

PROOF. Let $S \in \mathcal{P}\mathfrak{A}$, $X \in \mathfrak{Z}_0$, $x \in \text{atoms}^{\mathfrak{A}}$.

$$\begin{aligned} \left\langle \bigsqcup \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S \right\rangle X &= \\ \bigsqcup_{\mathcal{B} \in S} \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle X &= \\ \begin{cases} \bigsqcup S & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} \neq \perp^{\mathfrak{A}} \\ \perp^{\mathfrak{B}} & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} = \perp^{\mathfrak{A}} \end{cases} &= \\ \langle \mathcal{A} \times^{\text{FCD}} \bigsqcup S \rangle X. & \end{aligned}$$

Thus $\sqcup \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S = \mathcal{A} \times^{\text{FCD}} \sqcup S$ by theorem 1615.

$$\begin{aligned} \langle \sqcup \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S \rangle x &= \\ \sqcup_{\mathcal{B} \in S} \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x &= \\ \begin{cases} \sqcup S & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} \neq \perp^{\mathfrak{A}} \\ \perp^{\mathfrak{B}} & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} = \perp^{\mathfrak{A}} \end{cases} &= \\ \langle \mathcal{A} \times^{\text{FCD}} \sqcup S \rangle x. & \end{aligned}$$

Thus $\sqcap \langle \mathcal{A} \times^{\text{FCD}} \rangle^* S = \mathcal{A} \times^{\text{FCD}} \sqcap S$ by theorem 1648.

If $\mathcal{A} \neq \perp^{\mathfrak{A}}$ then obviously $\mathcal{A} \times^{\text{FCD}} \mathcal{X} \sqsubseteq \mathcal{A} \times^{\text{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$, because $\text{im}(\mathcal{A} \times^{\text{FCD}} \mathcal{X}) = \mathcal{X}$ and $\text{im}(\mathcal{A} \times^{\text{FCD}} \mathcal{Y}) = \mathcal{Y}$. \square

PROPOSITION 1672. Let \mathfrak{A} be a meet-semilattice with least element and \mathfrak{B} be a poset with least element. If a is an atom of \mathfrak{A} , $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $f|_a = a \times^{\text{FCD}} \langle f \rangle a$.

PROOF. Let $\mathcal{X} \in \mathfrak{A}$.

$$\mathcal{X} \sqcap a \neq \perp^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \sqcap a = \perp^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \perp^{\mathfrak{B}}.$$

\square

PROPOSITION 1673. $f \circ (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathcal{A} \times^{\text{FCD}} \langle f \rangle \mathcal{B}$ for elements $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{B} \in \mathfrak{B}$ of some posets $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ with least elements and $f \in \text{pFCD}(\mathfrak{B}, \mathfrak{C})$.

PROOF. Let $\mathcal{X} \in \mathfrak{A}, \mathcal{Y} \in \mathfrak{B}$.

$$\begin{aligned} \langle f \circ (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \rangle \mathcal{X} &= \left(\begin{cases} \langle f \rangle \mathcal{B} & \text{if } \mathcal{X} \neq \mathcal{A} \\ \perp & \text{if } \mathcal{X} \simeq \mathcal{A} \end{cases} \right) = \langle \mathcal{A} \times^{\text{FCD}} \langle f \rangle \mathcal{B} \rangle \mathcal{X}; \\ \langle (f \circ (\mathcal{A} \times^{\text{FCD}} \mathcal{B}))^{-1} \rangle \mathcal{Y} &= \\ \langle (\mathcal{B} \times^{\text{FCD}} \mathcal{A}) \circ f^{-1} \rangle \mathcal{Y} &= \\ \left(\begin{cases} \mathcal{A} & \text{if } \langle f^{-1} \rangle \mathcal{Y} \neq \mathcal{B} \\ \perp & \text{if } \langle f^{-1} \rangle \mathcal{Y} \simeq \mathcal{B} \end{cases} \right) &= \\ \left(\begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \neq \langle f \rangle \mathcal{B} \\ \perp & \text{if } \mathcal{Y} \simeq \langle f \rangle \mathcal{B} \end{cases} \right) &= \\ \langle \langle f \rangle \mathcal{B} \times^{\text{FCD}} \mathcal{A} \rangle \mathcal{Y} &= \\ \langle (\mathcal{A} \times^{\text{FCD}} \langle f \rangle \mathcal{B})^{-1} \rangle \mathcal{Y}. & \end{aligned}$$

\square

20.9. Category of pointfree functors

I will define the category pFCD of pointfree functors:

- The class of objects are small posets.
- The set of morphisms from \mathfrak{A} to \mathfrak{B} is $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$.
- The composition is the composition of pointfree functors.
- Identity morphism for an object \mathfrak{A} is $(\mathfrak{A}, \mathfrak{A}, \text{id}_{\mathfrak{A}}, \text{id}_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

The *category of pointfree functor quintuples* is defined as follows:

- Objects are pairs $(\mathfrak{A}, \mathcal{A})$ where \mathfrak{A} is a small meet-semilattice and $\mathcal{A} \in \mathfrak{A}$.

- The morphisms from an object $(\mathfrak{A}, \mathcal{A})$ to an object $(\mathfrak{B}, \mathcal{B})$ are tuples $(\mathfrak{A}, \mathfrak{B}, \mathcal{A}, \mathcal{B}, f)$ where $f \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and

$$\forall x \in \mathfrak{A} : \langle f \rangle x \sqsubseteq \mathcal{A}, \quad \forall y \in \mathfrak{B} : \langle f^{-1} \rangle y \sqsubseteq \mathcal{B}. \quad (28)$$

- The composition is defined by the formula

$$(\mathfrak{B}, \mathfrak{C}, \mathcal{B}, \mathcal{C}, g) \circ (\mathfrak{A}, \mathfrak{B}, \mathcal{A}, \mathcal{B}, f) = (\mathfrak{A}, \mathfrak{C}, \mathcal{A}, \mathcal{C}, g \circ f).$$

- Identity morphism for an object $(\mathfrak{A}, \mathcal{A})$ is $\text{id}_{\mathcal{A}}^{\mathbf{pFCD}(\mathfrak{A})}$. (Note: this is defined only for meet-semilattices.)

To prove that it is really a category is trivial.

PROPOSITION 1674. For strongly separated and bounded \mathfrak{A} and \mathfrak{B} formula (28) is equivalent to each of the following:

- 1°. $\text{dom } f \sqsubseteq \mathcal{A} \wedge \text{im } f \sqsubseteq \mathcal{B}$;
- 2°. $f \sqsubseteq \mathcal{A} \times^{\mathbf{FCD}} \mathcal{B}$.

PROOF. Because $\langle f \rangle x \sqsubseteq \text{im } f$, $\langle f^{-1} \rangle y \sqsubseteq \text{dom } f$, and theorem 1661. \square

20.10. Atomic pointfree functors

THEOREM 1675. Let $\mathfrak{A}, \mathfrak{B}$ be atomic bounded separable meet-semilattices. An $f \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is an atom of the poset $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff there exist $a \in \text{atoms}^{\mathfrak{A}}$ and $b \in \text{atoms}^{\mathfrak{B}}$ such that $f = a \times^{\mathbf{FCD}} b$.

PROOF.

\Rightarrow . Let f be an atom of the poset $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$. Let's get elements $a \in \text{atoms dom } f$ and $b \in \text{atoms } \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp a \Rightarrow \langle a \times^{\mathbf{FCD}} b \rangle \mathcal{X} = \perp^{\mathfrak{B}} \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \langle a \times^{\mathbf{FCD}} b \rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}.$$

So $\langle a \times^{\mathbf{FCD}} b \rangle \mathcal{X} \sqsubseteq \langle f \rangle \mathcal{X}$ and similarly $\langle b \times^{\mathbf{FCD}} a \rangle \mathcal{Y} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}$ for every $\mathcal{Y} \in \mathfrak{B}$ thus $a \times^{\mathbf{FCD}} b \sqsubseteq f$; because f is atomic we have $f = a \times^{\mathbf{FCD}} b$.

\Leftarrow . Let $a \in \text{atoms}^{\mathfrak{A}}$, $b \in \text{atoms}^{\mathfrak{B}}$, $f \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$. If $b \not\asymp \langle f \rangle a$ then $\neg(a [f] b)$, $f \sqcap (a \times^{\mathbf{FCD}} b) = \perp^{\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})}$ (by corollary 1664 because \mathfrak{A} and \mathfrak{B} are bounded meet-semilattices); if $b \sqsubseteq \langle f \rangle a$, then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp a \Rightarrow \langle a \times^{\mathbf{FCD}} b \rangle \mathcal{X} = \perp^{\mathfrak{B}} \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \langle a \times^{\mathbf{FCD}} b \rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}$$

that is $\langle a \times^{\mathbf{FCD}} b \rangle \mathcal{X} \sqsubseteq \langle f \rangle \mathcal{X}$ and likewise $\langle b \times^{\mathbf{FCD}} a \rangle \mathcal{Y} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}$ for every $\mathcal{Y} \in \mathfrak{B}$, so $f \sqsupseteq a \times^{\mathbf{FCD}} b$. Consequently $f \sqcap (a \times^{\mathbf{FCD}} b) = \perp^{\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})} \vee f \sqsupseteq a \times^{\mathbf{FCD}} b$; that is $a \times^{\mathbf{FCD}} b$ is an atomic pointfree functor. \square

THEOREM 1676. Let $\mathfrak{A}, \mathfrak{B}$ be atomic bounded separable meet-semilattices. Then $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is atomic.

PROOF. Let $f \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \neq \perp^{\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})}$. Then $\text{dom } f \neq \perp^{\mathfrak{A}}$, thus exists $a \in \text{atoms dom } f$. So $\langle f \rangle a \neq \perp^{\mathfrak{B}}$ thus exists $b \in \text{atoms } \langle f \rangle a$. Finally the atomic pointfree functor $a \times^{\mathbf{FCD}} b \sqsubseteq f$. \square

PROPOSITION 1677. Let $\mathfrak{A}, \mathfrak{B}$ be starrish bounded separable lattices. $\text{atoms}(f \sqcup g) = \text{atoms } f \cup \text{atoms } g$ for every $f, g \in \mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROOF.

$$\begin{aligned}
(a \times^{\text{FCD}} b) \sqcap (f \sqcup g) \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})} &\Leftrightarrow (\text{corollary 1664}) \Leftrightarrow \\
&a [f \sqcup g] b \Leftrightarrow (\text{theorem 1632}) \Leftrightarrow \\
&a [f] b \vee a [g] b \Leftrightarrow (\text{corollary 1664}) \Leftrightarrow \\
&(a \times^{\text{FCD}} b) \sqcap f \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})} \vee (a \times^{\text{FCD}} b) \sqcap g \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})}
\end{aligned}$$

for every $a \in \text{atoms}^{\mathfrak{A}}$ and $b \in \text{atoms}^{\mathfrak{B}}$. \square

THEOREM 1678. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a co-frame.

PROOF. Theorems 1616 and 533. \square

COROLLARY 1679. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a co-brouwerian lattice.

PROPOSITION 1680. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be atomic bounded separable meet-semilattices, and $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$, $g \in \text{pFCD}(\mathfrak{B}, \mathfrak{C})$. Then

$$\text{atoms}(g \circ f) = \left\{ \frac{x \times^{\text{FCD}} z}{\begin{array}{l} x \in \text{atoms}^{\mathfrak{A}}, z \in \text{atoms}^{\mathfrak{C}}, \\ \exists y \in \text{atoms}^{\mathfrak{B}} : (x \times^{\text{FCD}} y \in \text{atoms } f \wedge y \times^{\text{FCD}} z \in \text{atoms } g) \end{array}} \right\}.$$

PROOF.

$$\begin{aligned}
(x \times^{\text{FCD}} z) \sqcap (g \circ f) \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{C})} &\Leftrightarrow \\
&x [g \circ f] z \Leftrightarrow \\
&\exists y \in \text{atoms}^{\mathfrak{B}} : (x [f] y \wedge y [g] z) \Leftrightarrow \\
\exists y \in \text{atoms}^{\mathfrak{B}} : ((x \times^{\text{FCD}} y) \sqcap f \neq \perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})}) &\wedge (y \times^{\text{FCD}} z) \sqcap g \neq \perp^{\text{pFCD}(\mathfrak{B}, \mathfrak{C})}
\end{aligned}$$

(were used corollary 1664 and theorem 1655). \square

THEOREM 1681. Let f be a pointfree funcoid between atomic bounded separable meet-semilattices \mathfrak{A} and \mathfrak{B} .

- 1°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists F \in \text{atoms } f : \mathcal{X} [F] \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$;
- 2°. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{A}$ provided that \mathfrak{B} is a complete lattice.

PROOF.

1°.

$$\begin{aligned}
&\exists F \in \text{atoms } f : \mathcal{X} [F] \mathcal{Y} \Leftrightarrow \\
&\exists a \in \text{atoms}^{\mathfrak{A}}, b \in \text{atoms}^{\mathfrak{B}} : (a \times^{\text{FCD}} b \neq f \wedge \mathcal{X} [a \times^{\text{FCD}} b] \mathcal{Y}) \Leftrightarrow \\
&\exists a \in \text{atoms}^{\mathfrak{A}}, b \in \text{atoms}^{\mathfrak{B}} : (a \times^{\text{FCD}} b \neq f \wedge a \times^{\text{FCD}} b \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \Leftrightarrow \\
&\exists F \in \text{atoms } f : (F \neq f \wedge F \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \Leftrightarrow \\
&\hspace{10em} (\text{by theorem 1676}) \\
&\hspace{10em} f \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y} \Leftrightarrow \\
&\hspace{10em} \mathcal{X} [f] \mathcal{Y}.
\end{aligned}$$

2°. Let $\mathcal{Y} \in \mathfrak{B}$. Suppose $\mathcal{Y} \neq \langle f \rangle \mathcal{X}$. Then $\mathcal{X} [f] \mathcal{Y}$; $\exists F \in \text{atoms } f : \mathcal{X} [F] \mathcal{Y}$; $\exists F \in \text{atoms } f : \mathcal{Y} \neq \langle F \rangle \mathcal{X}$; and (taking into account that \mathfrak{B} is strongly separable by theorem 225) $\mathcal{Y} \neq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \subseteq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ by strong separability. The contrary $\langle f \rangle \mathcal{X} \supseteq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ is obvious. \square

20.11. Complete pointfree functors

DEFINITION 1682. Let \mathfrak{A} and \mathfrak{B} be posets. A pointfree functor $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is *complete*, when for every $S \in \mathcal{P}\mathfrak{A}$ whenever both $\bigsqcup S$ and $\bigsqcup \langle \langle f \rangle \rangle^* S$ are defined we have

$$\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S.$$

DEFINITION 1683. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be filtrators. I will call a *co-complete pointfree functor* a pointfree functor $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ such that $\langle f \rangle X \in \mathfrak{Z}_1$ for every $X \in \mathfrak{Z}_0$.

PROPOSITION 1684. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Co-complete pointfree functors $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ bijectively correspond to functions $\mathfrak{Z}_1^{\mathfrak{Z}_0}$ preserving finite joins, where the bijection is $f \mapsto \langle f \rangle|_{\mathfrak{Z}_0}$.

PROOF. It follows from the theorem 1616. \square

THEOREM 1685. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ be a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.

Let $(\mathfrak{B}, \mathfrak{Z}_1)$ be a star-separable filtrator.

The following conditions are equivalent for every pointfree functor $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$:

- 1°. f^{-1} is co-complete;
- 2°. $\forall S \in \mathcal{P}\mathfrak{A}, J \in \mathfrak{Z}_1 : (\bigsqcup^{\mathfrak{A}} S [f] J \Rightarrow \exists I \in S : I [f] J)$;
- 3°. $\forall S \in \mathcal{P}\mathfrak{Z}_0, J \in \mathfrak{Z}_1 : (\bigsqcup^{\mathfrak{Z}_0} S [f] J \Rightarrow \exists I \in S : I [f] J)$;
- 4°. f is complete;
- 5°. $\forall S \in \mathcal{P}\mathfrak{Z}_0 : \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$.

PROOF. First note that the theorem 583 applies to the filtrator $(\mathfrak{A}, \mathfrak{Z}_0)$.

3° \Rightarrow 1°. For every $S \in \mathcal{P}\mathfrak{Z}_0, J \in \mathfrak{Z}_1$

$$\bigsqcup^{\mathfrak{Z}_0} S \cap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq \perp^{\mathfrak{A}} \Rightarrow \exists I \in S : I \cap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq \perp^{\mathfrak{A}}, \quad (29)$$

consequently by the theorem 583 we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$.

1° \Rightarrow 2°. For every $S \in \mathcal{P}\mathfrak{A}, J \in \mathfrak{Z}_1$ we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$, consequently

$$\forall S \in \mathcal{P}\mathfrak{A}, J \in \mathfrak{Z}_1 : \left(\bigsqcup^{\mathfrak{A}} S \neq \langle f^{-1} \rangle J \Rightarrow \exists I \in S : I \neq \langle f^{-1} \rangle J \right).$$

From this follows 2°.

$2^\circ \Rightarrow 4^\circ$. Let $\langle f \rangle \sqcup^{\mathfrak{Z}_0} S$ and $\sqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$ be defined. We have $\langle f \rangle \sqcup^{\mathfrak{A}} S = \langle f \rangle \sqcup^{\mathfrak{Z}_0} S$.

$$\begin{aligned} J \cap^{\mathfrak{B}} \langle f \rangle \sqcup^{\mathfrak{A}} S \neq \perp^{\mathfrak{B}} &\Leftrightarrow \\ \sqcup^{\mathfrak{A}} S [f] J &\Leftrightarrow \\ \exists I \in S : I [f] J &\Leftrightarrow \\ \exists I \in S : J \cap^{\mathfrak{B}} \langle f \rangle I \neq \perp^{\mathfrak{B}} &\Leftrightarrow \\ J \cap^{\mathfrak{B}} \sqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S \neq \perp^{\mathfrak{B}} & \end{aligned}$$

(used theorem 583). Thus $\langle f \rangle \sqcup^{\mathfrak{A}} S = \sqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$ by star-separability of $(\mathfrak{B}, \mathfrak{Z}_1)$.

$5^\circ \Rightarrow 3^\circ$. Let $\langle f \rangle \sqcup^{\mathfrak{Z}_0} S$ be defined. Then $\sqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$ is also defined because $\langle f \rangle \sqcup^{\mathfrak{Z}_0} S = \sqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$. Then

$$\sqcup^{\mathfrak{Z}_0} S [f] J \Leftrightarrow J \cap^{\mathfrak{B}} \langle f \rangle \sqcup^{\mathfrak{Z}_0} S \neq \perp^{\mathfrak{B}} \Leftrightarrow J \cap^{\mathfrak{B}} \sqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S \neq \perp^{\mathfrak{B}}$$

what by theorem 583 is equivalent to $\exists I \in S : J \cap^{\mathfrak{B}} \langle f \rangle I \neq \perp^{\mathfrak{B}}$ that is $\exists I \in S : I [f] J$.

$2^\circ \Rightarrow 3^\circ$, $4^\circ \Rightarrow 5^\circ$. By join-closedness of the core of $(\mathfrak{A}, \mathfrak{Z}_0)$. □

THEOREM 1686. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If R is a set of co-complete pointfree functors in $\mathbf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\sqcup R$ is a co-complete pointfree functor.

PROOF. Let R be a set of co-complete pointfree functors. Then for every $X \in \mathfrak{Z}_0$

$$\left\langle \sqcup R \right\rangle X = \sqcup_{f \in R}^{\mathfrak{B}} \langle f \rangle X = \sqcup_{f \in R}^{\mathfrak{Z}_1} \langle f \rangle X \in \mathfrak{Z}_1$$

(used the theorems 1630 and 534). □

Let \mathfrak{A} and \mathfrak{B} be posets with least elements. I will denote $\mathbf{ComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathbf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ the sets of complete and co-complete functors correspondingly from a poset \mathfrak{A} to a poset \mathfrak{B} .

PROPOSITION 1687.

- 1°. Let $f \in \mathbf{ComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $g \in \mathbf{ComplpFCD}(\mathfrak{B}, \mathfrak{C})$ where \mathfrak{A} and \mathfrak{C} are posets with least elements and \mathfrak{B} is a complete lattice. Then $g \circ f \in \mathbf{ComplpFCD}(\mathfrak{A}, \mathfrak{C})$.
- 2°. Let $f \in \mathbf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $g \in \mathbf{CoComplpFCD}(\mathfrak{B}, \mathfrak{C})$ where $(\mathfrak{A}, \mathfrak{Z}_0)$, $(\mathfrak{B}, \mathfrak{Z}_1)$, $(\mathfrak{C}, \mathfrak{Z}_2)$ are filtrators. Then $g \circ f \in \mathbf{CoComplpFCD}(\mathfrak{A}, \mathfrak{C})$.

PROOF.

1°. Let $\sqcup S$ and $\sqcup \langle \langle g \circ f \rangle \rangle^* S$ be defined. Then

$$\langle g \circ f \rangle \sqcup S = \langle g \rangle \langle f \rangle \sqcup S = \langle g \rangle \sqcup \langle \langle f \rangle \rangle^* S = \sqcup \langle \langle g \rangle \rangle^* \langle \langle f \rangle \rangle^* S = \sqcup \langle \langle g \circ f \rangle \rangle^* S.$$

2°. $\langle g \circ f \rangle \mathfrak{Z}_0 = \langle g \rangle \langle f \rangle \mathfrak{Z}_0 \in \mathfrak{Z}_2$ because $\langle f \rangle \mathfrak{Z}_0 \in \mathfrak{Z}_1$. □

PROPOSITION 1688. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathbf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ (with induced order) is a complete lattice.

PROOF. Follows from the theorem 1686. \square

THEOREM 1689. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators where \mathfrak{Z}_0 and \mathfrak{Z}_1 are boolean lattices. Let R be a set of pointfree functors from \mathfrak{A} to \mathfrak{B} .

$g \circ (\bigsqcup R) = \bigsqcup_{g \in R} (g \circ f) = \bigsqcup \langle g \circ \rangle^* R$ if g is a complete pointfree functor from \mathfrak{B} .

PROOF. For every $X \in \mathfrak{A}$

$$\begin{aligned} \langle g \circ (\bigsqcup R) \rangle X &= \\ \langle g \rangle \langle \bigsqcup R \rangle X &= \\ \langle g \rangle \bigsqcup_{f \in R} \langle f \rangle X &= \\ \bigsqcup_{f \in R} \langle g \rangle \langle f \rangle X &= \\ \bigsqcup_{f \in R} \langle g \circ f \rangle X &= \\ \langle \bigsqcup_{f \in R} (g \circ f) \rangle X &= \\ \langle \bigsqcup \langle g \circ \rangle^* R \rangle X. & \end{aligned}$$

So $g \circ (\bigsqcup R) = \bigsqcup \langle g \circ \rangle^* R$. \square

20.12. Completion and co-completion

DEFINITION 1690. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices and \mathfrak{Z}_1 is a complete atomistic lattice.

Co-completion of a pointfree functor $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is pointfree functor $\text{CoCompl } f$ defined by the formula (for every $X \in \mathfrak{Z}_0$)

$$\langle \text{CoCompl } f \rangle X = \text{Cor} \langle f \rangle X.$$

PROPOSITION 1691. Above defined co-completion always exists.

PROOF. Existence of $\text{Cor} \langle f \rangle X$ follows from completeness of \mathfrak{Z}_1 .

We may apply the theorem 1616 because

$$\text{Cor} \langle f \rangle (X \sqcup^{\mathfrak{Z}_0} Y) = \text{Cor} (\langle f \rangle X \sqcup^{\mathfrak{B}} \langle f \rangle Y) = \text{Cor} \langle f \rangle X \sqcup^{\mathfrak{Z}_1} \text{Cor} \langle f \rangle Y$$

by theorem 603. \square

OBVIOUS 1692. Co-completion is always co-complete.

PROPOSITION 1693. For above defined always $\text{CoCompl } f \sqsubseteq f$.

PROOF. By proposition 542. \square

20.13. Monovalued and injective pointfree functors

DEFINITION 1694. Let \mathfrak{A} and \mathfrak{B} be posets. Let $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

The pointfree functor f is:

- *monovalued* when $f \circ f^{-1} \sqsubseteq 1_{\mathfrak{B}}^{\text{pFCD}}$.
- *injective* when $f^{-1} \circ f \sqsubseteq 1_{\mathfrak{A}}^{\text{pFCD}}$.

Monovaluedness is dual of injectivity.

PROPOSITION 1695. Let \mathfrak{A} and \mathfrak{B} be posets. Let $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

The pointfree functor f is:

- monovalued iff $f \circ f^{-1} \sqsubseteq \text{id}_{\text{im } f}^{\text{pFCD}(\mathfrak{B})}$, if \mathfrak{A} has greatest element and \mathfrak{B} is a strongly separable meet-semilattice;
- injective iff $f^{-1} \circ f \sqsubseteq \text{id}_{\text{dom } f}^{\text{pFCD}(\mathfrak{A})}$, if \mathfrak{B} has greatest element and \mathfrak{A} is a strongly separable meet-semilattice.

PROOF. It's enough to prove $f \circ f^{-1} \sqsubseteq 1_{\mathfrak{B}}^{\text{pFCD}} \Leftrightarrow f \circ f^{-1} \sqsubseteq \text{id}_{\text{im } f}^{\text{pFCD}(\mathfrak{B})}$. $\text{im } f$ is defined because \mathfrak{A} has greatest element. $\text{id}_{\text{im } f}^{\text{pFCD}(\mathfrak{B})}$ is defined because \mathfrak{B} is a meet-semilattice.

\Leftarrow . Obvious.

\Rightarrow . Let $f \circ f^{-1} \sqsubseteq 1_{\mathfrak{B}}^{\text{pFCD}}$. Then $\langle f \circ f^{-1} \rangle x \sqsubseteq x$; $\langle f \circ f^{-1} \rangle x \sqsubseteq \text{im } f$ (proposition 1603). Thus $\langle f \circ f^{-1} \rangle x \sqsubseteq x \sqcap \text{im } f = \langle \text{id}_{\text{im } f}^{\text{pFCD}(\mathfrak{B})} \rangle x$.

$$\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x \text{ and } \langle (f \circ f^{-1})^{-1} \rangle x = \langle f \circ f^{-1} \rangle x \sqsubseteq \text{im } f. \text{ Thus}$$

$$\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x \sqcap \text{im } f = \langle \text{id}_{\text{im } f}^{\text{pFCD}(\mathfrak{B})} \rangle x.$$

$$\text{Thus } f \circ f^{-1} \sqsubseteq \text{id}_{\text{im } f}^{\text{pFCD}(\mathfrak{B})}.$$

□

THEOREM 1696. Let \mathfrak{A} be an atomistic meet-semilattice with least element, \mathfrak{B} be an atomistic bounded meet-semilattice. The following statements are equivalent for every $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$:

- 1°. f is monovalued.
- 2°. $\forall a \in \text{atoms}^{\mathfrak{A}} : \langle f \rangle a \in \text{atoms}^{\mathfrak{B}} \cup \{\perp^{\mathfrak{B}}\}$.
- 3°. $\forall i, j \in \mathfrak{B} : \langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j$.

PROOF.

$2^\circ \Rightarrow 3^\circ$. Let $a \in \text{atoms}^{\mathfrak{A}}$, $\langle f \rangle a = b$. Then because $b \in \text{atoms}^{\mathfrak{B}} \cup \{\perp^{\mathfrak{B}}\}$

$$(i \sqcap j) \sqcap b \neq \perp^{\mathfrak{B}} \Leftrightarrow i \sqcap b \neq \perp^{\mathfrak{B}} \wedge j \sqcap b \neq \perp^{\mathfrak{B}};$$

$$a [f] i \sqcap j \Leftrightarrow a [f] i \wedge a [f] j;$$

$$i \sqcap j [f^{-1}] a \Leftrightarrow i [f^{-1}] a \wedge j [f^{-1}] a;$$

$$a \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle (i \sqcap j) \neq \perp^{\mathfrak{A}} \Leftrightarrow a \sqcap \langle f^{-1} \rangle i \neq \perp^{\mathfrak{A}} \wedge a \sqcap \langle f^{-1} \rangle j \neq \perp^{\mathfrak{A}};$$

$$a \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle (i \sqcap j) \neq \perp^{\mathfrak{A}} \Leftrightarrow a \sqcap \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j \neq \perp^{\mathfrak{A}};$$

$$\langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j.$$

$3^\circ \Rightarrow 1^\circ$. $\langle f^{-1} \rangle a \sqcap \langle f^{-1} \rangle b = \langle f^{-1} \rangle (a \sqcap b) = \langle f^{-1} \rangle \perp^{\mathfrak{B}} = \perp^{\mathfrak{A}}$ (by proposition 1602) for every two distinct $a, b \in \text{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg(\langle f^{-1} \rangle a [f] b)$; $b \sqcap \langle f \rangle \langle f^{-1} \rangle a = \perp^{\mathfrak{B}}$; $b \sqcap \langle f \circ f^{-1} \rangle a = \perp^{\mathfrak{B}}$; $\neg(a [f \circ f^{-1}] b)$. So $a [f \circ f^{-1}] b \Rightarrow a = b$ for every $a, b \in \text{atoms}^{\mathfrak{B}}$. This is possible only (corollary 1650 and the fact that \mathfrak{B} is atomic) when $f \circ f^{-1} \sqsubseteq 1_{\mathfrak{B}}^{\text{pFCD}}$.

$\neg 2^\circ \Rightarrow \neg 1^\circ$. Suppose $\langle f \rangle a \notin \text{atoms}^{\mathfrak{B}} \cup \{\perp^{\mathfrak{B}}\}$ for some $a \in \text{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p \neq q$ such that $\langle f \rangle a \supseteq p \wedge \langle f \rangle a \supseteq q$. Consequently $p \sqcap \langle f \rangle a \neq \perp^{\mathfrak{B}}$; $a \sqcap \langle f^{-1} \rangle p \neq \perp^{\mathfrak{A}}$; $a \sqsubseteq \langle f^{-1} \rangle p$; $\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \supseteq \langle f \rangle a \supseteq q$ (by proposition 1603 because \mathfrak{B} is separable by proposition 234 and thus strongly separable by theorem 225); $\langle f \circ f^{-1} \rangle p \not\sqsubseteq p$ and $\langle f \circ f^{-1} \rangle p \neq \perp^{\mathfrak{B}}$. So it cannot be $f \circ f^{-1} \sqsubseteq 1_{\mathfrak{B}}^{\text{pFCD}}$.

□

THEOREM 1697. The following is equivalent for primary filtrators $(\mathfrak{A}, \mathfrak{J}_0)$ and $(\mathfrak{B}, \mathfrak{J}_1)$ over boolean lattices and pointfree funcoids $f : \mathfrak{A} \rightarrow \mathfrak{B}$:

- 1°. f is monovalued.
- 2°. f is metamonovalued.

3°. f is weakly metamonovalued.

PROOF.

2° \Rightarrow 3°. Obvious.

1° \Rightarrow 2°.

$$\langle (\prod G) \circ f \rangle x = \langle (\prod G) \rangle \langle f \rangle x = \prod_{g \in G} \langle g \rangle \langle f \rangle x = \prod_{g \in G} \langle g \circ f \rangle x = \left\langle \prod_{g \in G} (g \circ f) \right\rangle x$$

for every atomic filter object $x \in \text{atoms}^{\mathfrak{A}}$. Thus $(\prod G) \circ f = \prod_{g \in G} (g \circ f)$.

3° \Rightarrow 1°. Take $g = a \times^{\text{FCD}} y$ and $h = b \times^{\text{FCD}} y$ for arbitrary atomic filter objects $a \neq b$ and y . We have $g \sqcap h = \perp$; thus $(g \circ f) \sqcap (h \circ f) = (g \sqcap h) \circ f = \perp$ and thus impossible $x [f] a \wedge x [f] b$ as otherwise $x [g \circ f] y$ and $x [h \circ f] y$ so $x [(g \circ f) \sqcap (h \circ f)] y$. Thus f is monovalued. \square

THEOREM 1698. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. A pointfree funcoid $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ is monovalued iff

$$\forall I, J \in \mathfrak{Z}_1 : \langle f^{-1} \rangle (I \sqcap^{\mathfrak{Z}_1} J) = \langle f^{-1} \rangle I \sqcap \langle f^{-1} \rangle J.$$

PROOF. \mathfrak{A} and \mathfrak{B} are complete lattices (corollary 518).

$(\mathfrak{B}, \mathfrak{Z}_1)$ is a filtrator with separable core by theorem 537.

$(\mathfrak{B}, \mathfrak{Z}_1)$ is binarily meet-closed by corollary 536.

\mathfrak{A} and \mathfrak{B} are starrish by corollary 531.

$(\mathfrak{A}, \mathfrak{Z}_0)$ is with separable core by theorem 537.

We are under conditions of theorem 1615 for the pointfree funcoid f^{-1} .

\Rightarrow . Obvious (taking into account that $(\mathfrak{B}, \mathfrak{Z}_1)$ is binarily meet-closed).

\Leftarrow .

$$\begin{aligned} \langle f^{-1} \rangle (\mathcal{I} \sqcap \mathcal{J}) &= \\ \prod \langle \langle f^{-1} \rangle \rangle^* \text{up}^{\mathfrak{Z}_1} (\mathcal{I} \sqcap \mathcal{J}) &= \\ \prod \langle \langle f^{-1} \rangle \rangle^* \left\{ \frac{I \sqcap^{\mathfrak{Z}_1} J}{I \in \text{up } \mathcal{I}, J \in \text{up } \mathcal{J}} \right\} &= \\ \prod \left\{ \frac{\langle f^{-1} \rangle (I \sqcap^{\mathfrak{Z}_1} J)}{I \in \text{up } \mathcal{I}, J \in \text{up } \mathcal{J}} \right\} &= \\ \prod \left\{ \frac{\langle f^{-1} \rangle I \sqcap \langle f^{-1} \rangle J}{I \in \text{up } \mathcal{I}, J \in \text{up } \mathcal{J}} \right\} &= \\ \prod \left\{ \frac{\langle f^{-1} \rangle I}{I \in \text{up } \mathcal{I}} \right\} \sqcap \prod \left\{ \frac{\langle f^{-1} \rangle J}{J \in \text{up } \mathcal{J}} \right\} &= \\ \langle f^{-1} \rangle \mathcal{I} \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle \mathcal{J} & \end{aligned}$$

(used theorem 1615, corollary 521, theorem 1604). \square

PROPOSITION 1699. Let \mathfrak{A} be an atomistic meet-semilattice with least element, \mathfrak{B} be an atomistic bounded meet-semilattice. Then if f, g are pointfree funcoids from \mathfrak{A} to \mathfrak{B} , $f \sqsubseteq g$ and g is monovalued then $g|_{\text{dom } f} = f$.

PROOF. Obviously $g|_{\text{dom } f} \sqsupseteq f$. Suppose for contrary that $g|_{\text{dom } f} \sqsubset f$. Then there exists an atom $a \in \text{atoms dom } f$ such that $\langle g|_{\text{dom } f} \rangle a \neq \langle f \rangle a$ that is $\langle g \rangle a \sqsubset \langle f \rangle a$ what is impossible. \square

20.14. Elements closed regarding a pointfree funcoid

Let \mathfrak{A} be a poset. Let $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{A})$.

DEFINITION 1700. Let's call *closed* regarding a pointfree funcoid f such element $a \in \mathfrak{A}$ that $\langle f \rangle a \sqsubseteq a$.

PROPOSITION 1701. If i and j are closed (regarding a pointfree funcoid $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{A})$), S is a set of closed elements (regarding f), then

- 1°. $i \sqcup j$ is a closed element, if \mathfrak{A} is a separable starrish join-semilattice;
- 2°. $\prod S$ is a closed element if \mathfrak{A} is a strongly separable complete lattice.

PROOF. $\langle f \rangle (i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j \sqsubseteq i \sqcup j$ (theorem 1604), $\langle f \rangle \prod S \sqsubseteq \prod \langle \langle f \rangle \rangle^* S \sqsubseteq \prod S$ (used strong separability of \mathfrak{A} twice). Consequently the elements $i \sqcup j$ and $\prod S$ are closed. \square

PROPOSITION 1702. If S is a set of elements closed regarding a complete point-free funcoid f with strongly separable destination which is a complete lattice, then the element $\prod S$ is also closed regarding our funcoid.

PROOF. $\langle f \rangle \prod S = \prod \langle \langle f \rangle \rangle^* S \sqsubseteq \prod S$. \square

20.15. Connectedness regarding a pointfree funcoid

Let \mathfrak{A} be a poset with least element. Let $\mu \in \text{pFCD}(\mathfrak{A}, \mathfrak{A})$.

DEFINITION 1703. An element $a \in \mathfrak{A}$ is called *connected* regarding a pointfree funcoid μ over \mathfrak{A} when

$$\forall x, y \in \mathfrak{A} \setminus \{\perp^{\mathfrak{A}}\} : (x \sqcup y = a \Rightarrow x [\mu] y).$$

PROPOSITION 1704. Let $(\mathfrak{A}, \mathfrak{B})$ be a co-separable filtrator with finitely join-closed core. An $A \in \mathfrak{B}$ is connected regarding a funcoid μ iff

$$\forall X, Y \in \mathfrak{B} \setminus \{\perp^{\mathfrak{B}}\} : (X \sqcup^{\mathfrak{B}} Y = A \Rightarrow X [\mu] Y).$$

PROOF.

\Rightarrow . Obvious.

\Leftarrow . Follows from co-separability. \square

OBVIOUS 1705. For \mathfrak{A} being a set of filters over a boolean lattice, an element $a \in \mathfrak{A}$ is connected regarding a pointfree funcoid μ iff it is connected regarding the funcoid $\mu \sqcap (a \times^{\text{FCD}} a)$.

EXERCISE 1706. Consider above without requirement of existence of least element.

20.16. Boolean funcoids

I call *boolean funcoids* pointfree funcoids between boolean lattices.

PROPOSITION 1707. Every pointfree funcoid, whose source is a complete and completely starrish and whose destination is complete and completely starrish and separable, is complete.

PROOF. It's enough to prove $\langle f \rangle \prod S = \prod \langle \langle f \rangle \rangle^* S$ for our pointfree funcoid f for every $S \in \mathcal{P} \text{Src } f$.

Really, $Y \neq \langle f \rangle \prod S \Leftrightarrow \prod S \neq \langle f^{-1} \rangle Y \Leftrightarrow \exists X \in S : X \neq \langle f^{-1} \rangle Y \Leftrightarrow \exists X \in S : Y \neq \langle f \rangle X \Leftrightarrow Y \neq \prod \langle \langle f \rangle \rangle^* S$ for every $Y \in \text{Dst } f$ and thus we have $\langle f \rangle \prod S = \prod \langle \langle f \rangle \rangle^* S$ because $\text{Dst } f$ is separable. \square

REMARK 1708. It seems that this theorem can be generalized for non-complete lattices.

COROLLARY 1709. Every boolean funcoid is complete and co-complete.

PROOF. Using proposition 226 and corollary 89. \square

THEOREM 1710. Let $\mathfrak{A}, \mathfrak{B}$ be complete boolean lattices.

A function $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ is equal to the component $\langle f \rangle$ of a pointtree funcoid $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff α is preserving all joins (= lower adjoint).

PROOF. Let $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and preserves all joins. Then $\alpha \in \mathcal{F}(\mathfrak{B})^{\mathfrak{A}}$ (We equate principal filters of the set $\mathcal{F}\mathfrak{A}$ of filters on \mathfrak{A} with elements of \mathfrak{A}). Thus (theorem 1616) $\alpha = \langle g \rangle^*$ for some $g \in \text{pFCD}(\mathcal{F}\mathfrak{A}, \mathcal{F}\mathfrak{B})$.

$\langle g^{-1} \rangle \in \mathcal{F}(\mathfrak{A})^{\mathcal{F}(\mathfrak{B})}$.

Let $y \in \mathfrak{B}$. We need to prove $\langle g^{-1} \rangle y \in \mathfrak{A}$ that is $\bigsqcup S \neq \langle g^{-1} \rangle y \Leftrightarrow \exists x \in S : \langle g^{-1} \rangle y \neq x$ for every $S \in \mathcal{P}\mathfrak{A}$.

Really, $\bigsqcup S \neq \langle g^{-1} \rangle y \Leftrightarrow y \neq \langle g \rangle \bigsqcup S \Leftrightarrow y \neq \bigsqcup \langle \langle g \rangle \rangle^* S \Leftrightarrow \exists x \in S : y \neq \langle g \rangle x \Leftrightarrow \exists x \in S : \langle g^{-1} \rangle y \neq x$.

Take $\beta = \langle g^{-1} \rangle^*$. We have $\beta \in \mathfrak{A}^{\mathfrak{B}}$.

$x \neq \beta y \Leftrightarrow x \neq \langle g^{-1} \rangle y \Leftrightarrow y \neq \langle g \rangle x \Leftrightarrow y \neq \alpha x$.

So $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ is a pointfree funcoid.

The other direction: Let now $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$. We need to prove that it preserves all joins. But it was proved above. \square

CONJECTURE 1711. Let $\mathfrak{A}, \mathfrak{B}$ be boolean lattices.

A function $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ is equal to the component $\langle f \rangle$ of a pointfree funcoid $f \in \text{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff α is a lower adjoint.

It is tempting to conclude that $\langle f \rangle$ is a lower adjoint to $\langle f^{-1} \rangle$. But that's false:

We should disprove that $\langle f \rangle X \sqsubseteq Y \Leftrightarrow X \sqsubseteq \langle f^{-1} \rangle Y$.

For a counter-example, take $f = \{0\} \times \mathbb{N}$. Then our condition takes form $Y = \mathbb{N} \Leftrightarrow X \sqsubseteq \{0\}$ for $X \ni 0, Y \ni 0$ what obviously does not hold.

20.17. Binary relations are pointfree funcoids

Below for simplicity we will equate $\mathcal{T}A$ with $\mathcal{P}A$.

THEOREM 1712. Pointfree funcoids f between powerset posets $\mathcal{T}A$ and $\mathcal{T}B$ bijectively (moreover this bijection is an order-isomorphism) correspond to morphisms $p \in \mathbf{Rel}(A, B)$ by the formulas:

$$\langle f \rangle = \langle p \rangle^*, \quad \langle f^{-1} \rangle = \langle p^{-1} \rangle^*; \quad (30)$$

$$(x, y) \in \text{GR } p \Leftrightarrow y \in \langle f \rangle \{x\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}. \quad (31)$$

PROOF. Suppose $p \in \mathbf{Rel}(A, B)$ and prove that there is a pointfree funcoid f conforming to (30). Really, for every $X \in \mathcal{T}A, Y \in \mathcal{T}B$

$$Y \neq \langle f \rangle X \Leftrightarrow Y \neq \langle p \rangle^* X \Leftrightarrow Y \neq \langle p \rangle X \Leftrightarrow$$

$$X \neq \langle p^{-1} \rangle Y \Leftrightarrow X \neq \langle p^{-1} \rangle^* Y \Leftrightarrow X \neq \langle f^{-1} \rangle Y.$$

Now suppose $f \in \text{pFCD}(\mathcal{T}A, \mathcal{T}B)$ and prove that the relation defined by the formula (31) exists. To prove it, it's enough to show that $y \in \langle f \rangle \{x\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}$. Really,

$$y \in \langle f \rangle \{x\} \Leftrightarrow \{y\} \neq \langle f \rangle \{x\} \Leftrightarrow \{x\} \neq \langle f^{-1} \rangle \{y\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}.$$

It remains to prove that functions defined by (30) and (31) are mutually inverse. (That these functions are monotone is obvious.)

Let $p_0 \in \mathbf{Rel}(A, B)$ and $f \in \mathbf{pFCD}(\mathcal{T}A, \mathcal{T}B)$ corresponds to p_0 by the formula (30); let $p_1 \in \mathbf{Rel}(A, B)$ corresponds to f by the formula (31). Then $p_0 = p_1$ because

$$(x, y) \in \mathbf{GR} p_0 \Leftrightarrow y \in \langle p_0 \rangle^* \{x\} \Leftrightarrow y \in \langle f \rangle \{x\} \Leftrightarrow (x, y) \in \mathbf{GR} p_1.$$

Let now $f_0 \in \mathbf{pFCD}(\mathcal{T}A, \mathcal{T}B)$ and $p \in \mathbf{Rel}(A, B)$ corresponds to f_0 by the formula (31); let $f_1 \in \mathbf{pFCD}(\mathcal{T}A, \mathcal{T}B)$ corresponds to p by the formula (30). Then $(x, y) \in \mathbf{GR} p \Leftrightarrow y \in \langle f_0 \rangle \{x\}$ and $\langle f_1 \rangle = \langle p \rangle^*$; thus

$$y \in \langle f_1 \rangle \{x\} \Leftrightarrow y \in \langle p \rangle^* \{x\} \Leftrightarrow (x, y) \in \mathbf{GR} p \Leftrightarrow y \in \langle f_0 \rangle \{x\}.$$

So $\langle f_0 \rangle = \langle f_1 \rangle$. Similarly $\langle f_0^{-1} \rangle = \langle f_1^{-1} \rangle$. \square

PROPOSITION 1713. The bijection defined by the theorem 1712 preserves composition and identities, that is is a functor between categories \mathbf{Rel} and $(A, B) \mapsto \mathbf{pFCD}(\mathcal{T}A, \mathcal{T}B)$.

PROOF. Let $\langle f \rangle = \langle p \rangle^*$ and $\langle g \rangle = \langle q \rangle^*$. Then $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle = \langle q \rangle^* \circ \langle p \rangle^* = \langle q \circ p \rangle^*$. Likewise $\langle (g \circ f)^{-1} \rangle = \langle (q \circ p)^{-1} \rangle^*$. So it preserves composition.

Let $p = 1_{\mathbf{Rel}}^A$ for some set A . Then $\langle f \rangle = \langle p \rangle^* = \langle 1_{\mathbf{Rel}}^A \rangle^* = \text{id}_{\mathcal{T}A}$ and likewise $\langle f^{-1} \rangle = \text{id}_{\mathcal{T}A}$, that is f is an identity pointfree funcoid. So it preserves identities. \square

PROPOSITION 1714. The bijection defined by theorem 1712 preserves reversal.

PROOF. $\langle f^{-1} \rangle = \langle p^{-1} \rangle^*$. \square

PROPOSITION 1715. The bijection defined by theorem 1712 preserves monoaluedness and injectivity.

PROOF. Because it is a functor which preserves reversal. \square

PROPOSITION 1716. The bijection defined by theorem 1712 preserves domain an image.

PROOF. $\text{im } f = \langle f \rangle \top = \langle p \rangle^* \top = \text{im } p$, likewise for domain. \square

PROPOSITION 1717. The bijection defined by theorem 1712 maps cartesian products to corresponding funcoidal products.

PROOF. $\langle A \times^{\text{FCD}} B \rangle X = \begin{cases} B & \text{if } X \not\asymp A \\ \perp & \text{if } X \asymp A \end{cases} = \langle A \times B \rangle^* X$. Likewise $\langle (A \times^{\text{FCD}} B)^{-1} \rangle Y = \langle (A \times B)^{-1} \rangle^* Y$. \square

»»» > master

Alternative representations of binary relations

THEOREM 1718. Let A and B be fixed sets. The diagram at the figure 11 is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity. All “parallel” arrows are mutually inverse.

For a Galois connection f I denote f_0 the lower adjoint and f_1 the upper adjoint. For simplicity, in the diagram I equate $\mathcal{P}A$ and $\mathcal{T}A$.

PROOF. First, note that despite we use the notation Ψ_i^{-1} , it is not yet proved that Ψ_i^{-1} is the inverse of Ψ_i . We will prove it below.

Now prove a list of claims. First concentrate on the upper “triangle” of the diagram (the lower one will be considered later).

Claim: $\left\{ \frac{(x,y)}{y \in f_0\{x\}} \right\} = \left\{ \frac{(x,y)}{x \in f_1\{y\}} \right\}$ when f is an antitone Galois connection between $\mathcal{P}A$ and $\mathcal{P}B$.

Proof: $y \in f_0\{x\} \Leftrightarrow \{y\} \subseteq f_0\{x\} \Leftrightarrow \{x\} \subseteq f_1\{y\} \Leftrightarrow x \in f_1\{y\}$. ■

Claim: $(X \mapsto \prod_{x \in \mathcal{T}X \setminus \{\perp\}} \langle f \rangle x, Y \mapsto \prod_{y \in \mathcal{T}Y \setminus \{\perp\}} \langle f^{-1} \rangle y) = (X \mapsto \prod_{x \in X} \langle f \rangle \{x\}, Y \mapsto \prod_{y \in Y} \langle f^{-1} \rangle \{y\})$ when f is a pointfree funcoid between $\mathcal{P}A$ and $\mathcal{P}B$.

Proof: It is enough to prove $\prod_{x \in \mathcal{T}X \setminus \{\perp\}} \langle f \rangle x = \prod_{x \in X} \langle f \rangle \{x\}$ (the rest follows from symmetry). $\prod_{x \in \mathcal{T}X \setminus \{\perp\}} \langle f \rangle x \subseteq \prod_{x \in X} \langle f \rangle \{x\}$ because $\mathcal{T}X \setminus \{\perp\} \supseteq \left\{ \frac{\{x\}}{x \in X} \right\}$. $\prod_{x \in \mathcal{T}X \setminus \{\perp\}} \langle f \rangle x \supseteq \prod_{x \in X} \langle f \rangle \{x\}$ because if $x \in \mathcal{T}X \setminus \{\perp\}$ then we can take $x' \in x$ that is $\{x'\} \subseteq x$ and thus $\langle f \rangle x \supseteq \langle f \rangle \{x'\}$, so $\prod_{x \in \mathcal{T}X \setminus \{\perp\}} \langle f \rangle x \supseteq \prod_{x \in \mathcal{T}X \setminus \{\perp\}} \langle f \rangle \{x'\} \supseteq \prod_{x \in X} \langle f \rangle \{x\}$. ■

Claim: $(\mathcal{P}A, \mathcal{P}B, X \mapsto \bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} f_0x, Y \mapsto \bigsqcup_{y \in \mathcal{T}Y \setminus \{\perp\}} f_1y) = (\mathcal{P}A, \mathcal{P}B, X \mapsto \bigsqcup_{x \in X} f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1\{y\})$ when f is an antitone Galois connection between $\mathcal{P}A$ and $\mathcal{P}B$.

Proof: It is enough to prove $\bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} f_0x = \bigsqcup_{x \in X} f_0\{x\}$ (the rest follows from symmetry). We have $\bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} f_0x \supseteq \bigsqcup_{x \in X} f_0\{x\}$ because $\{x\} \in \mathcal{T}X \setminus \{\perp\}$. Let $x \in \mathcal{T}X \setminus \{\perp\}$. Take $x' \in X$. We have $f_0x \subseteq f_0\{x'\}$ and thus $f_0x \subseteq \bigsqcup_{x \in X} f_0\{x\}$. So $\bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} f_0x \subseteq \bigsqcup_{x \in X} f_0\{x\}$. ■

Claim: $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$.

Proof: $\Psi_2 \Psi_1 f = (\mathcal{P}A, \mathcal{P}B, X \mapsto \left\{ \frac{y}{\exists x \in X: (x,y) \in \Psi_1 f} \right\}, Y \mapsto \left\{ \frac{x}{\exists y \in Y: (x,y) \in \Psi_1 f} \right\}) = (\mathcal{P}A, \mathcal{P}B, X \mapsto \left\{ \frac{y}{\exists x \in X: y \in f_0\{x\}} \right\}, Y \mapsto \left\{ \frac{x}{\exists y \in Y: x \in f_1\{y\}} \right\}) = (\mathcal{P}A, \mathcal{P}B, X \mapsto \bigsqcup_{x \in X} f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1\{y\}) = \Psi_3^{-1} f$. ■

Claim: $\Psi_3 = \Psi_1^{-1} \circ \Psi_2^{-1}$.

Proof: $\Psi_1^{-1} \Psi_2^{-1} f = (X \mapsto \left\{ \frac{y \in B}{\forall x \in X: \{x\} [f] \{y\}} \right\}, Y \mapsto \left\{ \frac{x \in A}{\forall y \in Y: \{x\} [f] \{y\}} \right\}) = (X \mapsto \left\{ \frac{y \in B}{\forall x \in X: y \in \langle f \rangle \{x\}} \right\}, Y \mapsto \left\{ \frac{x \in A}{\forall y \in Y: x \in \langle f^{-1} \rangle \{y\}} \right\}) = (X \mapsto \prod_{x \in X} \langle f \rangle \{x\}, Y \mapsto \prod_{y \in Y} \langle f^{-1} \rangle \{y\}) = \Psi_3 f$. ■

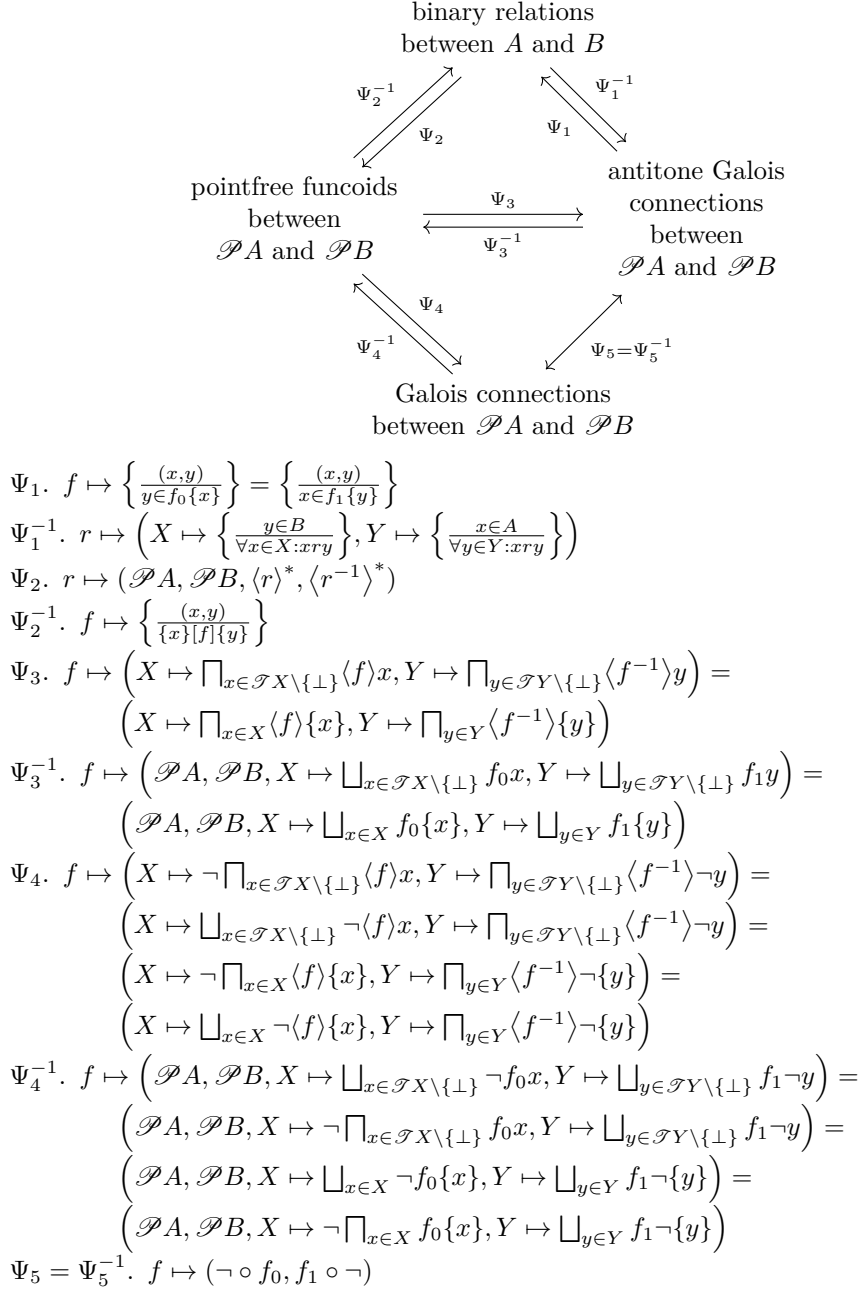


FIGURE 11

Claim: Ψ_1 maps antitone Galois connections between $\mathcal{P}A$ and $\mathcal{P}B$ into binary relations between A and B .

Proof: Obvious. ■

Claim: Ψ_1^{-1} maps binary relations between A and B into antitone Galois connections between $\mathcal{P}A$ and $\mathcal{P}B$.

Proof: We need to prove $Y \subseteq \left\{ \frac{y \in B}{\forall x \in X: x r y} \right\} \Leftrightarrow X \subseteq \left\{ \frac{x \in A}{\forall y \in Y: x r y} \right\}$. After we equivalently rewrite it:

$$\forall y \in Y \forall x \in X : x r y \Leftrightarrow \forall x \in X \forall y \in Y : x r y$$

it becomes obvious. ■

Claim: Ψ_2 maps binary relations between A and B into pointfree funcoids between $\mathcal{P}A$ and $\mathcal{P}B$.

Proof: We need to prove that $f = (\mathcal{P}A, \mathcal{P}B, \langle f \rangle, \langle f^{-1} \rangle)$ is a pointfree funcoids that is $Y \not\prec \langle f \rangle X \Leftrightarrow X \not\prec \langle f^{-1} \rangle Y$. Really, for every $X \in \mathcal{T}A, Y \in \mathcal{T}B$

$$\begin{aligned} Y \not\prec \langle f \rangle X &\Leftrightarrow Y \not\prec \langle r \rangle^* X \Leftrightarrow Y \not\prec \langle r \rangle X \Leftrightarrow \\ &X \not\prec \langle r^{-1} \rangle Y \Leftrightarrow X \not\prec \langle r^{-1} \rangle^* Y \Leftrightarrow X \not\prec \langle f^{-1} \rangle Y. \end{aligned}$$

Claim: Ψ_2^{-1} maps pointfree funcoids between $\mathcal{P}A$ and $\mathcal{P}B$ into binary relations between A and B . ■

Proof: Suppose $f \in \text{pFCD}(\mathcal{T}A, \mathcal{T}B)$ and prove that the relation defined by the formula Ψ_2^{-1} exists. To prove it, it's enough to show that $y \in \langle f \rangle \{x\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}$. Really,

$$y \in \langle f \rangle \{x\} \Leftrightarrow \{y\} \not\prec \langle f \rangle \{x\} \Leftrightarrow \{x\} \not\prec \langle f^{-1} \rangle \{y\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}.$$

Claim: Ψ_3 maps pointfree funcoids between $\mathcal{P}A$ and $\mathcal{P}B$ into antitone Galois connections between $\mathcal{P}A$ and $\mathcal{P}B$. ■

Proof: Because $\Psi_3 = \Psi_1^{-1} \circ \Psi_2^{-1}$. ■

Claim: Ψ_3^{-1} maps antitone Galois connections between $\mathcal{P}A$ and $\mathcal{P}B$ into pointfree funcoids between $\mathcal{P}A$ and $\mathcal{P}B$.

Proof: Because $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$. ■

Claim: Ψ_2 and Ψ_2^{-1} are mutually inverse.

Proof: Let $r_0 \in \mathcal{P}(A \times B)$ and $f \in \text{pFCD}(\mathcal{T}A, \mathcal{T}B)$ corresponds to r_0 by the formula Ψ_2 ; let $r_1 \in \mathcal{P}(A \times B)$ corresponds to f by the formula Ψ_2^{-1} . Then $r_0 = r_1$ because

$$(x, y) \in r_0 \Leftrightarrow y \in \langle r_0 \rangle^* \{x\} \Leftrightarrow y \in \langle f \rangle \{x\} \Leftrightarrow (x, y) \in r_1.$$

Let now $f_0 \in \text{pFCD}(\mathcal{T}A, \mathcal{T}B)$ and $r \in \mathcal{P}(A \times B)$ corresponds to f_0 by the formula Ψ_2^{-1} ; let $f_1 \in \text{pFCD}(\mathcal{T}A, \mathcal{T}B)$ corresponds to r by the formula Ψ_2 . Then $(x, y) \in r \Leftrightarrow y \in \langle f_0 \rangle \{x\}$ and $\langle f_1 \rangle = \langle r \rangle^*$; thus

$$y \in \langle f_1 \rangle \{x\} \Leftrightarrow y \in \langle r \rangle^* \{x\} \Leftrightarrow (x, y) \in r \Leftrightarrow y \in \langle f_0 \rangle \{x\}.$$

So $\langle f_0 \rangle = \langle f_1 \rangle$. Similarly $\langle f_0^{-1} \rangle = \langle f_1^{-1} \rangle$. ■

Claim: Ψ_1 and Ψ_1^{-1} are mutually inverse.

Proof: Let $r_0 \in \mathcal{P}(A \times B)$ and $f \in \mathcal{T}A \otimes \mathcal{T}B$ corresponds to r_0 by the formula Ψ_1^{-1} ; let $r_1 \in \mathcal{P}(A \times B)$ corresponds to f by the formula Ψ_1 . Then $r_0 = r_1$ because

$$(x, y) \in r_1 \Leftrightarrow y \in f_0 \{x\} \Leftrightarrow y \in \left\{ \frac{y \in B}{x r_0 y} \right\} \Leftrightarrow x r_0 y.$$

Let now $f_0 \in \mathcal{T}A \otimes \mathcal{T}B$ and $r \in \mathcal{P}(A \times B)$ corresponds to f_0 by the formula Ψ_1 ; let $f_1 \in \mathcal{T}A \otimes \mathcal{T}B$ corresponds to r by the formula Ψ_1^{-1} . Then $f_0 = f_1$ because

$$\begin{aligned} f_{10}X &= \left\{ \frac{y \in B}{\forall x \in X : x r y} \right\} = \left\{ \frac{y \in B}{\forall x \in X : y \in f_{00} \{x\}} \right\} = \\ &\prod_{x \in X} f_{00} \{x\} = (\text{obvious 142}) = f_{00}X. \end{aligned}$$

Claim: Ψ_3 and Ψ_3^{-1} are mutually inverse. ■

Proof: Because $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$ and $\Psi_3 = \Psi_1^{-1} \circ \Psi_2^{-1}$ and that Ψ_2^{-1} is the inverse of Ψ_2 and Ψ_3^{-1} is the inverse of Ψ_3 were proved above. ■

Now switch to the lower “triangle”:

Claim: $(X \mapsto \bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} \neg f_0 x, Y \mapsto \bigsqcup_{y \in \mathcal{T}Y \setminus \{\perp\}} f_1 \neg y) = (X \mapsto \bigsqcup_{x \in X} \neg f_0 \{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1 \neg \{y\})$.

Proof: It is enough to prove $\bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} \neg f_0 x = \bigsqcup_{x \in X} \neg f_0 \{x\}$ for a Galois connection f (the rest follows from symmetry).

$\bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} \neg f_0 x \supseteq \bigsqcup_{x \in X} \neg f_0 \{x\}$ because $\{x\} \in \mathcal{T}X \setminus \{\perp\}$. If $x \in \mathcal{T}X \setminus \{\perp\}$ then there exists $x' \in \{x\}$ and thus $\neg f_0 \{x'\} \supseteq \neg f_0 x$. Thus $\neg f_0 x \sqsubseteq \bigsqcup_{x \in X} \neg f_0 \{x\}$ and so $\bigsqcup_{x \in \mathcal{T}X \setminus \{\perp\}} \neg f_0 x \sqsubseteq \bigsqcup_{x \in X} \neg f_0 \{x\}$. ■

Claim: Ψ_5 is self-inverse.

Proof: Obvious. ■

Claim: $\Psi_4 = \Psi_5 \circ \Psi_3$.

Proof: Easily follows from symmetry. ■

Claim: $\Psi_4^{-1} = \Psi_3^{-1} \circ \Psi_5^{-1}$.

Proof: Easily follows from symmetry. ■

Claim: Ψ_4 and Ψ_4^{-1} are mutually inverse.

Proof: From two above claims and the fact that Ψ_3^{-1} is the inverse of Ψ_3 and Ψ_5^{-1} is the inverse of Ψ_5 proved above. ■

Note that now we have proved that Ψ_i and Ψ_i^{-1} are mutually inverse for all $i = 1, 2, 3, 4, 5$.

Claim: For every path of the diagram on figure 12 started with the circled node, the corresponding morphism is with which the node is labeled.

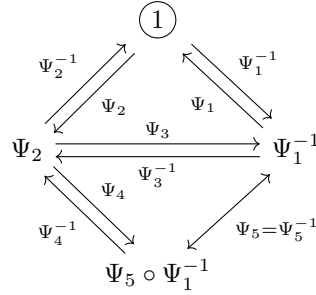


FIGURE 12

Proof: Take into account that $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$, $\Psi_4 = \Psi_5 \circ \Psi_3$ and thus also $\Psi_4 \circ \Psi_2 = \Psi_5 \circ \Psi_1^{-1}$. Now prove it by induction on path length. ■

Claim: Every cycle in the diagram at figure 11 is identity.

Proof: For cycles starting at the top node it follows from the previous claim. For arbitrary cycles it follows from theorem 195. ■

Claim: The diagram at figure 11 is commutative.

Proof: From the previous claim. ■

□

PROPOSITION 1719. We equate the set of binary relations between A and B with $\mathbf{Rld}(A, B)$. Ψ_2 and Ψ_2^{-1} from the diagram at figure 11 preserve composition and identities (that are functors between categories \mathbf{Rel} and $(A, B) \mapsto \mathbf{pFCD}(\mathcal{T}A, \mathcal{T}B)$) and also reversal ($f \mapsto f^{-1}$).

PROOF. Let $\langle f \rangle = \langle p \rangle^*$ and $\langle g \rangle = \langle q \rangle^*$. Then $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle = \langle q \rangle^* \circ \langle p \rangle^* = \langle q \circ p \rangle^*$. Likewise $\langle (g \circ f)^{-1} \rangle = \langle (q \circ p)^{-1} \rangle^*$. So Φ_2 preserves composition.

Let $p = 1_{\mathbf{Rel}}^A$ for some set A . Then $\langle f \rangle = \langle p \rangle^* = \langle 1_{\mathbf{Rel}}^A \rangle^* = \text{id}_{\mathcal{P}A}$ and likewise $\langle f^{-1} \rangle = \text{id}_{\mathcal{P}A}$, that is f is an identity pointfree funcoid. So Φ_2 preserves identities.

That Φ_2^{-1} preserves composition and identities follows from the fact that it is an isomorphism.

That it preserves reversal follows from the formula $\langle f^{-1} \rangle = \langle p^{-1} \rangle^*$. \square

PROPOSITION 1720. The bijections Ψ_2 and Ψ_2^{-1} from the diagram at figure 11 preserves monovaluedness and injectivity.

PROOF. Because it is a functor which preserves reversal. \square

PROPOSITION 1721. The bijections Ψ_2 and Ψ_2^{-1} from the diagram at figure 11 preserves domain and image.

PROOF. $\text{im } f = \langle f \rangle \top = \langle p \rangle^* \top = \text{im } p$, likewise for domain. \square

PROPOSITION 1722. The bijections Ψ_2 and Ψ_2^{-1} from the diagram at figure 11 maps cartesian products to corresponding funcoidal products.

PROOF. $\langle A \times^{\text{FCD}} B \rangle X = \begin{cases} B & \text{if } X \not\asymp A \\ \perp & \text{if } X \asymp A \end{cases} = \langle A \times B \rangle^* X$. Likewise $\langle (A \times^{\text{FCD}} B)^{-1} \rangle Y = \langle (A \times B)^{-1} \rangle^* Y$. \square

Let Φ map a pointfree funcoid whose first component is c into the Galois connection whose lower adjoint is c . Then Φ is an isomorphism (theorem 1710) and Φ^{-1} maps a Galois connection whose lower adjoint is c into the pointfree funcoid whose first component is c .

Informally speaking, Φ replaces a relation r with its complement relations $\neg r$. Formally:

PROPOSITION 1723.

- 1°. For every path P in the diagram at figure 11 from binary relations between A and B to pointfree funcoids between $\mathcal{P}A$ and $\mathcal{P}B$ and every path Q in the diagram at figure 11 from Galois connections between $\mathcal{P}A$ and $\mathcal{P}B$ to binary relations between A and B , we have $Q\Phi Pr = \neg r$.
- 2°. For every path Q in the diagram at figure 11 from binary relations between A and B to pointfree funcoids between $\mathcal{P}A$ and $\mathcal{P}B$ and every path P in the diagram at figure 11 from Galois connections between $\mathcal{P}A$ and $\mathcal{P}B$ to binary relations between A and B , we have $P\Phi^{-1}Qr = \neg r$.

PROOF. We will prove only the second ($P \circ \Phi^{-1} \circ Q = \neg$), because the first ($Q \circ \Phi \circ P = \neg$) can be obtained from it by inverting the morphisms (and variable replacement).

Because the diagram is commutative, it is enough to prove it for some fixed P and Q . For example, we will prove $\Psi_2^{-1}\Phi^{-1}\Psi_4\Psi_2r = \neg r$.

$$\Psi_4\Psi_2r = \left(X \mapsto \neg \prod_{x \in X} \langle r \rangle^* \{x\}, Y \mapsto \prod_{y \in Y} \langle r \rangle^* \neg \{y\} \right).$$

$$\Phi^{-1}\Psi_4\Psi_2r \text{ is pointfree funcoid } f \text{ with } \langle f \rangle = X \mapsto \neg \prod_{x \in X} \langle r \rangle^* \{x\}.$$

$\Psi_2^{-1}\Phi^{-1}\Psi_4\Psi_2r$ is the relation consisting of (x, y) such that $\{x\} [f] \{y\}$ what is equivalent to: $\{y\} \neq \langle f \rangle \{x\}$; $\{y\} \neq \neg \langle r \rangle^* \{x\}$; $\{y\} \not\sqsubseteq \langle r \rangle^* \{x\}$; $y \notin \langle r \rangle^* \{x\}$.

So $\Psi_2^{-1}\Phi^{-1}\Psi_4\Psi_2r = \neg r$. \square

PROPOSITION 1724. Φ and Φ^{-1} preserve composition.

PROOF. By definitions of compositions and the fact that both pointfree funcoids and Galois connections are determined by the first component. \square

Part 4

Staroids and multifuncoids

Disjoint product

I remind that $\coprod X = \bigcup_{i \in \text{dom } X} (i, X_i)$ for every indexed family X of sets.

OBVIOUS 1725. $\coprod X \in \mathbf{Set}(\text{dom } X, \text{im } X)$.

DEFINITION 1726. I will call disjoint join of an indexed family \mathcal{X} of filters the following reloid: $\coprod \mathcal{X} = \bigsqcup_{i \in \text{dom } f} (\{i\} \times^{\text{RLD}} \mathcal{X}_i)$.

22.1. Some funcoids

PROPOSITION 1727. $\langle x \mapsto (i, x) \rangle \mathcal{X} = \{i\} \times^{\text{RLD}} \mathcal{X}$ for every filter \mathcal{X} .

PROOF. $\langle x \mapsto (i, x) \rangle \mathcal{X} = \prod \left\{ \frac{\langle x \mapsto (i, x) \rangle^* X}{X \in \text{up } \mathcal{X}} \right\} = \prod \left\{ \frac{\{i\} \times X}{X \in \text{up } \mathcal{X}} \right\} = \{i\} \times^{\text{RLD}} \mathcal{X}$. \square

PROPOSITION 1728. $\langle (x \mapsto (i, x))^{-1} \rangle \mathcal{X} = \text{im}(\mathcal{X}|_{\{i\}})$ for a filter \mathcal{X} on the set $\mathcal{U} \cup \{\mathcal{U}\}$ where \mathcal{U} is a Grothendieck universe.

PROOF. $\langle (x \mapsto (i, x))^{-1} \rangle \mathcal{X} = \prod \left\{ \frac{\langle (x \mapsto (i, x))^{-1} \rangle^* X}{X \in \text{up } \mathcal{X}} \right\} = \prod \left\{ \frac{\left\{ \frac{x}{(i, x) \in X} \right\}}{X \in \text{up } \mathcal{X}} \right\} = \prod \left\{ \frac{x \in \text{im } X|_{\{i\}}}{X \in \text{up } \mathcal{X}} \right\} = \text{im} \prod \left\{ \frac{x \in X|_{\{i\}}}{X \in \text{up } \mathcal{X}} \right\} = \text{im}(\mathcal{X}|_{\{i\}})$. \square

22.2. Cartesian product of funcoids

22.2.1. Definition and basic properties.

DEFINITION 1729. *Cartesian product* of an indexed family f of funcoids is a funcoid

$$\prod^{(J)} f = \bigsqcup_{i \in \text{dom } f} ((x \mapsto (i, x)) \circ f_i \circ (x \mapsto (i, x))^{-1}).$$

PROPOSITION 1730. $\left\langle \prod^{(J)} f \right\rangle \mathcal{X} = \prod_{i \in \text{dom } f} \langle f_i \circ (x \mapsto (i, x))^{-1} \rangle \mathcal{X}$.

PROOF.

$$\begin{aligned} \prod_{i \in \text{dom } f} \langle f_i \circ (x \mapsto (i, x))^{-1} \rangle \mathcal{X} &= \\ \bigsqcup_{i \in \text{dom } f} (\{i\} \times^{\text{RLD}} \langle f_i \circ (x \mapsto (i, x))^{-1} \rangle \mathcal{X}) &= \\ \bigsqcup_{i \in \text{dom } f} (\langle x \mapsto (i, x) \rangle \langle f_i \circ (x \mapsto (i, x))^{-1} \rangle \mathcal{X}) &= \\ \left\langle \prod^{(J)} f \right\rangle \mathcal{X}. & \end{aligned}$$

\square

22.2.2. Projections.

THEOREM 1731. f_i can be restored from the value of $\prod^{(J)} f = f_i$.

PROOF. $f_i = (x \mapsto (i, x)^{-1}) \circ \prod^{(J)} f \circ (x \mapsto (i, x)^{-1})$ (taken into account that $x \mapsto (i, x)^{-1}$ is a principal funcoid). \square

22.3. Arrow product of funcoids

DEFINITION 1732. *Arrow product* of an indexed family f of funcoids is a funcoid

$$\prod^{\rightarrow} f = \bigsqcup_{i \in \text{dom } f} ((x \mapsto (i, x)) \circ f_i.$$

PROPOSITION 1733. $\langle \prod^{\rightarrow} f \rangle \mathcal{X} = \prod_{i \in \text{dom } f} \langle f_i \rangle \mathcal{X}$.

PROOF.

$$\prod_{i \in \text{dom } f} \langle f_i \rangle \mathcal{X} =$$

$$\begin{aligned} & \bigsqcup_{i \in \text{dom } f} (\{i\} \times^{\text{RLD}} \langle f_i \rangle \mathcal{X}) = \\ & \bigsqcup_{i \in \text{dom } f} (\langle x \mapsto (i, x) \rangle \langle f_i \rangle \mathcal{X}) = \end{aligned}$$

$$\left\langle \prod^{\rightarrow} f \right\rangle \mathcal{X}.$$

\square

22.3.1. Projections.

DEFINITION 1734. *Arrow projections* $\pi_i^{\rightarrow} = (x \mapsto (i, x))^{-1}$.

THEOREM 1735. $\pi_i^{\rightarrow} \circ \prod^{\rightarrow} f = f_i$

PROOF. Because π_i^{\rightarrow} is a principal funcoid, we have

$$\pi_i^{\rightarrow} \circ \prod^{\rightarrow} f = \bigsqcup_{j \in \text{dom } f} ((x \mapsto (i, x))^{-1} \circ (x \mapsto (j, x)) \circ f_j).$$

But $(x \mapsto (i, x))^{-1} \circ (x \mapsto (j, x))$ is the identity if $i = j$ or empty otherwise. So $\pi_i^{\rightarrow} \circ \prod^{\rightarrow} f = f_i$. \square

22.4. Cartesian product of reloids**22.4.1. Definition and basic properties.**

DEFINITION 1736. *Cartesian product* of an indexed family f of reloids is a reloid

$$\prod^{(J)} f = \bigsqcup_{i \in \text{dom } f} ((x \mapsto (i, x)) \circ f_i \circ (x \mapsto (i, x)^{-1}).$$

CONJECTURE 1737. $\prod^{(J)} (g \circ f) = \prod^{(J)} g \circ \prod^{(J)} f$.

22.4.2. Projections.

THEOREM 1738. f_i can be restored from the value of $\prod^{(J)} f = f_i$.

PROOF. $f_i = (x \mapsto (i, x)^{-1}) \circ \prod^{(J)} f \circ (x \mapsto (i, x)^{-1})$. \square

22.5. Arrow product of reloids

DEFINITION 1739. *Arrow product* of an indexed family f of reloids is a reloid

$$\prod_{\rightarrow} f = \bigsqcup_{i \in \text{dom } f} ((x \mapsto (i, x)) \circ f_i).$$

22.5.1. Projections.

DEFINITION 1740. *Arrow projections* $\pi_i^{\rightarrow} = (x \mapsto (i, x))^{-1}$.

PROPOSITION 1741. $\pi_i^{\rightarrow} \circ \prod_{\rightarrow} f = f_i$.

PROOF. Because $x \mapsto (i, x)$ is a principal funcoid, we have $\pi_i^{\rightarrow} \circ \prod_{\rightarrow} f = \pi_i^{\rightarrow} \circ \bigsqcup_{i \in \text{dom } f} ((x \mapsto (i, x)) \circ f_i) = \bigsqcup_{j \in \text{dom } f} ((x \mapsto (i, x))^{-1} \circ (x \mapsto (j, x)) \circ f_j$.

But $(x \mapsto (i, x))^{-1} \circ (x \mapsto (j, x))$ is the identity if $i = j$ or empty otherwise. So $\pi_i^{\rightarrow} \circ \prod_{\rightarrow} f = f_i$. \square

Multifuncoids and staroids

23.1. Product of two funcoids

23.1.1. Definition.

DEFINITION 1742. I will call a *quasi-invertible category* a partially ordered dagger category such that it holds

$$g \circ f \not\prec h \Leftrightarrow g \not\prec h \circ f^\dagger \quad (32)$$

for every morphisms $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$, $h \in \text{Hom}(A, C)$, where A, B, C are objects of this category.

Inverting this formula, we get $f^\dagger \circ g^\dagger \not\prec h^\dagger \Leftrightarrow g^\dagger \not\prec f \circ h^\dagger$. After replacement of variables, this gives: $f^\dagger \circ g \not\prec h \Leftrightarrow g \not\prec f \circ h$.

EXERCISE 1743. Prove that every ordered groupoid is quasi-invertible category if we define the dagger as the inverse morphism.

As it follows from above, the categories **Rel** of binary relations (proposition 283), FCD of funcoids (theorem 882) and RLD of reloids (theorem 1004) are quasi-invertible (taking $f^\dagger = f^{-1}$). Moreover the category of pointfree funcoids between lattices of filters on boolean lattices is quasi-invertible (theorem 1657).

DEFINITION 1744. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree funcoid $\text{Hom}(\text{Src } f, \text{Src } g) \rightarrow \text{Hom}(\text{Dst } f, \text{Dst } g)$ defined by the formulas (for every $a \in \text{Hom}(\text{Src } f, \text{Src } g)$ and $b \in \text{Hom}(\text{Dst } f, \text{Dst } g)$):

$$\langle f \times^{(C)} g \rangle a = g \circ a \circ f^\dagger \quad \text{and} \quad \langle (f \times^{(C)} g)^{-1} \rangle b = g^\dagger \circ b \circ f.$$

We need to prove that it is really a pointfree funcoid that is that

$$b \not\prec \langle f \times^{(C)} g \rangle a \Leftrightarrow a \not\prec \langle (f \times^{(C)} g)^{-1} \rangle b.$$

This formula means $b \not\prec g \circ a \circ f^\dagger \Leftrightarrow a \not\prec g^\dagger \circ b \circ f$ and can be easily proved applying formula (32) twice.

PROPOSITION 1745. $a [f \times^{(C)} g] b \Leftrightarrow a \circ f^\dagger \not\prec g^\dagger \circ b$.

PROOF. From the definition. □

PROPOSITION 1746. $a [f \times^{(C)} g] b \Leftrightarrow f [a \times^{(C)} b] g$.

PROOF. $f [a \times^{(C)} b] g \Leftrightarrow f \circ a^\dagger \not\prec b^\dagger \circ g \Leftrightarrow a \circ f^\dagger \not\prec g^\dagger \circ b \Leftrightarrow a [f \times^{(C)} g] b$. □

THEOREM 1747. $(f \times^{(C)} g)^{-1} = f^\dagger \times^{(C)} g^\dagger$.

PROOF. For every morphisms $a \in \text{Hom}(\text{Src } f, \text{Src } g)$ and $b \in \text{Hom}(\text{Dst } f, \text{Dst } g)$ we have:

$$\begin{aligned} \langle (f \times^{(C)} g)^{-1} \rangle b &= g^\dagger \circ b \circ f = \langle f^\dagger \times^{(C)} g^\dagger \rangle b. \\ \langle ((f \times^{(C)} g)^{-1})^{-1} \rangle a &= \langle f \times^{(C)} g \rangle a = g \circ a \circ f^\dagger = \langle (f^\dagger \times^{(C)} g^\dagger)^{-1} \rangle a. \end{aligned} \quad \square$$

THEOREM 1748. Let f, g be pointfree funcoids between filters on boolean lattices. Then for every filters $\mathcal{A}_0 \in \mathcal{F}(\text{Src } f)$, $\mathcal{B}_0 \in \mathcal{F}(\text{Src } g)$

$$\langle f \times^{(C)} g \rangle (\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\text{FCD}} \langle g \rangle \mathcal{B}_0.$$

PROOF. For every atom $a_1 \times^{\text{FCD}} b_1$ ($a_1 \in \text{atoms}^{\text{Dst } f}$, $b_1 \in \text{atoms}^{\text{Dst } g}$) (see theorem 1675) of the lattice of funcoids we have:

$$\begin{aligned} a_1 \times^{\text{FCD}} b_1 \neq \langle f \times^{(C)} g \rangle (\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) &\Leftrightarrow \\ \mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0 [f \times^{(C)} g] a_1 \times^{\text{FCD}} b_1 &\Leftrightarrow \\ (\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \circ f^{-1} \neq g^{-1} \circ (a_1 \times^{\text{FCD}} b_1) &\Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0 \neq a_1 \times^{\text{FCD}} \langle g^{-1} \rangle b_1 &\Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \neq a_1 \wedge \langle g^{-1} \rangle b_1 \neq \mathcal{B}_0 &\Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \neq a_1 \wedge \langle g \rangle \mathcal{B}_0 \neq b_1 &\Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \times^{\text{FCD}} \langle g \rangle \mathcal{B}_0 \neq a_1 \times^{\text{FCD}} b_1. & \end{aligned}$$

Thus $\langle f \times^{(C)} g \rangle (\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\text{FCD}} \langle g \rangle \mathcal{B}_0$ because the lattice $\text{pFCD}(\mathcal{F}(\text{Dst } f), \mathcal{F}(\text{Dst } g))$ is atomically separable (corollary 1666). \square

COROLLARY 1749. $\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0 [f \times^{(C)} g] \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$ for every $\mathcal{A}_0 \in \mathcal{F}(\text{Src } f)$, $\mathcal{A}_1 \in \mathcal{F}(\text{Dst } f)$, $\mathcal{B}_0 \in \mathcal{F}(\text{Src } g)$, $\mathcal{B}_1 \in \mathcal{F}(\text{Dst } g)$ where $\text{Src } f, \text{Dst } f, \text{Src } g, \text{Dst } g$ are boolean lattices.

PROOF.

$$\begin{aligned} \mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0 [f \times^{(C)} g] \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 &\Leftrightarrow \\ \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 \neq \langle f \times^{(C)} g \rangle \mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0 &\Leftrightarrow \\ \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 \neq \langle f \rangle \mathcal{A}_0 \times^{\text{FCD}} \langle g \rangle \mathcal{B}_0 &\Leftrightarrow \\ \mathcal{A}_1 \neq \langle f \rangle \mathcal{A}_0 \wedge \mathcal{B}_1 \neq \langle g \rangle \mathcal{B}_0 &\Leftrightarrow \\ \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1. & \end{aligned}$$

\square

23.2. Definition of staroids

It follows from the above theorem 831 that funcoids are essentially the same as relations δ between sets A and B , such that $\left\{ \frac{Y \in \mathcal{P} B}{\exists X \in \mathcal{P} A: X \delta Y} \right\}$ and $\left\{ \frac{X \in \mathcal{P} A}{\exists Y \in \mathcal{P} B: X \delta Y} \right\}$ are free stars. This inspires the below definition of staroids (switching from two sets X and Y to a (potentially infinite) family of posets).

Whilst I have (mostly) thoroughly studied basic properties of funcoids, *staroids* (defined below) are yet much a mystery. For example, we do not know whether the set of staroids on powersets is atomic.

Let n be a set. As an example, n may be an ordinal, n may be a natural number, considered as a set by the formula $n = \{0, \dots, n-1\}$. Let $\mathfrak{A} = \mathfrak{A}_{i \in n}$ be a family of posets indexed by the set n .

DEFINITION 1750. I will call an *anchored relation* a pair $f = (\text{form } f, \text{GR } f)$ of a family $\text{form}(f)$ of relational structures indexed by some index set and a relation $\text{GR}(f) \in \mathcal{P} \prod \text{form}(f)$. I call $\text{GR}(f)$ the *graph* of the anchored relation f . I denote $\text{Anch}(\mathfrak{A})$ the set of anchored relations of the form \mathfrak{A} .

DEFINITION 1751. *Infinitary anchored relation* is such an anchored relation whose arity is infinite; *finitary anchored relation* is such an anchored relation whose arity is finite.

DEFINITION 1752. An anchored relation *on powersets* is an anchored relation f such that every $(\text{form } f)_i$ is a powerset.

I will denote $\text{arity } f = \text{dom form } f$.

DEFINITION 1753. $[f]^*$ is the relation between typed elements $\mathfrak{T}(\text{form } f)_i$ (for $i \in \text{arity } f$) defined by the formula $L \in [f]^* \Leftrightarrow \mathfrak{T} \circ L \in \text{GR } f$.

Every set of anchored relations of the same form constitutes a poset by the formula $f \sqsubseteq g \Leftrightarrow \text{GR } f \subseteq \text{GR } g$.

DEFINITION 1754. An anchored relation is an *anchored relation between posets* when every $(\text{form } f)_i$ is a poset.

DEFINITION 1755. $(\text{val } f)_i L = \left\{ \frac{X \in (\text{form } f)_i}{L \cup \{(i, X)\} \in \text{GR } f} \right\}$.

PROPOSITION 1756. f can be restored knowing $\text{form}(f)$ and $(\text{val } f)_i$ for some $i \in \text{arity } f$.

PROOF.

$$\begin{aligned} \text{GR } f &= \left\{ \frac{K \in \prod \text{form } f}{K \in \text{GR } f} \right\} = \\ &= \left\{ \frac{L \cup \{(i, X)\}}{L \in \prod (\text{form } f)|_{(\text{arity } f) \setminus \{i\}}, X \in (\text{form } f)_i, L \cup \{(i, X)\} \in \text{GR } f} \right\} = \\ &= \left\{ \frac{L \cup \{(i, X)\}}{L \in \prod (\text{form } f)|_{(\text{arity } f) \setminus \{i\}}, X \in (\text{val } f)_i L} \right\}. \end{aligned}$$

□

DEFINITION 1757. A *prestaroid* is an anchored relation f between posets such that $(\text{val } f)_i L$ is a free star for every $i \in \text{arity } f$, $L \in \prod (\text{form } f)|_{(\text{arity } f) \setminus \{i\}}$.

DEFINITION 1758. A *staroid* is a prestaroid whose graph is an upper set (on the poset $\prod \text{form}(f)$).

DEFINITION 1759. A *(pre)staroid on power sets* is such a (pre)staroid f that every $(\text{form } f)_i$ is a lattice of all subsets of some set.

PROPOSITION 1760. If $L \in \prod \text{form } f$ and $L_i = \perp^{(\text{form } f)_i}$ for some $i \in \text{arity } f$ then $L \notin \text{GR } f$ if f is a prestaroid.

PROOF. Let $K = L|_{(\text{arity } f) \setminus \{i\}}$. We have $\perp \notin (\text{val } f)_i K$; $K \cup \{(i, \perp)\} \notin \text{GR } f$; $L \notin \text{GR } f$. □

Next we will define *completary staroids*. First goes the general case, next simpler case for the special case of join-semilattices instead of arbitrary posets.

DEFINITION 1761. A *completary staroid* is an anchored relation between posets conforming to the formulas:

- 1°. $\forall K \in \prod \text{form } f : (K \supseteq L_0 \wedge K \supseteq L_1 \Rightarrow K \in \text{GR } f)$ is equivalent to $\exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)} i) \in \text{GR } f$ for every $L_0, L_1 \in \prod \text{form } f$.
- 2°. If $L \in \prod \text{form } f$ and $L_i = \perp^{(\text{form } f)_i}$ for some $i \in \text{arity } f$ then $L \notin \text{GR } f$.

LEMMA 1762. Every graph of completary staroid is an upper set.

PROOF. Let f be a completary staroid. Let $L_0 \sqsubseteq L_1$ for some $L_0, L_1 \in \prod \text{form } f$ and $L_0 \in \text{GR } f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n : L_{c(i)} i = \lambda i \in n : L_0 i = L_0 \in \text{GR } f$ and thus $L_1 \in \text{GR } f$ because $L_1 \supseteq L_0 \wedge L_1 \supseteq L_1$. □

PROPOSITION 1763. An anchored relation f between posets whose form is a family of join-semilattices is a completary staroid iff both:

- 1°. $L_0 \sqcup L_1 \in \text{GR } f \Leftrightarrow \exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)}i) \in \text{GR } f$ for every $L_0, L_1 \in \prod \text{form } f$.
- 2°. If $L \in \prod \text{form } f$ and $L_i = \perp^{(\text{form } f)_i}$ for some $i \in \text{arity } f$ then $L \notin \text{GR } f$.

PROOF. Let the formulas 1° and 2° hold. Then f is an upper set: Let $L_0 \sqsubseteq L_1$ for some $L_0, L_1 \in \prod \text{form } f$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n : L_{c(i)}i = \lambda i \in n : L_0i = L_0 \in \text{GR } f$ and thus $L_1 = L_0 \sqcup L_1 \in \text{GR } f$.

Thus to finish the proof it is enough to show that

$$L_0 \sqcup L_1 \in \text{GR } f \Leftrightarrow \forall K \in \prod \text{form } f : (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \text{GR } f)$$

under condition that $\text{GR } f$ is an upper set. But this equivalence is obvious in both directions. \square

PROPOSITION 1764. Every completary staroid is a staroid.

PROOF. Let f be a completary staroid.

Let $i \in \text{arity } f$, $K \in \prod_{i \in (\text{arity } f) \setminus \{i\}} (\text{form } f)_i$. Let $L_0 = K \cup \{(i, X_0)\}$, $L_1 = K \cup \{(i, X_1)\}$ for some $X_0, X_1 \in \mathfrak{A}_i$.

Let

$$\forall Z \in \mathfrak{A}_i : (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow Z \in (\text{val } f)_i K);$$

then

$$\forall Z \in \mathfrak{A}_i : (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow K \cup \{(i, Z)\} \in \text{GR } f).$$

If $z \sqsupseteq L_0 \wedge z \sqsupseteq L_1$ then $z \sqsupseteq K \cup \{(i, z_i)\}$, thus taking into account that $\text{GR } f$ is an upper set,

$$\forall z \in \prod \mathfrak{A} : (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow K \cup \{(i, z_i)\} \in \text{GR } f).$$

$$\forall z \in \prod \mathfrak{A} : (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow z \in \text{GR } f).$$

Thus, by the definition of completary staroid, $L_0 \in \text{GR } f \vee L_1 \in \text{GR } f$ that is

$$X_0 \in (\text{val } f)_i K \vee X_1 \in (\text{val } f)_i K.$$

So $(\text{val } f)_i K$ is a free star (taken into account that $z_i = \perp^{(\text{form } f)_i} \Rightarrow z \notin \text{GR } f$ and that $(\text{val } f)_i K$ is an upper set). \square

EXERCISE 1765. Write a simplified proof for the case if every $(\text{form } f)_i$ is a join-semilattice.

LEMMA 1766. Every finitary prestaroid is completary.

PROOF.

$$\begin{aligned}
& \exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)}i) \in \text{GR } f \Leftrightarrow \\
& \exists c \in \{0, 1\}^{n-1} : \left(\left(\{(n-1, L_0(n-1))\} \cup (\lambda i \in n-1 : L_{c(i)}i) \right) \in \text{GR } f \vee \right. \\
& \quad \left. \left(\{(n-1, L_1(n-1))\} \cup (\lambda i \in n-1 : L_{c(i)}i) \right) \in \text{GR } f \right) \Leftrightarrow \\
& \exists c \in \{0, 1\}^{n-1} : \left(L_0(n-1) \in (\text{val } f)_{n-1}(\lambda i \in n-1 : L_{c(i)}i) \vee \right. \\
& \quad \left. L_1(n-1) \in (\text{val } f)_{n-1}(\lambda i \in n-1 : L_{c(i)}i) \right) \Leftrightarrow \\
& \exists c \in \{0, 1\}^{n-1} \forall K \in (\text{form } f)_i : \left(\begin{array}{l} K \sqsupseteq L_0(n-1) \vee K \sqsupseteq L_1(n-1) \Rightarrow \\ K \in (\text{val } f)_{n-1}(\lambda i \in n-1 : L_{c(i)}i) \end{array} \right) \Leftrightarrow \\
& \exists c \in \{0, 1\}^{n-1} \forall K \in (\text{form } f)_i : \left(\begin{array}{l} K \sqsupseteq L_0(n-1) \vee K \sqsupseteq L_1(n-1) \Rightarrow \\ \{(n-1, K)\} \cup (\lambda i \in n-1 : L_{c(i)}i) \in \text{GR } f \end{array} \right) \Leftrightarrow \\
& \quad \dots \\
& \forall K \in \prod \text{form } f : (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \text{GR } f). \quad \square
\end{aligned}$$

EXERCISE 1767. Prove the simpler special case of the above theorem when the form is a family of join-semilattices.

THEOREM 1768. For finite arity the following are the same:

- 1°. prestaroids;
- 2°. staroids;
- 3°. completary staroids.

PROOF. f is a finitary prestaroid $\Rightarrow f$ is a finitary completary staroid.
 f is a finitary completary staroid $\Rightarrow f$ is a finitary staroid.
 f is a finitary staroid $\Rightarrow f$ is a finitary prestaroid. \square

DEFINITION 1769. We will denote the set of staroids of a form \mathfrak{A} as $\text{Strd}(\mathfrak{A})$.

23.3. Upgrading and downgrading a set regarding a filtrator

Let fix a filtrator $(\mathfrak{A}, \mathfrak{B})$.

DEFINITION 1770. $\Downarrow f = f \cap \mathfrak{B}$ for every $f \in \mathcal{P}\mathfrak{A}$ (downgrading f).

DEFINITION 1771. $\Uparrow f = \left\{ \frac{L \in \mathfrak{A}}{\text{up } L \subseteq f} \right\}$ for every $f \in \mathcal{P}\mathfrak{B}$ (upgrading f).

OBVIOUS 1772. $a \in \Uparrow f \Leftrightarrow \text{up } a \subseteq f$ for every $f \in \mathcal{P}\mathfrak{B}$ and $a \in \mathfrak{A}$.

PROPOSITION 1773. $\Downarrow \Uparrow f = f$ if f is an upper set for every $f \in \mathcal{P}\mathfrak{B}$.

PROOF. $\Downarrow \Uparrow f = \Uparrow f \cap \mathfrak{B} = \left\{ \frac{L \in \mathfrak{B}}{\text{up } L \subseteq f} \right\} = \left\{ \frac{L \in \mathfrak{B}}{L \subseteq f} \right\} = f \cap \mathfrak{B} = f. \quad \square$

23.3.1. Upgrading and downgrading staroids. Let fix a family $(\mathfrak{A}, \mathfrak{B})$ of filtrators.

For a graph f of an anchored relation between posets define $\Downarrow f$ and $\Uparrow f$ taking the filtrator of $(\prod \mathfrak{A}, \prod \mathfrak{B})$.

For a anchored relation between posets f define:

$$\begin{aligned}
& \text{form } \Downarrow f = \mathfrak{B} \quad \text{and} \quad \text{GR } \Downarrow f = \Downarrow \text{GR } f; \\
& \text{form } \Uparrow f = \mathfrak{A} \quad \text{and} \quad \text{GR } \Uparrow f = \Uparrow \text{GR } f.
\end{aligned}$$

Below we will show that under certain conditions upgraded staroid is a staroid, see theorem 1798.

PROPOSITION 1774. $(\text{val } \Downarrow f)_i L = (\text{val } f)_i L \cap \mathfrak{Z}_i$ for every $L \in \prod \mathfrak{Z}_{(\text{arity } f) \setminus \{i\}}$.

PROOF. $(\text{val } \Downarrow f)_i L = \left\{ \frac{X \in \mathfrak{Z}_i}{L \cup \{(i, X)\} \in \text{GR } f \cap \prod \mathfrak{Z}} \right\} = \left\{ \frac{X \in \mathfrak{Z}_i}{L \cup \{(i, X)\} \in \text{GR } f} \right\} = (\text{val } f)_i L \cap \mathfrak{Z}_i. \quad \square$

PROPOSITION 1775. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be binarily join-closed filtrators with both the base and the core being join-semilattices. If f is a staroid of the form \mathfrak{A} , then $\Downarrow f$ is a staroid of the form \mathfrak{Z} .

PROOF. Let f be a staroid.

We need to prove that $(\text{val } \Downarrow f)_i L$ is a free star. It follows from the last proposition and the fact that it is binarily join-closed. \square

PROPOSITION 1776. Let each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ for $i \in n$ (where n is an index set) be a binarily join-closed filtrator, such that each \mathfrak{A}_i and each \mathfrak{Z}_i are join-semilattices. If f is a completary staroid of the form \mathfrak{A} then $\Downarrow f$ is a completary staroid of the form \mathfrak{Z} .

PROOF.

$$\begin{aligned} L_0 \sqcup^{\mathfrak{Z}} L_1 \in \text{GR } \Downarrow f &\Leftrightarrow L_0 \sqcup^{\mathfrak{Z}} L_1 \in \text{GR } f \Leftrightarrow L_0 \sqcup^{\mathfrak{A}} L_1 \in \text{GR } f \Leftrightarrow \\ &\exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)} i) \in \text{GR } f \Leftrightarrow \\ &\exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)} i) \in \text{GR } \Downarrow f \end{aligned}$$

for every $L_0, L_1 \in \prod \mathfrak{Z}$. \square

23.4. Principal staroids

DEFINITION 1777. The *staroid generated* by an anchored relation F is the staroid $f = \uparrow^{\text{Strd}} F$ on powersets such that $\uparrow \circ L \in \text{GR } f \Leftrightarrow \prod L \not\neq F$ and $(\text{form } f)_i = \mathcal{T}(\text{form } F)_i$ for every $L \in \prod_{i \in \text{arity } f} \mathcal{T}(\text{form } F)_i$.

REMARK 1778. Below we will prove that staroid generated by an anchored relation is a staroid and moreover a completary staroid.

DEFINITION 1779. A *principal staroid* is a staroid generated by some anchored relation.

PROPOSITION 1780. Every principal staroid is a completary staroid.

PROOF. That $L \notin \text{GR } f$ if $L_i = \perp^{(\text{form } f)_i}$ for some $i \in \text{arity } f$ is obvious. It remains to prove

$$\prod (L_0 \sqcup L_1) \not\neq F \Leftrightarrow \exists c \in \{0, 1\}^{\text{arity } f} : \prod_{i \in n} L_{c(i)} i \not\neq F.$$

Really

$$\begin{aligned}
& \prod (L_0 \sqcup L_1) \not\asymp F \Leftrightarrow \\
& \exists x \in \prod (L_0 \sqcup L_1) : x \in F \Leftrightarrow \\
& \exists x \in \prod_{i \in \text{arity } f} (\text{form } f)_i \forall i \in \text{arity } f : (x_i \in L_0 i \sqcup L_1 i \wedge x \in F) \Leftrightarrow \\
& \exists x \in \prod_{i \in \text{arity } f} (\text{form } f)_i \forall i \in \text{arity } f : ((x_i \in L_0 i \vee x_i \in L_1 i) \wedge x \in F) \Leftrightarrow \\
& \exists x \in \prod_{i \in \text{arity } f} (\text{form } f)_i \left(\exists c \in \{0, 1\}^{\text{arity } f} : x \in \prod_{i \in \text{arity } f} L_{c(i)} i \wedge x \in F \right) \Leftrightarrow \\
& \exists c \in \{0, 1\}^{\text{arity } f} : \prod_{i \in n} L_{c(i)} i \not\asymp F.
\end{aligned}$$

□

DEFINITION 1781. The *upgraded staroid generated* by an anchored relation F is the anchored relation $\uparrow\uparrow\text{Strd } F$.

PROPOSITION 1782. $\uparrow\text{Strd } F = \downarrow\uparrow\uparrow\text{Strd } F$.

PROOF. Because GR $\uparrow\text{Strd } F$ is an upper set. □

EXAMPLE 1783. There is such anchored relation f that $\uparrow\uparrow f$ is not a completary staroid. This also proves existence of non-completary staroids (but not on powersets).

PROOF. (based on an ANDREAS BLASS's proof) Take f the set of functions $x : \mathbb{N} \rightarrow \mathbb{N}$ where x_0 is an arbitrary natural number and $x_i = \begin{cases} 0 & \text{if } n \leq x_0 \\ 1 & \text{if } n > x_0 \end{cases}$ for $i = 1, 2, 3, \dots$. Thus f is the graph of a staroid of the form $\lambda i \in \mathbb{N} : \mathcal{P}\mathbb{N}$ (on powersets).

Let $\mathcal{L}_0(0) = \mathcal{L}_1(0) = \Omega(\mathbb{N})$, $\mathcal{L}_0(i) = \uparrow\{0\}$ and $\mathcal{L}_1(i) = \uparrow\{1\}$ for $i > 0$.

Let $X \in \text{up}(\mathcal{L}_0 \sqcup \mathcal{L}_1)$ that is $X \in \text{up } \mathcal{L}_0 \cap \text{up } \mathcal{L}_1$.

X_0 contains all but finitely many elements of \mathbb{N} .

For $i > 0$ we have $\{0, 1\} \subseteq X_i$.

Evidently, $\prod X$ contains an element of f , that is $\text{up}(\mathcal{L}_0 \sqcup \mathcal{L}_1) \in f$ what means $\mathcal{L}_0 \sqcup \mathcal{L}_1 \in \uparrow\uparrow f$.

Now consider any fixed $c \in \{0, 1\}^{\mathbb{N}}$. There is at most one $k \in \mathbb{N}$ such that the sequence $x = \llbracket k, c(1), c(2), \dots \rrbracket$ (i.e. c with $c(0)$ replaced by k) is in f . Let $Q = \mathbb{N} \setminus \{k\}$ if there is such a k and $Q = \mathbb{N}$ otherwise.

Take $Y_i = \begin{cases} Q & \text{if } i = 0 \\ \{c(i)\} & \text{if } i > 0 \end{cases}$ for $i = 0, 1, 2, \dots$. We have $Y \in \text{up}(\lambda i \in \mathbb{N} : \mathcal{L}_{c(i)}(i))$ for every $c \in \{0, 1\}^{\mathbb{N}}$.

But evidently $\prod Y$ does not contain an element of f . Thus, $\prod Y \asymp f$ that is $Y \notin f$; $\text{up } Y \notin f$; $Y \notin \text{GR } \uparrow\uparrow f$ what is impossible if $\uparrow\uparrow f$ is completary. □

EXAMPLE 1784. There exists such an (infinite) set N and N -ary relation f that $\mathcal{P} \in \text{GR } \uparrow\uparrow f$ but there is no indexed family $a \in \prod_{i \in N} \text{atoms } \mathcal{P}_i$ of atomic filters such that $a \in \text{GR } \uparrow\uparrow f$ that is $\forall A \in \text{up } a : f \not\asymp \prod A$.

PROOF. Take $\mathcal{L}_0, \mathcal{L}_1$ and f from the proof of example 1783. Take $\mathcal{P} = \mathcal{L}_0 \sqcup \mathcal{L}_1$. If $a \in \prod_{i \in N} \text{atoms } \mathcal{P}_i$ then there exists $c \in \{0, 1\}^{\mathbb{N}}$ such that $a_i \sqsubseteq \mathcal{L}_{c(i)}(i)$ (because $\mathcal{L}_{c(i)}(i) \neq \perp$). Then from that example it follows that $(\lambda i \in N : \mathcal{L}_{c(i)}(i)) \notin \text{GR } \uparrow\uparrow f$ and thus $a \notin \text{GR } \uparrow\uparrow f$. □

CONJECTURE 1785. Filtrators of staroids on powersets are join-closed.

23.5. Multifuncoids

DEFINITION 1786. Let $(\mathfrak{A}_i, \mathfrak{B}_i)$ (where $i \in n$ for an index set n) be an indexed family of filtrators.

I call a *mult* f of the form $(\mathfrak{A}_i, \mathfrak{B}_i)$ the triple $f = (\text{base } f, \text{core } f, \langle f \rangle^*)$ of n -indexed families of posets $\text{base } f$ and $\text{core } f$ and $\langle f \rangle^*$ of functions where for every $i \in n$

$$\langle f \rangle_i^* : \prod (\text{core } f)_i |_{(\text{dom } \mathfrak{A}) \setminus \{i\}} \rightarrow (\text{base } f)_i.$$

I call $(\text{base } f, \text{core } f)$ the *form* of the mult f .

REMARK 1787. I call it *mult* because it comprises multiple functions $\langle f \rangle_i^*$.

DEFINITION 1788. A *mult on powersets* is a mult such that every $((\text{base } f)_i, (\text{core } f)_i)$ is a powerset filtrator.

DEFINITION 1789. I will call a *relational mult* a mult f such that every $(\text{base } f)_i$ is a set and for every $i, j \in n$ and $L \in \prod \text{core } f$

$$L_i \in \langle f \rangle_i^* L |_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \in \langle f \rangle_j^* L |_{(\text{dom } L) \setminus \{j\}}.$$

I denote arity $f = n$.

DEFINITION 1790. *Prestaroidal mult* is a relational mult of the form $(\mathfrak{A}, \lambda i \in \text{dom } \mathfrak{A} : \mathfrak{S}(\mathfrak{A}_i))$ (where \mathfrak{A} is a poset), that is such that $\langle f \rangle_i^* L$ is a free star for every $i \in n$ and $L \in \prod_{i \in (\text{dom } L) \setminus \{i\}} \text{core } f_i$.

DEFINITION 1791. I will call a *multifuncoid* a mult f such that $(\text{core } f)_i \subseteq (\text{base } f)_i$ (thus having a filtrator $((\text{base } f)_i, (\text{core } f)_i)$) for each $i \in n$ and for every $i, j \in n$ and $L \in \prod \text{core } f$

$$L_i \not\in \langle f \rangle_i^* L |_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \not\in \langle f \rangle_j^* L |_{(\text{dom } L) \setminus \{j\}}. \quad (33)$$

I denote the set of multifuncoids for a family $(\mathfrak{A}, \mathfrak{B})$ of filtrators as $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ or just $\text{pFCD}(\mathfrak{A})$ when \mathfrak{B} is clear from context.

DEFINITION 1792. To every multifuncoid f corresponds an anchored relation g by the formula (with arbitrary $i \in \text{arity } f$)

$$L \in \text{GR } g \Leftrightarrow L_i \not\in \langle f \rangle_i^* L |_{(\text{dom } L) \setminus \{i\}}.$$

PROPOSITION 1793. Prestaroidal mults $\Lambda g = f$ of the form $(\mathfrak{B}, \lambda i \in \text{dom } \mathfrak{B} : \mathfrak{S}(\mathfrak{B}_i))$ bijectively correspond to pre-staroids g of the form \mathfrak{B} by the formulas (for every $K \in \prod \mathfrak{B}, i \in \text{dom } \mathfrak{B}, L \in \prod_{j \in (\text{dom } \mathfrak{A}) \setminus \{i\}} \mathfrak{B}_j, X \in \mathfrak{B}_i$)

$$K \in \text{GR } g \Leftrightarrow K_i \in \langle f \rangle_i^* K |_{(\text{dom } L) \setminus \{i\}}; \quad (34)$$

$$X \in \langle f \rangle_i^* L \Leftrightarrow L \cup \{(i, X)\} \in \text{GR } g. \quad (35)$$

PROOF. If f is a prestaroidal mult, then obviously formula (34) defines an anchored relation between posets. $(\text{val } g)_i = \langle f \rangle_i^* L$ is a free star. Thus g is a prestaroid.

If g is a prestaroid, then obviously formula (35) defines a relational mult. This mult is obviously prestaroidal.

It remains to prove that these correspondences are inverse of each other.

Let f_0 be a prestaroidal mult, g be the pre-staroid corresponding to f by formula (34), and f_1 be the prestaroidal mult corresponding to g by formula (35). Let's prove $f_0 = f_1$. Really,

$$X \in \langle f_1 \rangle_i^* L \Leftrightarrow L \cup \{(i, X)\} \in \text{GR } g \Leftrightarrow X \in \langle f_0 \rangle_i^* L.$$

Let now g_0 be a prestaroid, f be a prestaroidal mult corresponding to g_0 by formula (35), and g_1 be a prestaroid corresponding to f by formula (34). Let's prove $g_0 = g_1$. Really,

$$K \in \text{GR } g_1 \Leftrightarrow K_i \in \langle f \rangle_i^* K|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow K|_{(\text{dom } L) \setminus \{i\} \cup \{(i, K_i)\}} \in \text{GR } g_0 \Leftrightarrow K \in \text{GR } g_0.$$

□

DEFINITION 1794. I will denote $[f]^* = \text{GR } g$ for the prestaroidal mult f corresponding to anchored relation g .

PROPOSITION 1795. For a form $(\mathfrak{Z}, \lambda i \in \text{dom } \mathfrak{Z} : \mathfrak{S}(\mathfrak{Z}_i))$ where each \mathfrak{Z}_i is a boolean lattice, relational mults are the same as multifuncoids (if we equate poset elements with principal free stars).

PROOF.

$$\begin{aligned} (L_i \not\prec \langle f \rangle_i^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \not\prec \langle f \rangle_j^* L|_{(\text{dom } L) \setminus \{j\}} \Leftrightarrow \\ (L_i \in \partial \langle f \rangle_i^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \in \partial \langle f \rangle_j^* L|_{(\text{dom } L) \setminus \{j\}} \Leftrightarrow \\ (L_i \in \langle f \rangle_i^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \in \langle f \rangle_j^* L|_{(\text{dom } L) \setminus \{j\}}). \end{aligned}$$

□

THEOREM 1796. Fix some indexed family \mathfrak{Z} of join semi-lattices.

$$(\text{val } f)_j(L \cup \{(i, X \sqcup Y)\}) = (\text{val } f)_j(L \cup \{(i, X)\}) \sqcup (\text{val } f)_j(L \cup \{(i, Y)\})$$

for every prestaroid f of the form \mathfrak{Z} and $i, j \in \text{arity } f$, $i \neq j$, $L \in \prod_{k \in L \setminus \{i, j\}} \mathfrak{Z}_k$, $X, Y \in \mathfrak{Z}_i$.

PROOF. Let $i, j \in \text{arity } f$, $i \neq j$ and $L \in \prod_{k \in L \setminus \{i, j\}} \mathfrak{Z}_k$. Let $Z \in \mathfrak{Z}_i$.

$$\begin{aligned} Z \in (\text{val } f)_j(L \cup \{(i, X \sqcup Y)\}) \Leftrightarrow \\ L \cup \{(i, X \sqcup Y), (j, Z)\} \in \text{GR } f \Leftrightarrow \\ X \sqcup Y \in (\text{val } f)_i(L \cup \{(j, Z)\}) \Leftrightarrow \\ X \in (\text{val } f)_i(L \cup \{(j, Z)\}) \vee Y \in (\text{val } f)_i(L \cup \{(j, Z)\}) \Leftrightarrow \\ L \cup \{(i, X), (j, Z)\} \in \text{GR } f \vee L \cup \{(i, Y), (j, Z)\} \in \text{GR } f \Leftrightarrow \\ Z \in (\text{val } f)_j(L \cup \{(i, X)\}) \vee Z \in (\text{val } f)_j(L \cup \{(i, Y)\}) \Leftrightarrow \\ Z \in (\text{val } f)_j(L \cup \{(i, X)\}) \cup (\text{val } f)_j(L \cup \{(i, Y)\}) \Leftrightarrow \\ Z \in (\text{val } f)_j(L \cup \{(i, X)\}) \sqcup (\text{val } f)_j(L \cup \{(i, Y)\}) \end{aligned}$$

Thus $(\text{val } f)_j(L \cup \{(i, X \sqcup Y)\}) = (\text{val } f)_j(L \cup \{(i, X)\}) \sqcup (\text{val } f)_j(L \cup \{(i, Y)\})$. □

Let us consider the filtrator $(\prod_{i \in \text{arity } f} \mathfrak{S}((\text{form } f)_i), \prod_{i \in \text{arity } f} (\text{form } f)_i)$.

CONJECTURE 1797. A finitary anchored relation between join-semilattices is a staroid iff $(\text{val } f)_j(L \cup \{(i, X \sqcup Y)\}) = (\text{val } f)_j(L \cup \{(i, X)\}) \sqcup (\text{val } f)_j(L \cup \{(i, Y)\})$ for every $i, j \in \text{arity } f$ ($i \neq j$) and $X, Y \in (\text{form } f)_i$.

THEOREM 1798. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be a family of join-closed down-aligned filtrators whose both base and core are join-semilattices. Let f be a staroid of the form \mathfrak{Z} . Then $\uparrow\uparrow f$ is a staroid of the form \mathfrak{A} .

PROOF. First prove that $\uparrow\uparrow f$ is a prestaroid. We need to prove that $\perp \notin (\text{GR } \uparrow\uparrow f)_i$ (that is $\text{up } \perp \not\subseteq (\text{GR } f)_i$ that is $\perp \notin (\text{GR } f)_i$ what is true by the theorem conditions) and that for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{A}_i$ and $\mathcal{L} \in \prod_{i \in (\text{arity } f) \setminus \{i\}} \mathfrak{A}_i$ where $i \in \text{arity } f$

$$\mathcal{L} \cup \{(i, \mathcal{X} \sqcup \mathcal{Y})\} \in \text{GR } \uparrow\uparrow f \Leftrightarrow \mathcal{L} \cup \{(i, \mathcal{X})\} \in \text{GR } \uparrow\uparrow f \vee \mathcal{L} \cup \{(i, \mathcal{Y})\} \in \text{GR } \uparrow\uparrow f.$$

The reverse implication is obvious. Let $\mathcal{L} \cup \{(i, \mathcal{X} \sqcup \mathcal{Y})\} \in \text{GR} \uparrow\uparrow f$. Then for every $L \in \text{up } \mathcal{L}$ and $X \in \text{up } \mathcal{X}$, $Y \in \text{up } \mathcal{Y}$ we have $L \cup \{(i, X \sqcup^{\mathfrak{Z}} Y)\} \in \text{GR } f$ and thus

$$L \cup \{(i, X)\} \in \text{GR } f \vee L \cup \{(i, Y)\} \in \text{GR } f$$

consequently $\mathcal{L} \cup \{(i, \mathcal{X})\} \in \text{GR} \uparrow\uparrow f \vee \mathcal{L} \cup \{(i, \mathcal{Y})\} \in \text{GR} \uparrow\uparrow f$.

It is left to prove that $\uparrow\uparrow f$ is an upper set, but this is obvious. \square

There is a conjecture similar to the above theorems:

CONJECTURE 1799. $L \in \uparrow\uparrow [f]^* \Rightarrow \uparrow\uparrow [f]^* \cap \prod_{i \in \text{dom } \mathfrak{A}} \text{atoms } L_i \neq \emptyset$ for every multifuncooid f for the filtrator $(\mathcal{F}^n, \mathfrak{Z}^n)$.

CONJECTURE 1800. Let $(\mathfrak{A}, \mathfrak{Z})$ be a powerset filtrator, let n be an index set. Consider the filtrator $(\mathcal{F}^n, \mathfrak{Z}^n)$. Then if f is a completary staroid of the form \mathfrak{Z}^n , then $\uparrow\uparrow f$ is a completary staroid of the form \mathfrak{A}^n .

EXAMPLE 1801. There is such an anchored relation f that for some $k \in \text{dom } f$

$$\langle \uparrow\uparrow\uparrow f \rangle_k^* \mathcal{L} \neq \bigsqcup_{a \in \prod_{i \in (\text{dom } f) \setminus \{k\}} \text{atoms } \mathcal{L}_i} \langle \uparrow\uparrow\uparrow f \rangle_k^* a.$$

PROOF. Take $\mathcal{P} \in \text{GR } f$ from the counter-example 1784. We have

$$\forall a \in \prod_{i \in \text{dom } f} \text{atoms } \mathcal{P}_i : a \notin \text{GR } \mathcal{P}.$$

Take $k = 1$.

Let $\mathcal{L} = \mathcal{P}|_{(\text{dom } f) \setminus \{k\}}$. Then $a \notin \text{GR } \uparrow\uparrow\uparrow f$ and thus $a_k \asymp \langle \uparrow\uparrow\uparrow f \rangle_k^* a|_{(\text{dom } f) \setminus \{k\}}$.

Consequently $\mathcal{P}_k \asymp \langle \uparrow\uparrow\uparrow f \rangle_k^* a|_{(\text{dom } f) \setminus \{k\}}$ and thus $\mathcal{P}_k \asymp$

$\bigsqcup_{a \in \prod_{i \in (\text{dom } f) \setminus \{k\}} \text{atoms } \mathcal{L}_i} \langle \uparrow\uparrow\uparrow f \rangle_k^* a$ because \mathcal{P}_k is principal.

But $\mathcal{P}_k \not\asymp \langle \uparrow\uparrow\uparrow f \rangle_k^* \mathcal{L}$. Thus follows $\langle \uparrow\uparrow\uparrow f \rangle_k^* \mathcal{L} \neq \bigsqcup_{a \in \prod_{i \in (\text{dom } f) \setminus \{k\}} \text{atoms } \mathcal{L}_i} \langle \uparrow\uparrow\uparrow f \rangle_k^* a$. \square

23.6. Join of multifuncooids

Mults are ordered by the formula $f \sqsubseteq g \Leftrightarrow \langle f \rangle^* \sqsubseteq \langle g \rangle^*$ where \sqsubseteq in the right part of this formula is the product order. I will denote \sqcap , \sqcup , \sqcap , \sqcup (without an index) the order poset operations on the poset of mults.

REMARK 1802. To describe this, the definition of product order is used twice. Let f and g be mults of the same form $(\mathfrak{A}, \mathfrak{Z})$

$$\begin{aligned} \langle f \rangle^* \sqsubseteq \langle g \rangle^* &\Leftrightarrow \forall i \in \text{dom } \mathfrak{Z} : \langle f \rangle_i^* \sqsubseteq \langle g \rangle_i^*; \\ \langle f \rangle_i^* \sqsubseteq \langle g \rangle_i^* &\Leftrightarrow \forall L \in \prod_{i \in (\text{dom } \mathfrak{Z}) \setminus \{i\}} \mathfrak{Z} : \langle f \rangle_i^* L \sqsubseteq \langle g \rangle_i^* L. \end{aligned}$$

OBVIOUS 1803. $(\bigsqcup F)K = \bigsqcup_{f \in F} fK$ for every set F of mults of the same form \mathfrak{Z} and $K \in \prod \mathfrak{Z}$ whenever every $\bigsqcup_{f \in F} fK$ is defined.

THEOREM 1804. $f \sqcup^{\text{pFCD}(\mathfrak{A})} g = f \sqcup g$ for every multifuncooids f and g for the same indexed family of starrish join-semilattices filtrators.

PROOF. $\alpha_i x \stackrel{\text{def}}{=} \langle f_i \rangle^* x \sqcup \langle g_i \rangle^* x$. It is enough to prove that α is a multifuncooid. We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\text{dom } L) \setminus \{j\}}.$$

Really,

$$\begin{aligned}
& L_i \not\prec \alpha_i L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow \\
& L_i \not\prec \langle f_i \rangle^* L|_{(\text{dom } L) \setminus \{i\}} \sqcup \langle g_i \rangle^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow \\
& L_i \not\prec \langle f_i \rangle^* L|_{(\text{dom } L) \setminus \{i\}} \vee L_i \not\prec \langle g_i \rangle^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow \\
& L_j \not\prec \langle f_j \rangle^* L|_{(\text{dom } L) \setminus \{j\}} \vee L_j \not\prec \langle g_j \rangle^* L|_{(\text{dom } L) \setminus \{j\}} \Leftrightarrow \\
& L_j \not\prec \langle f_j \rangle^* L|_{(\text{dom } L) \setminus \{j\}} \sqcup \langle g_j \rangle^* L|_{(\text{dom } L) \setminus \{j\}} \Leftrightarrow \\
& L_j \not\prec \alpha_j L|_{(\text{dom } L) \setminus \{j\}}.
\end{aligned}$$

□

THEOREM 1805. $\bigsqcup^{\text{pFCD}(\mathfrak{A})} F = \bigsqcup F$ for every set F of multifunctors for the same indexed family of join infinite distributive complete lattices filtrators.

PROOF. $\alpha_i x \stackrel{\text{def}}{=} \bigsqcup_{f \in F} \langle f \rangle_i^* x$. It is enough to prove that α is a multifunctor. We need to prove:

$$L_i \not\prec \alpha_i L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow L_j \not\prec \alpha_j L|_{(\text{dom } L) \setminus \{j\}}.$$

Really,

$$\begin{aligned}
& L_i \not\prec \alpha_i L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow \\
& L_i \not\prec \bigsqcup_{f \in F} \langle f_i \rangle^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow \\
& \exists f \in F : L_i \not\prec \langle f_i \rangle^* L|_{(\text{dom } L) \setminus \{i\}} \Leftrightarrow \\
& \exists f \in F : L_j \not\prec \langle f_j \rangle^* L|_{(\text{dom } L) \setminus \{j\}} \Leftrightarrow \\
& L_j \not\prec \bigsqcup_{f \in F} \langle f_j \rangle^* L|_{(\text{dom } L) \setminus \{j\}} \Leftrightarrow \\
& L_j \not\prec \alpha_j L|_{(\text{dom } L) \setminus \{j\}}.
\end{aligned}$$

□

THEOREM 1806. If f, g are multifunctors for a primary filtrator $(\mathfrak{A}_i, \mathfrak{B}_i)$ where \mathfrak{B}_i are separable starrish posets, then $f \sqcup^{\text{pFCD}(\mathfrak{A})} g \in \text{pFCD}(\mathfrak{A})$.

PROOF. Let $A \in [f \sqcup^{\text{pFCD}(\mathfrak{A})} g]^*$ and $B \sqsupseteq A$. Then for every $k \in \text{dom } \mathfrak{A}$

$$A_k \not\prec \langle f \sqcup^{\text{pFCD}(\mathfrak{A})} g \rangle^* A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}; A_k \not\prec \langle f \sqcup g \rangle^* A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}; A_k \not\prec \langle f \rangle^* (A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) \sqcup \langle g \rangle^* (A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}).$$

$$\text{Thus } A_k \not\prec \langle f \rangle^* (A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) \vee A_k \not\prec \langle g \rangle^* (A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}); A \in [f]^* \vee A \in [g]^*; B \in [f]^* \vee B \in [g]^*; B_k \not\prec \langle f \rangle^* (B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) \vee B_k \not\prec \langle g \rangle^* (B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}});$$

$$B_k \not\prec \langle f \rangle^* (B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) \sqcup \langle g \rangle^* (B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}); B_k \not\prec \langle f \sqcup g \rangle^* B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} = \langle f \sqcup^{\text{pFCD}(\mathfrak{A})} g \rangle^* B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}.$$

$$\text{Thus } B \in [f \sqcup^{\text{pFCD}(\mathfrak{A})} g]^*.$$

□

THEOREM 1807. If F is a set of multifunctors for the same indexed family of join infinite distributive complete lattices filtrators, then $\bigsqcup^{\text{pFCD}(\mathfrak{A})} F \in \text{pFCD}(\mathfrak{A})$.

PROOF. Let $A \in \left[\bigsqcup^{\text{pFCD}(\mathfrak{A})} F \right]^*$ and $B \sqsupseteq A$. Then for every $k \in \text{dom } \mathfrak{A}$

$$A_k \not\prec \left\langle \bigsqcup^{\text{pFCD}(\mathfrak{A})} F \right\rangle^* A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} = \left\langle \bigsqcup F \right\rangle^* A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} = \bigsqcup_{f \in F} \langle f \rangle^* (A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}).$$

$$\text{Thus } \exists f \in F : A_k \not\prec \langle f \rangle^* (A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}); \exists f \in F : A \in [f]^*; B \in [f]^* \text{ for some } f \in F; \exists f \in F : B_k \not\prec \langle f \rangle^* (B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}); B_k \not\prec \bigsqcup_{f \in F} \langle f \rangle^* (B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) = \left\langle \bigsqcup^{\text{pFCD}(\mathfrak{A})} F \right\rangle^* B|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}. \text{ Thus } B \in \left[\bigsqcup^{\text{pFCD}(\mathfrak{A})} F \right]^*.$$

□

23.7. Infinite product of poset elements

Let A_i be a family of elements of a family \mathfrak{A}_i of posets. The *staroidal product* $\prod^{\text{Strd}(\mathfrak{A})} A$ is defined by the formula (for every $L \in \prod \mathfrak{A}$)

$$\text{form } \prod^{\text{Strd}(\mathfrak{A})} A = \mathfrak{A} \quad \text{and} \quad L \in \text{GR } \prod^{\text{Strd}(\mathfrak{A})} A \Leftrightarrow \forall i \in \text{dom } \mathfrak{A} : A_i \not\prec L_i.$$

PROPOSITION 1808. If \mathfrak{A}_i are powerset algebras, staroidal product of principal filters is essentially equivalent to Cartesian product. More precisely, $\prod_{i \in \text{dom } A}^{\text{Strd}} \uparrow^{\mathcal{F}} A_i = \uparrow \uparrow^{\text{Strd}} \prod A$ for an indexed family A of sets.

PROOF.

$$\begin{aligned} L \in \text{GR } \uparrow \uparrow^{\text{Strd}} \prod A &\Leftrightarrow \\ \text{up } L \subseteq \text{GR } \uparrow^{\text{Strd}} \prod A &\Leftrightarrow \\ \forall X \in \text{up } L : \prod X \not\prec \prod A &\Leftrightarrow \\ \forall X \in \text{up } L, i \in \text{dom } A : X_i \not\prec A_i &\Leftrightarrow \\ \forall i \in \text{dom } A : L_i \not\prec \uparrow^{\mathcal{F}} A_i &\Leftrightarrow \\ L \in \text{GR } \prod_{i \in \text{dom } A}^{\text{Strd}} \uparrow^{\mathcal{F}} A_i. & \end{aligned}$$

□

COROLLARY 1809. Staroidal product of principal filters is an upgraded principal staroid.

PROPOSITION 1810. $\prod^{\text{Strd}} a = \uparrow \downarrow \prod^{\text{Strd}} a$ if each $a_i \in \mathfrak{A}_i$ (for $i \in n$ where n is some index set) where each $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ is a filtrator with separable core.

PROOF.

$$\begin{aligned} \text{GR } \uparrow \downarrow \prod^{\text{Strd}} a &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\text{up } L \subseteq \mathfrak{Z} \cap \text{GR } \prod^{\text{Strd}} a} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\text{up } L \subseteq \text{GR } \prod^{\text{Strd}} a} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall K \in \text{up } L : K \in \text{GR } \prod^{\text{Strd}} a} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall K \in \text{up } L, i \in n : K_i \not\prec a_i} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in n, K \in \text{up } L : K_i \not\prec a_i} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in n : L_i \not\prec a_i} \right\} &= \\ \text{GR } \prod^{\text{Strd}} a & \end{aligned}$$

(taken into account that our filtrators are with a separable core). □

THEOREM 1811. Staroidal product is a completary staroid (if our posets are starrish join-semilattices).

PROOF. We need to prove

$$\forall i \in \text{dom } \mathfrak{A} : A_i \not\prec (L_0 i \sqcup L_1 i) \Leftrightarrow \exists c \in \{0, 1\}^n \forall i \in \text{dom } \mathfrak{A} : A_i \not\prec L_{c(i)} i.$$

Really,

$$\begin{aligned} \forall i \in \text{dom } \mathfrak{A} : A_i \not\prec (L_0 i \sqcup L_1 i) &\Leftrightarrow \forall i \in \text{dom } \mathfrak{A} : (A_i \not\prec L_0 i \vee A_i \not\prec L_1 i) \Leftrightarrow \\ &\exists c \in \{0, 1\}^{\text{dom } \mathfrak{A}} \forall i \in \text{dom } \mathfrak{A} : A_i \not\prec L_{c(i)} i. \end{aligned}$$

□

DEFINITION 1812. Let $(\mathfrak{A}_i, \mathfrak{B}_i)$ be an indexed family of filtrators and every \mathfrak{A}_i has least element.

Then for every $A \in \prod \mathfrak{A}$ *funcoidal product* is multifuncoid $\prod^{\text{FCD}(\mathfrak{A})} A$ defined by the formula (for every $L \in \prod \mathfrak{B}$):

$$\left\langle \prod_k^{\text{FCD}(\mathfrak{A})} A \right\rangle_k^* L = \begin{cases} A_k & \text{if } \forall i \in (\text{dom } \mathfrak{A}) \setminus \{k\} : A_i \not\prec L_i \\ \perp^{\mathfrak{A}} & \text{otherwise.} \end{cases}$$

$$\text{PROPOSITION 1813. } \text{GR } \prod^{\text{Strd}(\mathfrak{A})} A = \left[\prod^{\text{FCD}(\mathfrak{A})} A \right]^*.$$

PROOF.

$$\begin{aligned} L \in \text{GR } \prod^{\text{Strd}(\mathfrak{A})} A &\Leftrightarrow \\ \forall i \in \text{dom } \mathfrak{A} : A_i \not\prec L_i &\Leftrightarrow \\ \forall i \in (\text{dom } \mathfrak{A}) \setminus \{k\} : A_i \not\prec L_i \wedge L_k \not\prec A_k &\Leftrightarrow \\ L_k \not\prec \left\langle \prod_k^{\text{FCD}(\mathfrak{A})} A \right\rangle_k^* L|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} &\Leftrightarrow \\ L \in \left[\prod^{\text{FCD}(\mathfrak{A})} A \right]^* &. \end{aligned}$$

□

COROLLARY 1814. Funcoidal product is a completary multifuncoid.

PROOF. It is enough to prove that funcoidal product is a multifuncoid. Really,

$$\begin{aligned} L_i \not\prec \left\langle \prod_i^{\text{FCD}(\mathfrak{A})} A \right\rangle_i^* L|_{(\text{dom } \mathfrak{A}) \setminus \{i\}} &\Leftrightarrow \\ \forall i \in \text{dom } \mathfrak{A} : A_i \not\prec L_i &\Leftrightarrow L_j \not\prec \left\langle \prod_j^{\text{FCD}(\mathfrak{A})} A \right\rangle_j^* L|_{(\text{dom } \mathfrak{A}) \setminus \{j\}}. \end{aligned}$$

□

THEOREM 1815. If our each filtrator $(\mathfrak{A}_i, \mathfrak{B}_i)$ is with separable core and $A \in \prod \mathfrak{B}$, then $\uparrow \prod^{\text{Strd}(\mathfrak{B})} A = \prod^{\text{Strd}(\mathfrak{A})} A$.

PROOF.

$$\begin{aligned}
& \text{GR} \uparrow\uparrow \prod^{\text{Strd}(\mathfrak{Z})} A = \\
& \left\{ \frac{L \in \prod \mathfrak{A}}{\text{up } L \subseteq \prod^{\text{Strd}(\mathfrak{Z})} A} \right\} = \\
& \left\{ \frac{L \in \prod \mathfrak{A}}{\forall K \in \text{up } L, i \in \text{dom } \mathfrak{A} : A_i \not\leq K_i} \right\} = \\
& \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \text{dom } \mathfrak{A}, K \in \text{up } L_i : A_i \not\leq K} \right\} = \\
& \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \text{dom } \mathfrak{A} : A_i \not\leq L_i} \right\} = \\
& \text{GR} \prod^{\text{Strd}(\mathfrak{A})} A.
\end{aligned}$$

□

PROPOSITION 1816. Let $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ be a meet-closed filtrator, $A \in \prod \mathfrak{Z}$. Then $\downarrow\downarrow \prod^{\text{Strd}(\mathfrak{A})} A = \prod^{\text{Strd}(\mathfrak{Z})} A$.

PROOF.

$$\begin{aligned}
& \text{GR} \downarrow\downarrow \prod^{\text{Strd}(\mathfrak{A})} A = \\
& \downarrow\downarrow \text{GR} \prod^{\text{Strd}(\mathfrak{A})} A = \\
& \downarrow\downarrow \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \text{dom } \mathfrak{A} : A_i \not\leq L_i} \right\} = \\
& \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \text{dom } \mathfrak{A} : A_i \not\leq L_i} \right\} \cap \prod \mathfrak{Z} = \\
& \left\{ \frac{L \in \prod \mathfrak{Z}}{\forall i \in \text{dom } \mathfrak{A} : A_i \not\leq L_i} \right\} = \\
& \text{GR} \prod^{\text{Strd}(\mathfrak{Z})} A.
\end{aligned}$$

□

COROLLARY 1817. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a powerset filtrator and $A \in \prod \mathfrak{Z}$, then $\downarrow\downarrow \prod^{\text{Strd}(\mathfrak{A})} A$ is a principal staroid.

PROOF. Use the “obvious” fact above. □

THEOREM 1818. Let \mathcal{F} be a family of sets of filters on meet-semilattices with least elements. Let $a \in \prod \mathcal{F}$, $S \in \mathcal{P} \prod \mathcal{F}$, and every $\text{Pr}_i S$ be a generalized filter base, $\prod S = a$. Then

$$\prod^{\text{Strd}(\mathcal{F})} a = \prod_{A \in S} \prod^{\text{Strd}(\mathcal{F})} A.$$

PROOF. That $\prod^{\text{Strd}(\mathcal{F})} a$ is a lower bound for $\left\{ \prod_{A \in S}^{\text{Strd}(\mathcal{F})} A \right\}$ is obvious.

Let f be a lower bound for $\left\{ \prod_{A \in S}^{\text{Strd}(\mathcal{F})} A \right\}$. Thus $\forall A \in S : \text{GR } f \subseteq \text{GR} \prod^{\text{Strd}(\mathcal{F})} A$. Thus for every $A \in S$ we have $L \in \text{GR } f$ implies $\forall i \in \text{dom } \mathfrak{A} :$

$A_i \not\leq L_i$. Then, by properties of generalized filter bases, $\forall i \in \text{dom } \mathfrak{A} : a_i \not\leq L_i$ that is $L \in \text{GR } \prod^{\text{Strd}(\mathcal{F})} a$. So $f \sqsubseteq \prod^{\text{Strd}(\mathcal{F})} a$ and thus $\prod^{\text{Strd}(\mathcal{F})} a$ is the greatest lower bound of $\left\{ \prod_{A \in S}^{\text{Strd}(\mathcal{F})} A \right\}$. \square

CONJECTURE 1819. Let \mathcal{F} be a family of sets of filters on meet-semilattices with least elements. Let $a \in \prod \mathcal{F}$, $S \in \mathcal{P} \prod \mathcal{F}$ be a generalized filter base, $\prod S = a$, f is a staroid of the form $\prod \mathcal{F}$. Then

$$\prod^{\text{Strd}(\mathcal{F})} a \not\leq f \Leftrightarrow \forall A \in S : \prod^{\text{Strd}(3)} A \not\leq f.$$

23.8. On products of staroids

DEFINITION 1820. $\prod^{(D)} F = \left\{ \frac{\text{uncurry } z}{z \in \prod F} \right\}$ (*reindexation product*) for every indexed family F of relations.

DEFINITION 1821. *Reindexation product* of an indexed family F of anchored relations is defined by the formulas:

$$\text{form } \prod^{(D)} F = \text{uncurry}(\text{form } \circ F) \quad \text{and} \quad \text{GR } \prod^{(D)} F = \prod^{(D)} (\text{GR } \circ F).$$

OBVIOUS 1822.

$$\begin{aligned} 1^\circ. \text{ form } \prod^{(D)} F &= \left\{ \frac{((i,j), (\text{form } F_i)_j)}{i \in \text{dom } F, j \in \text{arity } F_i} \right\}; \\ 2^\circ. \text{ GR } \prod^{(D)} F &= \left\{ \frac{\left\{ \frac{((i,j), (zi)j)}{i \in \text{dom } F, j \in \text{arity } F_i} \right\}}{z \in \prod (\text{GR } \circ F)} \right\}. \end{aligned}$$

PROPOSITION 1823. $\prod^{(D)} F$ is an anchored relation if every F_i is an anchored relation.

PROOF. We need to prove $\text{GR } \prod^{(D)} F \in \mathcal{P} \prod \text{form} \left(\prod^{(D)} F \right)$ that is

$$\begin{aligned} \text{GR } \prod^{(D)} F &\subseteq \prod \text{form} \left(\prod^{(D)} F \right); & \left\{ \frac{\left\{ \frac{((i,j), (zi)j)}{i \in \text{dom } F, j \in \text{arity } F_i} \right\}}{z \in \prod (\text{GR } \circ F)} \right\} &\subseteq \\ \prod \left\{ \frac{((i,j), (\text{form } F_i)_j)}{i \in \text{dom } F, j \in \text{arity } F_i} \right\}; & & & \\ \forall z \in \prod (\text{GR } \circ F), i \in \text{dom } F, j \in \text{arity } F_i : (zi)j \in (\text{form } F_i)_j. & & & \\ \text{Really, } zi \in \text{GR } F_i \subseteq \prod (\text{form } F_i) & \text{ and thus } (zi)j \in (\text{form } F_i)_j. & & \square \end{aligned}$$

$$\text{OBVIOUS 1824. } \text{arity } \prod^{(D)} F = \prod_{i \in \text{dom } F} \text{arity } F_i = \left\{ \frac{(i,j)}{i \in \text{dom } F, j \in \text{arity } F_i} \right\}.$$

DEFINITION 1825. $f \times^{(D)} g = \prod^{(D)} \llbracket f, g \rrbracket$.

LEMMA 1826. $\prod^{(D)} F$ is an upper set if every F_i is an upper set.

PROOF. We need to prove that $\prod^{(D)} F$ is an upper set. Let $a \in \prod^{(D)} F$ and an anchored relation $b \sqsupseteq a$ of the same form as a . We have $a = \text{uncurry } z$ for some $z \in \prod F$ that is $a(i, j) = (zi)j$ for all $i \in \text{dom } F$ and $j \in \text{dom } F_i$ where $zi \in F_i$. Also $b(i, j) \sqsupseteq a(i, j)$. Thus $(\text{curry } b)i \sqsupseteq zi$; $\text{curry } b \in \prod F$ because every F_i is an upper set and so $b \in \prod^{(D)} F$. \square

PROPOSITION 1827. Let F be an indexed family of anchored relations and every $(\text{form } F)_i$ be a join-semilattice.

- 1°. $\prod^{(D)} F$ is a prestaroid if every F_i is a prestaroid.
- 2°. $\prod^{(D)} F$ is a staroid if every F_i is a staroid.
- 3°. $\prod^{(D)} F$ is a completary staroid if every F_i is a completary staroid.

PROOF.

1°. Let $q \in \text{arity } \prod^{(D)} F$ that is $q = (i, j)$ where $i \in \text{dom } F$, $j \in \text{arity } F_i$; let

$$L \in \prod \left(\left(\text{form } \prod^{(D)} F \right) \Big|_{(\text{arity } \prod^{(D)} F) \setminus \{q\}} \right)$$

that is $L_{(i', j')} \in \left(\text{form } \prod^{(D)} F \right)_{(i', j')}$ for every $(i', j') \in (\text{arity } \prod^{(D)} F) \setminus \{q\}$, that is $L_{(i', j')} \in (\text{form } F_{i'})_{j'}$. We have $X \in \left(\text{form } \prod^{(D)} F \right)_{(i, j)} \Leftrightarrow X \in (\text{form } F_i)_j$. So

$$\begin{aligned} \left(\text{val } \prod^{(D)} F \right)_{(i, j)} L &= \left\{ \frac{X \in (\text{form } F_i)_j}{L \cup \{(i, j), X\} \in \text{GR } \prod^{(D)} F} \right\} = \\ &= \left\{ \frac{X \in (\text{form } F_i)_j}{\exists z \in \prod (\text{GR } \circ F) : L \cup \{(i, j), X\} = \text{uncurry } z} \right\} = \\ &= \left\{ \frac{X \in (\text{form } F_i)_j}{\exists z \in \prod \left((\text{GR } \circ F) \Big|_{(\text{arity } \prod^{(D)} F) \setminus \{(i, j)\}} \right), v \in \text{GR } F_i : (L = \text{uncurry } z \wedge v_j = X)} \right\} = \\ &= \left\{ \frac{X \in (\text{form } F_i)_j}{\exists z \in \prod \left((\text{GR } \circ F) \Big|_{(\text{arity } \prod^{(D)} F) \setminus \{(i, j)\}} \right) : L = \text{uncurry } z \wedge \exists v \in \text{GR } F_i : v_j = X} \right\}. \end{aligned}$$

If $\exists z \in \prod \left((\text{GR } \circ F) \Big|_{(\text{arity } \prod^{(D)} F) \setminus \{(i, j)\}} \right) : L = \text{uncurry } z$ is false then $\left(\text{val } \prod^{(D)} F \right)_{(i, j)} L = \emptyset$ is a free star. We can assume it is true. So

$$\begin{aligned} \left(\text{val } \prod^{(D)} F \right)_{(i, j)} L &= \left\{ \frac{X \in (\text{form } F_i)_j}{\exists v \in \text{GR } F_i : v_j = X} \right\} = \\ &= \left\{ \frac{X \in (\text{form } F_i)_j}{\exists K \in (\text{form } F_i)_{(\text{arity } F_i) \setminus \{j\}} : K \cup \{(j, X)\} \in \text{GR } F_i} \right\} = \\ &= \left\{ \frac{X \in (\text{form } F_i)_j}{\exists K \in (\text{form } F_i)_{(\text{arity } F_i) \setminus \{j\}} : X \in (\text{val } F_i)_j K} \right\}. \end{aligned}$$

Thus

$$\begin{aligned} A \sqcup B \in \left(\text{val } \prod^{(D)} F \right)_{(i, j)} L &\Leftrightarrow \\ \exists K \in (\text{form } F_i)_{(\text{arity } F_i) \setminus \{j\}} : A \sqcup B \in (\text{val } F_i)_j K &\Leftrightarrow \\ \exists K \in (\text{form } F_i)_{(\text{arity } F_i) \setminus \{j\}} : (A \in (\text{val } F_i)_j K \vee B \in (\text{val } F_i)_j K) &\Leftrightarrow \\ \exists K \in (\text{form } F_i)_{(\text{arity } F_i) \setminus \{j\}} : A \in (\text{val } F_i)_j K \vee &\Leftrightarrow \\ \exists K \in (\text{form } F_i)_{(\text{arity } F_i) \setminus \{j\}} : B \in (\text{val } F_i)_j K &\Leftrightarrow \\ A \in \left(\text{val } \prod^{(D)} F \right)_{(i, j)} L \vee B \in \left(\text{val } \prod^{(D)} F \right)_{(i, j)} L. & \end{aligned}$$

Least element \perp is not in $\left(\text{val } \prod^{(D)} F \right)_{(i, j)} L$ because $K \cup \{(j, \perp)\} \notin \text{GR } F_i$.

2°. From the lemma.

3°. We need to prove

$$L_0 \sqcup L_1 \in \text{GR} \prod^{(D)} F \Leftrightarrow$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} : \left(\lambda i \in \text{arity} \prod^{(D)} F : L_{c(i)} i \right) \in \text{GR} \prod^{(D)} F$$

for every $L_0, L_1 \in \prod \text{form} \prod^{(D)} F$ that is $L_0, L_1 \in \prod \text{uncurry}(\text{form} \circ F)$.

$$\text{Really } L_0 \sqcup L_1 \in \text{GR} \prod^{(D)} F \Leftrightarrow L_0 \sqcup L_1 \in \left\{ \frac{\text{uncurry } z}{z \in \prod(\text{GR} \circ F)} \right\}.$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} : \left(\lambda i \in \text{arity} \prod^{(D)} F : L_{c(i)} i \right) \in \text{GR} \prod^{(D)} F \Leftrightarrow$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} : \left(\lambda i \in \text{arity} \prod^{(D)} F : L_{c(i)} i \right) \in \left\{ \frac{\text{uncurry } z}{z \in \prod(\text{GR} \circ F)} \right\} \Leftrightarrow$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} : \text{curry} \left(\lambda i \in \text{arity} \prod^{(D)} F : L_{c(i)} i \right) \in \prod(\text{GR} \circ F) \Leftrightarrow$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} : \text{curry} \left(\lambda(i, j) \in \text{arity} \prod^{(D)} F : L_{c(i, j)}(i, j) \right) \in \prod(\text{GR} \circ F) \Leftrightarrow$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} : (\lambda i \in \text{dom } F : (\lambda j \in \text{dom } F_i : L_{c(i, j)}(i, j))) \in \prod(\text{GR} \circ F) \Leftrightarrow$$

$$\exists c \in \{0, 1\}^{\text{arity} \prod^{(D)} F} \forall i \in \text{dom } F : (\lambda j \in \text{dom } F_i : L_{c(i, j)}(i, j)) \in \text{GR } F_i \Leftrightarrow$$

$$\forall i \in \text{dom } F \exists c \in \{0, 1\}^{\text{dom } F_i} : (\lambda j \in \text{dom } F_i : L_{c(j)}(i, j)) \in \text{GR } F_i \Leftrightarrow$$

$$\forall i \in \text{dom } F \exists c \in \{0, 1\}^{\text{dom } F_i} : (\lambda j \in \text{dom } F_i : (\text{curry}(L_{c(j)})i)j) \in \text{GR } F_i \Leftrightarrow$$

$$\forall i \in \text{dom } F : \text{curry}(L_0)i \sqcup \text{curry}(L_1)i \in \text{GR } F_i \Leftrightarrow$$

$$\forall i \in \text{dom } F : (\text{curry}(L_0) \sqcup \text{curry}(L_1))i \in \text{GR } F_i \Leftrightarrow$$

$$\forall i \in \text{dom } F : \text{curry}(L_0 \sqcup L_1)i \in \text{GR } F_i \Leftrightarrow$$

$$L_0 \sqcup L_1 \in \left\{ \frac{\text{uncurry } z}{z \in \prod(\text{GR} \circ F)} \right\} \Leftrightarrow$$

$$L_0 \sqcup L_1 \in \text{GR} \prod^{(D)} F.$$

□

For staroids it is defined *ordinated product* $\prod^{(\text{ord})}$ as defined in the section 3.7.4 above.

OBVIOUS 1828. If f and g are anchored relations and there exists a bijection φ from $\text{arity } g$ to $\text{arity } f$ such that $\left\{ \frac{F \circ \varphi}{F \in \text{GR } f} \right\} = \text{GR } g$, then:

1°. f is a prestaroid iff g is a prestaroid.

2°. f is a staroid iff g is a staroid.

3°. f is a completary staroid iff g is a completary staroid.

COROLLARY 1829. Let F be an indexed family of anchored relations and every $(\text{form } F)_i$ be a join-semilattice.

- 1°. $\prod^{(\text{ord})} F$ is a prestaroid if every F_i is a prestaroid.
- 2°. $\prod^{(\text{ord})} F$ is a staroid if every F_i is a staroid.
- 3°. $\prod^{(\text{ord})} F$ is a completary staroid if every F_i is a completary staroid.

PROOF. Use the fact that $\text{GR } \prod^{(\text{ord})} F = \left\{ \frac{F \circ (\bigoplus (\text{dom } \circ F))^{-1}}{F \in \text{GR } \prod^{(D)} f} \right\}$. □

DEFINITION 1830. $f \times^{(\text{ord})} g = \prod^{(\text{ord})} [f, g]$.

REMARK 1831. If f and g are binary funcoids, then $f \times^{(\text{ord})} g$ is ternary.

23.9. Star categories

DEFINITION 1832. A *precategory with star-morphisms* consists of

- 1°. a precategory C (*the base precategory*);
- 2°. a set M (*star-morphisms*);
- 3°. a function “arity” defined on M (how many objects are connected by this star-morphism);
- 4°. a function $\text{Obj}_m : \text{arity } m \rightarrow \text{Obj}(C)$ defined for every $m \in M$;
- 5°. a function (*star composition*) $(m, f) \mapsto \text{StarComp}(m, f)$ defined for $m \in M$ and f being an (arity m)-indexed family of morphisms of C such that $\forall i \in \text{arity } m : \text{Src } f_i = \text{Obj}_m i$ ($\text{Src } f_i$ is the source object of the morphism f_i) such that

such that it holds:

- 1°. $\text{StarComp}(m, f) \in M$;
- 2°. $\text{arity } \text{StarComp}(m, f) = \text{arity } m$;
- 3°. $\text{Obj}_{\text{StarComp}(m, f)} i = \text{Dst } f_i$;
- 4°. (*associativity law*)

$$\text{StarComp}(\text{StarComp}(m, f), g) = \text{StarComp}(m, \lambda i \in \text{arity } m : g_i \circ f_i).$$

The meaning of the set M is an extension of C having as morphisms things with arbitrary (possibly infinite) indexed set Obj_m of objects, not just two objects as morphisms of C have only source and destination.

DEFINITION 1833. I will call Obj_m the *form* of the star-morphism m .

(Having fixed a precategory with star-morphisms) I will denote $\text{StarHom}(P)$ the set of star-morphisms of the form P .

PROPOSITION 1834. The sets $\text{StarHom}(P)$ are disjoint (for different P).

PROOF. If two star-morphisms have different forms, they are clearly not equal. □

DEFINITION 1835. A *category with star-morphisms* is a precategory with star-morphisms whose base is a category and the following equality (*the law of composition with identity*) holds for every star-morphism m :

$$\text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = m.$$

DEFINITION 1836. A *partially ordered precategory with star-morphisms* is a category with star-morphisms, whose base precategory is a partially ordered precategory and every set $\text{StarHom}(X)$ is partially ordered for every X , such that:

$$m_0 \sqsubseteq m_1 \wedge f_0 \sqsubseteq f_1 \Rightarrow \text{StarComp}(m_0, f_0) \sqsubseteq \text{StarComp}(m_1, f_1)$$

for every $m_0, m_1 \in M$ such that $\text{Obj}_{m_0} = \text{Obj}_{m_1}$ and indexed families f_0 and f_1 of morphisms such that

$$\begin{aligned} \forall i \in \text{arity } m : \text{Src } f_0 i &= \text{Src } f_1 i = \text{Obj}_{m_0} i = \text{Obj}_{m_1} i; \\ \forall i \in \text{arity } m : \text{Dst } f_0 i &= \text{Dst } f_1 i. \end{aligned}$$

DEFINITION 1837. A *partially ordered category with star-morphisms* is a category with star-morphisms which is also a partially ordered precategory with star-morphisms.

DEFINITION 1838. A *quasi-invertible* precategory with star-morphisms is a partially ordered precategory with star-morphisms whose base precategory is a quasi-invertible precategory, such that for every index set n , star-morphisms a and b of arity n , and an n -indexed family f of morphisms of the base precategory it holds

$$b \not\prec \text{StarComp}(a, f) \Leftrightarrow a \not\prec \text{StarComp}(b, f^\dagger).$$

(Here $f^\dagger = \lambda i \in \text{dom } f : (f_i)^\dagger$.)

DEFINITION 1839. A *quasi-invertible* category with star-morphisms is a quasi-invertible precategory with star-morphisms which is a category with star-morphisms.

Each category with star-morphisms gives rise to a category (*abrupt category*, see a remark below why I call it “abrupt”), as described below. Below for simplicity I assume that the set M and the set of our indexed families of functions are disjoint. The general case (when they are not necessarily disjoint) may be easily elaborated by the reader.

- Objects are indexed (by arity m for some $m \in M$) families of objects of the category C and an (arbitrarily chosen) object None not in this set.
- There are the following disjoint sets of morphisms:
 - 1°. indexed (by arity m for some $m \in M$) families of morphisms of C ;
 - 2°. elements of M ;
 - 3°. the identity morphism 1_{None} on None .
- Source and destination of morphisms are defined by the formulas:
 - $\text{Src } f = \lambda i \in \text{dom } f : \text{Src } f_i$;
 - $\text{Dst } f = \lambda i \in \text{dom } f : \text{Dst } f_i$;
 - $\text{Src } m = \text{None}$;
 - $\text{Dst } m = \text{Obj}_m$.
- Compositions of morphisms are defined by the formulas:
 - $g \circ f = \lambda i \in \text{dom } f : g_i \circ f_i$ for our indexed families f and g of morphisms;
 - $f \circ m = \text{StarComp}(m, f)$ for $m \in M$ and a composable indexed family f ;
 - $m \circ 1_{\text{None}} = m$ for $m \in M$;
 - $1_{\text{None}} \circ 1_{\text{None}} = 1_{\text{None}}$.
- Identity morphisms for an object X are:
 - $\lambda i \in X : 1_{X_i}$ if $X \neq \text{None}$;
 - 1_{None} if $X = \text{None}$.

PROOF. We need to prove it is really a category.

We need to prove:

- 1°. Composition is associative.
- 2°. Composition with identities complies with the identity law.

Really:

$$\begin{aligned}
1^\circ. & (h \circ g) \circ f = \lambda i \in \text{dom } f : (h_i \circ g_i) \circ f_i = \lambda i \in \text{dom } f : h_i \circ (g_i \circ f_i) = h \circ (g \circ f); \\
& g \circ (f \circ m) = \text{StarComp}(\text{StarComp}(m, f), g) = \\
& \quad \text{StarComp}(m, \lambda i \in \text{arity } m : g_i \circ f_i) = \text{StarComp}(m, g \circ f) = (g \circ f) \circ m; \\
& f \circ (m \circ 1_{\text{None}}) = f \circ m = (f \circ m) \circ 1_{\text{None}}. \\
2^\circ. & m \circ 1_{\text{None}} = m; 1_{\text{Dst } m} \circ m = \text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = m.
\end{aligned}$$

□

REMARK 1840. I call the above defined category *abrupt category* because (excluding identity morphisms) it allows composition with an $m \in M$ only on the left (not on the right) so that the morphism m is “abrupt” on the right.

By $\llbracket x_0, \dots, x_{n-1} \rrbracket$ I denote an n -tuple.

DEFINITION 1841. Precategory with star morphisms *induced* by a dagger precategory C is:

- The base category is C .
- Star-morphisms are morphisms of C .
- $\text{arity } f = \{0, 1\}$.
- $\text{Obj}_m = \llbracket \text{Src } m, \text{Dst } m \rrbracket$.
- $\text{StarComp}(m, \llbracket f, g \rrbracket) = g \circ m \circ f^\dagger$.

Let prove it is really a precategory with star-morphisms.

PROOF. We need to prove the associativity law:

$$\text{StarComp}(\text{StarComp}(m, \llbracket f, g \rrbracket), \llbracket p, q \rrbracket) = \text{StarComp}(m, \llbracket p \circ f, q \circ g \rrbracket).$$

Really,

$$\begin{aligned}
\text{StarComp}(\text{StarComp}(m, \llbracket f, g \rrbracket), \llbracket p, q \rrbracket) &= \text{StarComp}(g \circ m \circ f^\dagger, \llbracket p, q \rrbracket) = \\
& q \circ g \circ m \circ f^\dagger \circ p^\dagger = q \circ g \circ m \circ (p \circ f)^\dagger = \text{StarComp}(m, \llbracket p \circ f, q \circ g \rrbracket).
\end{aligned}$$

□

DEFINITION 1842. Category with star morphisms *induced* by a dagger category C is the above defined precategory with star-morphisms.

That it is a category (the law of composition with identity) is trivial.

REMARK 1843. We can carry definitions (such as below defined cross-composition product) from categories with star-morphisms into plain dagger categories. This allows us to research properties of cross-composition product of indexed families of morphisms for categories with star-morphisms without separately considering the special case of dagger categories and just binary star-composition product.

23.9.1. Abrupt of quasi-invertible categories with star-morphisms.

DEFINITION 1844. The abrupt partially ordered precategory of a partially ordered precategory with star-morphisms is the abrupt precategory with the following order of morphisms:

- Indexed (by arity m for some $m \in M$) families of morphisms of C are ordered as function spaces of posets.
- Star-morphisms (which are morphisms $\text{None} \rightarrow \text{Obj}_m$ for some $m \in M$) are ordered in the same order as in the precategory with star-morphisms.
- Morphisms $\text{None} \rightarrow \text{None}$ which are only the identity morphism ordered by the unique order on this one-element set.

We need to prove it is a partially ordered precategory.

PROOF. It trivially follows from the definition of partially ordered precategory with star-morphisms. \square

23.10. Product of an arbitrary number of functors

In this section it will be defined a product of an arbitrary (possibly infinite) indexed family of functors.

23.10.1. Mapping a morphism into a pointfree functor.

DEFINITION 1845. Let's define the pointfree functor χf for every morphism f of a quasi-invertible category:

$$\langle \chi f \rangle a = f \circ a \quad \text{and} \quad \langle (\chi f)^{-1} \rangle b = f^\dagger \circ b.$$

We need to prove it is really a pointfree functor.

PROOF. $b \neq \langle \chi f \rangle a \Leftrightarrow b \neq f \circ a \Leftrightarrow a \neq f^\dagger \circ b \Leftrightarrow a \neq \langle (\chi f)^{-1} \rangle b$. \square

REMARK 1846. $\langle \chi f \rangle = (f \circ -)$ is the Hom-functor $\text{Hom}(f, -)$ and we can apply Yoneda lemma to it. (See any category theory book for definitions of these terms.)

OBVIOUS 1847. $\langle \chi(g \circ f) \rangle a = g \circ f \circ a$ for composable morphisms f and g or a quasi-invertible category.

23.10.2. General cross-composition product.

DEFINITION 1848. Let fix a quasi-invertible category with with star-morphisms. If f is an indexed family of morphisms from its base category, then the pointfree functor $\prod^{(C)} f$ (*cross-composition product* of f) from $\text{StarHom}(\lambda i \in \text{dom } f : \text{Src } f_i)$ to $\text{StarHom}(\lambda i \in \text{dom } f : \text{Dst } f_i)$ is defined by the formulas (for all star-morphisms a and b of these forms):

$$\left\langle \prod^{(C)} f \right\rangle a = \text{StarComp}(a, f) \quad \text{and} \quad \left\langle \left(\prod^{(C)} f \right)^{-1} \right\rangle b = \text{StarComp}(b, f^\dagger).$$

It is really a pointfree functor by the definition of quasi-invertible category with star-morphisms.

THEOREM 1849. $\left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) = \prod_{i \in n}^{(C)} (g_i \circ f_i)$ for every n -indexed families f and g of composable morphisms of a quasi-invertible category with star-morphisms.

PROOF. $\left\langle \prod_{i \in n}^{(C)} (g_i \circ f_i) \right\rangle a = \text{StarComp}(a, \lambda i \in n : g_i \circ f_i) = \text{StarComp}(\text{StarComp}(a, f), g)$ and

$$\left\langle \left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) \right\rangle a = \left\langle \prod^{(C)} g \right\rangle \left\langle \prod^{(C)} f \right\rangle a = \text{StarComp}(\text{StarComp}(a, f), g).$$

The rest follows from symmetry. \square

COROLLARY 1850. $\left(\prod^{(C)} f_{k-1} \right) \circ \dots \circ \left(\prod^{(C)} f_0 \right) = \prod_{i \in n}^{(C)} (f_{k-1} \circ \dots \circ f_0)$ for every n -indexed families f_0, \dots, f_{n-1} of composable morphisms of a quasi-invertible category with star-morphisms.

PROOF. By math induction. \square

23.10.3. Star composition of binary relations. First define *star composition* for an n -ary relation a and an n -indexed family f of binary relations as an n -ary relation complying with the formulas:

$$\begin{aligned} \text{Obj}_{\text{StarComp}(a,f)} &= \{*\}^n; \\ L \in \text{StarComp}(a,f) &\Leftrightarrow \exists y \in a \forall i \in n : y_i f_i L_i \end{aligned}$$

where $*$ is a unique object of the group of small binary relations considered as a category.

PROPOSITION 1851. $b \not\asymp \text{StarComp}(a,f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j f_j y_j$.

PROOF.

$$\begin{aligned} b \not\asymp \text{StarComp}(a,f) &\Leftrightarrow \exists y : (y \in b \wedge y \in \text{StarComp}(a,f)) \Leftrightarrow \\ &\exists y : (y \in b \wedge \exists x \in a \forall j \in n : x_j f_j y_j) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j f_j y_j. \end{aligned}$$

□

THEOREM 1852. The group of small binary relations considered as a category together with the set of all small n -ary relations (for every small n) and the above defined star-composition form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

- 1°. $\text{StarComp}(\text{StarComp}(m,f),g) = \text{StarComp}(m, \lambda i \in n : g_i \circ f_i)$;
- 2°. $\text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = m$;
- 3°. $b \not\asymp \text{StarComp}(a,f) \Leftrightarrow a \not\asymp \text{StarComp}(b, f^\dagger)$ (the rest is obvious).

Really,

- 1°. $L \in \text{StarComp}(a,f) \Leftrightarrow \exists y \in a \forall i \in n : y_i f_i L_i$.
Define the relation $R(f)$ by the formula $x R(f) y \Leftrightarrow \forall i \in n : x_i f_i y_i$. Obviously

$$R(\lambda i \in n : g_i \circ f_i) = R(g) \circ R(f).$$

$$L \in \text{StarComp}(a,f) \Leftrightarrow \exists y \in a : y R(f) L.$$

$$\begin{aligned} L \in \text{StarComp}(\text{StarComp}(a,f),g) &\Leftrightarrow \exists p \in \text{StarComp}(a,f) : p R(g) L \Leftrightarrow \\ &\exists p, y \in a : (y R(f) p \wedge p R(g) L) \Leftrightarrow \exists y \in a : y (R(g) \circ R(f)) L \Leftrightarrow \\ &\exists y \in a : y R(\lambda i \in n : g_i \circ f_i) L \Leftrightarrow L \in \text{StarComp}(a, \lambda i \in n : g_i \circ f_i) \end{aligned}$$

because $p \in \text{StarComp}(a,f) \Leftrightarrow \exists y \in a : y R(f) p$.

- 2°. Obvious.
- 3°. It follows from the proposition above.

□

OBVIOUS 1853. $\text{StarComp}(a \cup b, f) = \text{StarComp}(a, f) \cup \text{StarComp}(b, f)$ for n -ary relations a, b and an n -indexed family f of binary relations.

THEOREM 1854. $\langle \prod^{(C)} f \rangle \prod a = \prod_{i \in n} \langle f_i \rangle^* a_i$ for every family $f = f_{i \in n}$ of binary relations and $a = a_{i \in n}$ where a_i is a small set (for each $i \in n$).

PROOF.

$$\begin{aligned}
L \in \left\langle \prod^{(C)} f \right\rangle \prod a &\Leftrightarrow \\
L \in \text{StarComp}\left(\prod a, f\right) &\Leftrightarrow \\
\exists y \in \prod a \forall i \in n : y_i f_i L_i &\Leftrightarrow \\
\exists y \in \prod a \forall i \in n : \{y_i\} \neq \langle f_i^{-1} \rangle^* \{L_i\} &\Leftrightarrow \\
\forall i \in n \exists y \in a_i : \{y\} \neq \langle f_i^{-1} \rangle^* \{L_i\} &\Leftrightarrow \\
\forall i \in n : a_i \neq \langle f_i^{-1} \rangle^* \{L_i\} &\Leftrightarrow \\
\forall i \in n : \{L_i\} \neq \langle f_i \rangle^* a_i &\Leftrightarrow \\
\forall i \in n : L_i \in \langle f_i \rangle^* a_i &\Leftrightarrow \\
L \in \prod_{i \in n} \langle f_i \rangle^* a_i. &
\end{aligned}$$

□

23.10.4. Star composition of Rel-morphisms. Define *star composition* for an n -ary anchored relation a and an n -indexed family f of **Rel**-morphisms as an n -ary anchored relation complying with the formulas:

$$\begin{aligned}
\text{Obj}_{\text{StarComp}(a,f)} &= \lambda i \in \text{arity } a : \text{Dst } f_i; \\
\text{arity } \text{StarComp}(a, f) &= \text{arity } a;
\end{aligned}$$

$$L \in \text{GR } \text{StarComp}(a, f) \Leftrightarrow L \in \text{StarComp}(\text{GR } a, \text{GR } \circ f).$$

(Here I denote $\text{GR}(A, B, f) = f$ for every **Rel**-morphism f .)

PROPOSITION 1855.

$$b \neq \text{StarComp}(a, f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j \text{GR}(f_j) y_j.$$

PROOF. From the previous section. □

THEOREM 1856. Relations with above defined compositions form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

- 1°. $\text{StarComp}(\text{StarComp}(m, f), g) = \text{StarComp}(m, \lambda i \in \text{arity } m : g_i \circ f_i)$;
- 2°. $\text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = m$;
- 3°. $b \neq \text{StarComp}(a, f) \Leftrightarrow a \neq \text{StarComp}(b, f^\dagger)$

(the rest is obvious).

It follows from the previous section. □

PROPOSITION 1857. $\text{StarComp}(a \sqcup b, f) = \text{StarComp}(a, f) \sqcup \text{StarComp}(b, f)$ for an n -ary anchored relations a, b and an n -indexed family f of **Rel**-morphisms.

PROOF. It follows from the previous section. □

THEOREM 1858. Cross-composition product of a family of **Rel**-morphisms is a principal funcoid.

PROOF. By the proposition and symmetry $\prod^{(C)} f$ is a pointfree funcoid. Obviously it is a funcoid $\prod_{i \in n} \text{Src } f_i \rightarrow \prod_{i \in n} \text{Dst } f_i$. Its completeness (and dually co-completeness) is obvious. □

23.10.5. Cross-composition product of funcoids. Let a be an anchored relation of the form \mathfrak{A} and $\text{dom } \mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\text{Src } f_i = \mathfrak{A}_i$.

The star-composition of a with f is an anchored relation of the form $\lambda i \in \text{dom } \mathfrak{A} : \text{Dst } f_i$ defined by the formula

$$L \in \text{GR StarComp}(a, f) \Leftrightarrow (\lambda i \in n : \langle f_i^{-1} \rangle L_i) \in \text{GR } a.$$

THEOREM 1859. Let $\text{Src } f_i$ be separable starrish join-semilattice and $\text{Dst } f_i$ be a starrish join-semilattice for every $i \in n$ for a set n . Let form $a = \prod_{i \in n} (\text{Src } f_i)$.

1°. If a is a prestaroid then $\text{StarComp}(a, f)$ is a prestaroid.

2°. If a is a staroid and $\text{Src } f_i$ are strongly separable then $\text{StarComp}(a, f)$ is a staroid.

3°. If a is a completary staroid and then $\text{StarComp}(a, f)$ is a completary staroid.

PROOF. We have $\langle f_i^{-1} \rangle (X \sqcup Y) = \langle f_i^{-1} \rangle X \sqcup \langle f_i^{-1} \rangle Y$ by theorem 1604.

1°. Let $L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} (\text{form } f_i)$ for some $k \in n$ and $X, Y \in \text{form } f_k$. Then

$$\begin{aligned} X \sqcup Y \in \langle \text{StarComp}(a, f) \rangle_k^* L &\Leftrightarrow \\ \left(\lambda i \in \text{dom } f : \langle f_i^{-1} \rangle \left(\begin{array}{l} X \sqcup Y \quad \text{if } i = k \\ L_i \quad \text{if } i \neq k \end{array} \right)_i \right) \in \text{GR } a &\Leftrightarrow \\ \left(\lambda i \in \text{dom } f : \left(\begin{array}{l} \langle f_i^{-1} \rangle X \sqcup \langle f_i^{-1} \rangle Y \quad \text{if } i = k \\ \langle f_i^{-1} \rangle L_i \quad \text{if } i \neq k \end{array} \right)_i \right) \in \text{GR } a &\Leftrightarrow \\ \langle f_i^{-1} \rangle X \sqcup \langle f_i^{-1} \rangle Y \in \langle a \rangle_k^* (\lambda i \in (\text{dom } f) \setminus \{k\} : \langle f_i^{-1} \rangle L_i) &\Leftrightarrow \\ \langle f_i^{-1} \rangle X \in \langle a \rangle_k^* (\lambda i \in n \setminus \{k\} : \langle f_i^{-1} \rangle L_i) \vee \langle f_i^{-1} \rangle Y \in \langle a \rangle_k^* (\lambda i \in n \setminus \{k\} : \langle f_i^{-1} \rangle L_i) &\Leftrightarrow \\ \left(\lambda i \in \text{dom } f : \left(\begin{array}{l} \langle f_i^{-1} \rangle X \quad \text{if } i = k \\ \langle f_i^{-1} \rangle L_i \quad \text{if } i \neq k \end{array} \right)_i \right) \in \text{GR } a \vee &\Leftrightarrow \\ \left(\lambda i \in \text{dom } f : \left(\begin{array}{l} \langle f_i^{-1} \rangle Y \quad \text{if } i = k \\ \langle f_i^{-1} \rangle L_i \quad \text{if } i \neq k \end{array} \right)_i \right) \in \text{GR } a &\Leftrightarrow \\ \left(\lambda i \in \text{dom } f : \langle f_i^{-1} \rangle \left(\begin{array}{l} X \quad \text{if } i = k \\ L_i \quad \text{if } i \neq k \end{array} \right)_i \right) \in \text{GR } a \vee &\Leftrightarrow \\ \left(\lambda i \in \text{dom } f : \langle f_i^{-1} \rangle \left(\begin{array}{l} Y \quad \text{if } i = k \\ L_i \quad \text{if } i \neq k \end{array} \right)_i \right) \in \text{GR } a &\Leftrightarrow \\ X \in \langle \text{StarComp}(a, f) \rangle_k^* L \vee Y \in \langle \text{StarComp}(a, f) \rangle_k^* L. & \end{aligned}$$

Thus $\text{StarComp}(a, f)$ is a pre-staroid.

2°. $\langle f_i^{-1} \rangle$ are monotone functions by the proposition 1603. Thus $\langle f_i^{-1} \rangle X_i \sqsubseteq \langle f_i^{-1} \rangle Y_i$ if $X, Y \in \prod_{i \in (\text{arity } f) \setminus \{k\}} (\text{form } f_i)$ and $X \sqsubseteq Y$. So if a is a staroid and $X \in \text{GR StarComp}(a, f)$ then $(\lambda i \in \text{dom } f : \langle f_i^{-1} \rangle X_i) \in \text{GR } a$ then $(\lambda i \in \text{dom } f : \langle f_i^{-1} \rangle Y_i) \in \text{GR } a$ that is $Y \in \text{GR StarComp}(a, f)$.

3°.

$$\begin{aligned} L_0 \sqcup L_1 \in \text{GR StarComp}(a, f) &\Leftrightarrow \\ (\lambda i \in n : \langle f_i^{-1} \rangle (L_0 \sqcup L_1)_i) \in \text{GR } a &\Leftrightarrow \\ (\lambda i \in n : \langle f_i^{-1} \rangle L_{0i} \sqcup \langle f_i^{-1} \rangle L_{1i}) \in \text{GR } a &\Leftrightarrow \\ \exists c \in \{0, 1\} : (\lambda i \in n : \langle f_i^{-1} \rangle L_{c(i)} i) \in \text{GR } a &\Leftrightarrow \\ \exists c \in \{0, 1\} : (\lambda i \in n : L_{c(i)} i) \in \text{GR StarComp}(a, f). & \end{aligned}$$

□

CONJECTURE 1860. $b \not\prec^{\text{Anch}(\mathfrak{A})} \text{StarComp}(a, f) \Leftrightarrow \forall A \in \text{GR } a, B \in \text{GR } b, i \in n : A_i [f_i] B_i$ for anchored relations a and b on powersets.

It's consequence:

CONJECTURE 1861. $b \not\prec^{\text{Anch}(\mathfrak{A})} \text{StarComp}(a, f) \Leftrightarrow a \not\prec^{\text{Anch}(\mathfrak{A})} \text{StarComp}(b, f^\dagger)$ for anchored relations a and b on powersets.

CONJECTURE 1862. $b \not\prec^{\text{Strd}(\mathfrak{A})} \text{StarComp}(a, f) \Leftrightarrow a \not\prec^{\text{Strd}(\mathfrak{A})} \text{StarComp}(b, f^\dagger)$ for pre-staroids a and b on powersets.

PROPOSITION 1863. Anchored relations with objects being posets with above defined star-morphisms is a category with star morphisms.

PROOF. We need to prove:

1°. $\text{StarComp}(\text{StarComp}(m, f), g) = \text{StarComp}(m, \lambda i \in \text{arity } m : g_i \circ f_i)$;

2°. $\text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = m$.

(the rest is obvious). Really,

$$\begin{aligned} L \in \text{GR StarComp}(\text{StarComp}(m, f), g) &\Leftrightarrow \\ (\lambda i \in \text{arity } m : \langle g_i^{-1} \rangle L_i) \in \text{GR StarComp}(m, f) &\Leftrightarrow \\ (\lambda i \in n : \langle f_i^{-1} \rangle (\lambda j \in n : \langle g_j^{-1} \rangle L_j)_i) \in \text{GR } m &\Leftrightarrow \\ (\lambda i \in \text{arity } m : \langle f_i^{-1} \rangle \langle g_i^{-1} \rangle L_i) \in \text{GR } m &\Leftrightarrow \\ (\lambda i \in \text{arity } m : \langle (g_i \circ f_i)^{-1} \rangle L_i) \in \text{GR } m &\Leftrightarrow \\ L \in \text{GR StarComp}(m, \lambda i \in \text{arity } m : g_i \circ f_i) & \end{aligned}$$

and

$$\begin{aligned} L \in \text{GR StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) &\Leftrightarrow \\ (\lambda i \in n : \langle 1_{\text{Obj}_m i} \rangle L_i) \in \text{GR } m &\Leftrightarrow \\ (\lambda i \in \text{arity } m : \langle 1_{\text{Obj}_m i} \rangle L_i) \in \text{GR } m &\Leftrightarrow \\ (\lambda i \in \text{arity } m : L_i) \in \text{GR } m &\Leftrightarrow L \in \text{GR } m. \end{aligned}$$

□

CONJECTURE 1864. $\text{StarComp}(a \sqcup b, f) = \text{StarComp}(a, f) \sqcup \text{StarComp}(b, f)$ for anchored relations a, b of a form \mathfrak{A} , where every \mathfrak{A}_i is a distributive lattice, and an indexed family f of pointfree funcoids with $\text{Src } f_i = \mathfrak{A}_i$.

23.10.6. Cross-composition product of funcoids through atoms. Let a be a an anchored relation of the form \mathfrak{A} and $\text{dom } \mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\text{Src } f_i = \mathfrak{A}_i$.

The *atomary star-composition* of a with f is an anchored relation of the form $\lambda i \in \text{dom } \mathfrak{A} : \text{Dst } f_i$ defined by the formula

$$L \in \text{GR StarComp}^{(a)}(a, f) \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} \forall i \in n : y_i [f_i] L_i.$$

THEOREM 1865. Let $\text{Dst } f_i$ be a starrish join-semilattice for every $i \in n$.

1°. If a is a prestaroid then $\text{StarComp}^{(a)}(a, f)$ is a staroid.

2°. If a is a cometary staroid and then $\text{StarComp}^{(a)}(a, f)$ is a cometary staroid.

PROOF.

1°. First prove that $\text{StarComp}^{(a)}(a, f)$ is a prestaroid. We need to prove that $(\text{val StarComp}^{(a)}(a, f))_j L$ (for every $j \in n$) is a free star, that is

$$\left\{ \frac{X \in (\text{form } f)_j}{L \cup \{(j, X)\} \in \text{GR StarComp}^{(a)}(a, f)} \right\}$$

is a free star, that is the following is a free star

$$\left\{ \frac{X \in (\text{form } f)_j}{R(X)} \right\}$$

where $R(X) \Leftrightarrow \exists y \in \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} : (\forall i \in n \setminus \{j\} : y_i [f_i] L_i \wedge y_j [f_j] X \wedge y \in \text{GR } a)$.

$$R(X) \Leftrightarrow$$

$$\exists y \in \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} : (\forall i \in n \setminus \{j\} : y_i [f_i] L_i \wedge y_j [f_j] X \wedge y_j \in (\text{val } a)_j(y|_{n \setminus \{j\}})) \Leftrightarrow$$

$$\exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^{\mathfrak{A}_i}, y' \in \text{atoms}^{\mathfrak{A}_j} : \left(\forall i \in n \setminus \{j\} : y_i [f_i] L_i \wedge y' [f_j] X \wedge y' \in (\text{val } a)_j(y|_{n \setminus \{j\}}) \right) \Leftrightarrow$$

$$\exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^{\mathfrak{A}_i} \forall i \in n \setminus \{j\} : y_i [f_i] L_i \wedge$$

$$\exists y' \in \text{atoms}^{\mathfrak{A}_j} : (y' [f_j] X \wedge y' \in (\text{val } a)_j(y|_{n \setminus \{j\}})).$$

If $\exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^{\mathfrak{A}_i} \forall i \in n \setminus \{j\} : y_i [f_i] L_i$ is false our statement is obvious. We can assume it is true.

So it is enough to prove that

$$\left\{ \frac{X \in (\text{form } f)_j}{\exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^{\mathfrak{A}_i}, y' \in \text{atoms}^{\mathfrak{A}_j} : (y' [f_j] X \wedge y' \in (\text{val } a)_j(y|_{n \setminus \{j\}}))} \right\}$$

is a free star. That is

$$Q = \left\{ \frac{X \in (\text{form } f)_j}{\exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^{\mathfrak{A}_i}, y' \in (\text{atoms}^{\mathfrak{A}_j}) \cap (\text{val } a)_j(y|_{n \setminus \{j\}}) : y' [f_j] X} \right\}$$

is a free star. $\perp^{(\text{form } f)_j} \notin Q$ is obvious. That Q is an upper set is obvious. It remains to prove that $X_0 \sqcup X_1 \in Q \Rightarrow X_0 \in Q \vee X_1 \in Q$ for every $X_0, X_1 \in (\text{form } f)_j$. Let $X_0 \sqcup X_1 \in Q$. Then there exist $y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^{\mathfrak{A}_i}$, $y' \in (\text{atoms}^{\mathfrak{A}_j}) \cap (\text{val } a)_j(y|_{n \setminus \{j\}})$ such that $y' [f_j] X_0 \sqcup X_1$. Consequently (proposition 1605) $y' [f_j] X_0 \vee y' [f_j] X_1$. But then $X_0 \in Q \vee X_1 \in Q$.

To finish the proof we need to show that $\text{GR StarComp}^{(a)}(a, f)$ is an upper set, but this is obvious.

2°. Let a be a completary staroid. Let $L_0 \sqcup L_1 \in \text{GR StarComp}^{(a)}(a, f)$ that is $\exists y \in \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} : (\forall i \in n : y_i [f_i] L_0 i \sqcup L_1 i \wedge y \in \text{GR } a)$ that is $\exists c \in \{0, 1\}^n, y \in \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} : (\forall i \in n : y_i [f_i] L_{c(i)} i \wedge y \in \text{GR } a)$ (taken into account that $\text{Dst } f_i$ is starrish) that is $\exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)} i) \in \text{GR StarComp}^{(a)}(a, f)$. So $\text{StarComp}^{(a)}(a, f)$ is a completary staroid. □

LEMMA 1866. $b \not\prec^{\text{Anch}(\mathfrak{B})} \text{StarComp}^{(a)}(a, f) \Leftrightarrow \forall A \in \text{GR } a, B \in \text{GR } b, i \in n : A_i [f_i] B_i$ for anchored relations a and b , provided that $\text{Src } f_i$ are atomic posets.

PROOF.

$$\begin{aligned}
& b \not\asymp^{\text{Anch}(\mathfrak{B})} \text{StarComp}^{(a)}(a, f) \Leftrightarrow \\
& \exists x \in \text{Anch}(\mathfrak{B}) \setminus \{\perp\} : (x \sqsubseteq b \wedge x \sqsubseteq \text{StarComp}^{(a)}(a, f)) \Leftrightarrow \\
& \exists x \in \text{Anch}(\mathfrak{B}) \setminus \{\perp\} : (x \sqsubseteq b \wedge \forall B \in \text{GR } x : B \in \text{GR } \text{StarComp}^{(a)}(a, f)) \Leftrightarrow \\
& \exists x \in \text{Anch}(\mathfrak{B}) \setminus \{\perp\} : \\
& \left(x \sqsubseteq b \wedge \forall B \in \text{GR } x \exists A \in \prod_{i \in \text{dom } \mathfrak{B}} \text{atoms}^{\mathfrak{B}_i} : (\forall i \in n : A_i [f_i] B_i \wedge A \in \text{GR } a) \right) \Leftrightarrow \\
& \exists x \in \text{Anch}(\mathfrak{B}) \setminus \{\perp\} : (x \sqsubseteq b \wedge \forall B \in \text{GR } x, A \in \text{GR } a, i \in n : A_i [f_i] B_i) \Leftrightarrow \\
& \exists x \in \text{Anch}(\mathfrak{B}) : (x \sqsubseteq b \wedge \forall B \in \text{GR } x, A \in \text{GR } a, i \in n : A_i [f_i] B_i) \Leftrightarrow \\
& \forall B \in \text{GR } b, A \in \text{GR } a, i \in n : A_i [f_i] B_i.
\end{aligned}$$

□

DEFINITION 1867. I will denote the cross-composition product for the star-composition $\text{StarComp}^{(a)}$ as $\prod^{(a)}$.

THEOREM 1868. $a \left[\prod^{(a)} f \right] b \Leftrightarrow \forall A \in \text{GR } a, B \in \text{GR } b, i \in n : A_i [f_i] B_i$ for anchored relations a and b , provided that $\text{Src } f_i$ and $\text{Dst } f_i$ are atomic posets.

PROOF. From the lemma. □

CONJECTURE 1869. $b \not\asymp^{\text{Strd}(\mathfrak{B})} \text{StarComp}(a, f) \Leftrightarrow a \not\asymp^{\text{Strd}(\mathfrak{A})} \text{StarComp}(b, f^\dagger)$ for staroids a and b on indexed families \mathfrak{A} and \mathfrak{B} of filters on powersets.

THEOREM 1870. Anchored relations with objects being atomic posets and above defined compositions form a quasi-invertible precategory with star-morphisms.

REMARK 1871. It seems that this precategory with star-morphisms isn't a category with star-morphisms.

PROOF. We need to prove:

- 1°. $\text{StarComp}^{(a)}(\text{StarComp}^{(a)}(m, f), g) = \text{StarComp}^{(a)}(m, \lambda i \in \text{arity } m : g_i \circ f_i)$;
- 2°. $b \not\asymp \text{StarComp}^{(a)}(a, f) \Leftrightarrow a \not\asymp \text{StarComp}^{(a)}(b, f^\dagger)$

(the rest is obvious).

Really, let a be a star morphism and $\mathfrak{A}_i = (\text{Obj}_a)_i$ for every $i \in \text{arity } a$;

- 1°. $L \in \text{GR } \text{StarComp}^{(a)}(a, f) \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} \forall i \in n : y_i [f_i] L_i$.
Define the relation $R(f)$ by the formula $x R(f) y \Leftrightarrow \forall i \in n : x_i [f_i] y_i$.

Obviously

$$R(\lambda i \in n : g_i \circ f_i) = R(g) \circ R(f).$$

$$\begin{aligned}
L \in \text{GR StarComp}^{(a)}(a, f) &\Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{\mathfrak{A}_i} : y R(f) L. \\
L \in \text{GR StarComp}^{(a)}(\text{StarComp}(a, f), g) &\Leftrightarrow \\
\exists p \in \text{GR StarComp}^{(a)}(a, f) \cap \prod_{i \in n} \text{atoms}^{(\text{Dst } f)_i} : p R(g) L &\Leftrightarrow \\
\exists p \in \prod_{i \in n} \text{atoms}^{(\text{Dst } f)_i}, y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{(\text{Src } f)_i} : (y R(f) p \wedge p R(g) L) &\Leftrightarrow \\
\exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{(\text{Src } f)_i} : y (R(g) \circ R(f)) L &\Leftrightarrow \\
\exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{(\text{Src } f)_i} : y R(\lambda i \in n : g_i \circ f_i) L &\Leftrightarrow \\
\exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{(\text{Src } f)_i} y \forall i \in n : y_i [g_i \circ f_i] L_i &\Leftrightarrow \\
L \in \text{GR StarComp}^{(a)}(a, \lambda i \in n : g_i \circ f_i) &
\end{aligned}$$

because $p \in \text{GR StarComp}^{(a)}(a, f) \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{atoms}^{(\text{Src } f)_i} y : y R(f) p$.

2°. It follows from the lemma above. \square

THEOREM 1872. $\langle \prod^{(a)} f \rangle \prod^{\text{Strd}} a = \prod_{i \in n}^{\text{Strd}} \langle f_i \rangle a_i$ for every family $f = f_{i \in n}$ of pointfree funcoids between atomic posets and $a = a_{i \in n}$ where $a_i \in \text{Src } f_i$.

PROOF.

$$\begin{aligned}
L \in \text{GR} \left\langle \prod^{(a)} f \right\rangle \prod^{\text{Strd}} a &\Leftrightarrow \\
L \in \text{GR StarComp}^{(a)} \left(\prod^{\text{Strd}} a, f \right) &\Leftrightarrow \\
\exists y \in \prod_{i \in \text{dom } \mathfrak{A}} \text{atoms}^{\mathfrak{A}_i} \forall i \in n : (y_i [f_i] L_i \wedge y_i \neq a_i) &\Leftrightarrow \\
\forall i \in n \exists y \in \text{atoms}^{\mathfrak{A}_i} : (y [f_i] L_i \wedge y \neq a_i) &\Leftrightarrow \\
\forall i \in n : a_i [f_i] L_i &\Leftrightarrow \\
\forall i \in n : L_i \neq \langle f_i \rangle a_i &\Leftrightarrow \\
L \in \text{GR} \prod_{i \in n}^{\text{Strd}} \langle f_i \rangle a_i. &
\end{aligned}$$

\square

CONJECTURE 1873. $\text{StarComp}^{(a)}(a \sqcup b, f) = \text{StarComp}^{(a)}(a, f) \sqcup \text{StarComp}^{(a)}(b, f)$ for anchored relations a, b of a form \mathfrak{A} , where every \mathfrak{A}_i is a distributive lattice, and an indexed family f of pointfree funcoids with $\text{Src } f_i = \mathfrak{A}_i$.

23.10.7. Simple product of pointfree funcoids.

DEFINITION 1874. Let f be an indexed family of pointfree funcoids with every $\text{Src } f_i$ and $\text{Dst } f_i$ (for all $i \in \text{dom } f$) being a poset with least element. *Simple product* of f is

$$\prod^{(S)} f = \left(\lambda x \in \prod_{i \in \text{dom } f} \text{Src } f_i : \lambda i \in \text{dom } f : \langle f_i \rangle x_i, \lambda y \in \prod_{i \in \text{dom } f} \text{Dst } f_i : \lambda i \in \text{dom } f : \langle f_i^{-1} \rangle y_i \right).$$

PROPOSITION 1875. Simple product is a pointfree funcoid

$$\prod^{(S)} f \in \text{pFCD} \left(\prod_{i \in \text{dom } f} \text{Src } f_i, \prod_{i \in \text{dom } f} \text{Dst } f_i \right).$$

PROOF. Let $x \in \prod_{i \in \text{dom } f} \text{Src } f_i$ and $y \in \prod_{i \in \text{dom } f} \text{Dst } f_i$. Then (take into account that $\text{Src } f_i$ and $\text{Dst } f_i$ are posets with least elements)

$$\begin{aligned} y \not\prec \left(\lambda x \in \prod_{i \in \text{dom } f} \text{Src } f_i : \lambda i \in \text{dom } f : \langle f_i \rangle x_i \right) x &\Leftrightarrow \\ y \not\prec \lambda i \in \text{dom } f : \langle f_i \rangle x_i &\Leftrightarrow \\ \exists i \in \text{dom } f : y_i \not\prec \langle f_i \rangle x_i &\Leftrightarrow \\ \exists i \in \text{dom } f : x_i \not\prec \langle f_i^{-1} \rangle y_i &\Leftrightarrow \\ x \not\prec \lambda i \in \text{dom } f : \langle f_i^{-1} \rangle y_i &\Leftrightarrow \\ x \not\prec \left(\lambda y \in \prod_{i \in \text{dom } f} \text{Dst } f_i : \lambda i \in \text{dom } f : \langle f_i^{-1} \rangle y_i \right) y. & \end{aligned}$$

□

OBVIOUS 1876. $\langle \prod^{(S)} f \rangle x = \lambda i \in \text{dom } f : \langle f_i \rangle x_i$ for $x \in \prod \text{Src } f_i$.

OBVIOUS 1877. $\left(\langle \prod^{(S)} f \rangle x \right)_i = \langle f_i \rangle x_i$ for $x \in \prod \text{Src } f_i$.

PROPOSITION 1878. f_i can be restored if we know $\prod^{(S)} f$ if f_i is a family of pointfree funcoids between posets with least elements.

PROOF. Let's restore the value of $\langle f_i \rangle x$ where $i \in \text{dom } f$ and $x \in \text{Src } f_i$.

Let $x'_i = x$ and $x'_j = \perp$ for $j \neq i$.

Then $\langle f_i \rangle x = \langle f_i \rangle x'_i = \left(\langle \prod^{(S)} f \rangle x' \right)_i$.

We have restored the value of $\langle f_i \rangle$. Restoring the value of $\langle f_i^{-1} \rangle$ is similar. □

REMARK 1879. In the above proposition it is not required that f_i are non-zero.

PROPOSITION 1880. $\left(\prod^{(S)} g \right) \circ \left(\prod^{(S)} f \right) = \prod_{i \in n}^{(S)} (g_i \circ f_i)$ for n -indexed families f and g of composable pointfree funcoids between posets with least elements.

PROOF.

$$\begin{aligned} \left\langle \prod_{i \in n}^{(S)} (g_i \circ f_i) \right\rangle x &= \lambda i \in \text{dom } f : \langle g_i \circ f_i \rangle x_i = \lambda i \in \text{dom } f : \langle g_i \rangle \langle f_i \rangle x_i = \\ \left\langle \prod^{(S)} g \right\rangle \lambda i \in \text{dom } f : \langle f_i \rangle x_i &= \left\langle \prod^{(S)} g \right\rangle \left\langle \prod^{(S)} f \right\rangle x = \left\langle \left(\prod^{(S)} g \right) \circ \left(\prod^{(S)} f \right) \right\rangle x. \end{aligned}$$

Thus $\left\langle \prod_{i \in n}^{(S)} (g_i \circ f_i) \right\rangle = \left\langle \left(\prod^{(S)} g \right) \circ \left(\prod^{(S)} f \right) \right\rangle$.

$\left\langle \left(\prod_{i \in n}^{(S)} (g_i \circ f_i) \right)^{-1} \right\rangle = \left\langle \left(\left(\prod^{(S)} g \right) \circ \left(\prod^{(S)} f \right) \right)^{-1} \right\rangle$ is similar. □

COROLLARY 1881. $\left(\prod^{(S)} f_{k-1} \right) \circ \dots \circ \left(\prod^{(S)} f_0 \right) = \prod_{i \in n}^{(S)} (f_{k-1} \circ \dots \circ f_0)$ for every n -indexed families f_0, \dots, f_{n-1} of composable pointfree funcoids between posets with least elements.

23.11. Multireloids

DEFINITION 1882. I will call a *multireloid* of the form $A = A_{i \in n}$, where every A_i is a set, a pair (f, A) where f is a filter on the set $\prod A$.

DEFINITION 1883. I will denote $\text{Obj}(f, A) = A$ and $\text{GR}(f, A) = f$ for every multireloid (f, A) .

I will denote $\text{RLD}(A)$ the set of multireloids of the form A .

The multireloid $\uparrow^{\text{RLD}(A)} F$ for a relation F is defined by the formulas:

$$\text{Obj} \uparrow^{\text{RLD}(A)} F = A \quad \text{and} \quad \text{GR} \uparrow^{\text{RLD}(A)} F = \uparrow^{\prod A} F.$$

For an anchored relation f I define $\text{Obj} \uparrow f = \text{form } f$ and $\text{GR} \uparrow f = \uparrow^{\prod \text{form } f} \text{GR } f$.

Let a be a multireloid of the form A and $\text{dom } A = n$.

Let every f_i be a reloid with $\text{Src } f_i = A_i$.

The star-composition of a with f is a multireloid of the form $\lambda i \in \text{dom } A : \text{Dst } f_i$ defined by the formulas:

$$\begin{aligned} \text{arity StarComp}(a, f) &= n; \\ \text{GR StarComp}(a, f) &= \prod^{\text{RLD}(A)} \left\{ \frac{\text{GR StarComp}(A, F)}{A \in \text{GR } a, F \in \prod_{i \in n} \text{GR } f_i} \right\}; \\ \text{Obj}_m \text{StarComp}(a, f) &= \lambda i \in n : \text{Dst } f_i. \end{aligned}$$

THEOREM 1884. Multireloids with above defined compositions form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

- 1°. $\text{StarComp}(\text{StarComp}(m, f), g) = \text{StarComp}(m, \lambda i \in \text{arity } m : g_i \circ f_i)$;
- 2°. $\text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = m$;
- 3°. $b \neq \text{StarComp}(a, f) \Leftrightarrow a \neq \text{StarComp}(b, f^\dagger)$

(the rest is obvious).

Really,

- 1°. Using properties of generalized filter bases,

$$\begin{aligned} &\text{StarComp}(\text{StarComp}(a, f), g) = \\ &\prod^{\text{RLD}} \left\{ \frac{\text{StarComp}(B, G)}{B \in \text{GR StarComp}(a, f), G \in \prod_{i \in n} \text{GR } g_i} \right\} = \\ &\prod^{\text{RLD}} \left\{ \frac{\text{StarComp}(\text{StarComp}(A, F), G)}{A \in \text{GR } a, F \in \prod_{i \in n} \text{GR } f_i, G \in \prod_{i \in n} \text{GR } g_i} \right\} = \\ &\prod^{\text{RLD}} \left\{ \frac{\text{StarComp}(A, G \circ F)}{A \in \text{GR } a, F \in \prod_{i \in n} \text{GR } f_i, G \in \prod_{i \in n} \text{GR } g_i} \right\} = \\ &\prod^{\text{RLD}} \left\{ \frac{\text{StarComp}(A, H)}{A \in \text{GR } a, H \in \prod_{i \in n} \text{GR}(g_i \circ f_i)} \right\} = \\ &\text{StarComp}(a, \lambda i \in \text{arity } n : g_i \circ f_i). \end{aligned}$$

2°.

$$\begin{aligned}
& \text{StarComp}(m, \lambda i \in \text{arity } m : 1_{\text{Obj}_m i}) = \\
& \prod^{\text{RLD}(A)} \left\{ \frac{\text{StarComp}(A, H)}{A \in \text{GR } m, H \in \prod_{i \in \text{arity } m} \text{GR } 1_{\text{Obj}_m i}} \right\} = \\
& \prod^{\text{RLD}(A)} \left\{ \frac{\text{StarComp}(A, \lambda i \in \text{arity } m : H_i)}{A \in \text{GR } m, H \in \prod_{i \in \text{arity } m} \text{GR } 1_{\text{Obj}_m i}} \right\} = \\
& \prod^{\text{RLD}(A)} \left\{ \frac{\text{StarComp}(A, \lambda i \in \text{arity } m : 1_{X_i})}{A \in \text{GR } m, X \in \prod_{i \in \text{arity } m} \text{Obj}_m i} \right\} = \\
& \prod^{\text{RLD}(A)} \left\{ \frac{(A \cap \prod X)}{A \in \text{GR } m, X \in \prod_{i \in \text{arity } m} \text{Obj}_m i} \right\} = \\
& \prod^{\text{RLD}(A)} \left\{ \frac{A}{A \in \text{GR } m} \right\} = m.
\end{aligned}$$

3°. Using properties of generalized filter bases,

$$\begin{aligned}
& b \not\prec \text{StarComp}(a, f) \Leftrightarrow \\
& \forall A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i : B \not\prec \text{StarComp}(A, F) \Leftrightarrow \\
& \forall A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i : B \not\prec \left\langle \prod_{i \in n}^{(C)} F \right\rangle A \Leftrightarrow \\
& \forall A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i : A \not\prec \left\langle \left(\prod_{i \in n}^{(C)} F \right)^{-1} \right\rangle B \Leftrightarrow \\
& \forall A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i : A \not\prec \text{StarComp}(B, F^\dagger) \Leftrightarrow \\
& a \not\prec \text{StarComp}(b, f^\dagger).
\end{aligned}$$

□

DEFINITION 1885. Let f be a multireloid of the form A . Then for $i \in \text{dom } A$

$$\text{Pr}_i^{\text{RLD}} f = \prod_i^{\mathcal{F}} \langle \text{Pr}_i \rangle^* \text{GR } f.$$

PROPOSITION 1886. $\text{up Pr}_i^{\text{RLD}} f = \langle \text{Pr}_i \rangle^* \text{GR } f$ for every multireloid f and $i \in \text{arity } f$.PROOF. It's enough to show that $\langle \text{Pr}_i \rangle^* \text{GR } f$ is a filter.That $\langle \text{Pr}_i \rangle^* \text{GR } f$ is an upper set is obvious.Let $X, Y \in \langle \text{Pr}_i \rangle^* \text{GR } f$. Then there exist $F, G \in \text{GR } f$ such that $X = \text{Pr}_i F$, $Y = \text{Pr}_i G$. Then $X \cap Y \supseteq \text{Pr}_i(F \cap G) \in \langle \text{Pr}_i \rangle^* \text{GR } f$. Thus $X \cap Y \in \langle \text{Pr}_i \rangle^* \text{GR } f$.

□

DEFINITION 1887. $\prod^{\text{RLD}} \mathcal{X} = \prod_{X \in \text{up } \prod \mathcal{X}}^{\text{RLD}(\lambda i \in \text{dom } \mathcal{X} : \text{Base}(\mathcal{X}_i))} \prod X$ for every indexed family \mathcal{X} of filters on powersets.PROPOSITION 1888. $\text{Pr}_k^{\text{RLD}} \prod^{\text{RLD}} x = x_k$ for every indexed family x of proper filters.PROOF. $\text{up Pr}_k^{\text{RLD}} \prod^{\text{RLD}} x = \langle \text{Pr}_k \rangle^* \prod^{\text{RLD}} x = \text{up } x_k$.

□

CONJECTURE 1889. $\text{GR StarComp}(a \sqcup b, f) = \text{GR StarComp}(a, f) \sqcup \text{GR StarComp}(b, f)$ if f is a reloid and a, b are multireloids of the same form, composable with f .

THEOREM 1890. $\prod^{\text{RLD}} A = \sqcup \left\{ \frac{\prod^{\text{RLD}} a}{a \in \prod_{i \in \text{dom } A} \text{atoms } A_i} \right\}$ for every indexed family A of filters on powersets.

PROOF. Obviously $\prod^{\text{RLD}} A \supseteq \sqcup \left\{ \frac{\prod^{\text{RLD}} a}{a \in \prod_{i \in \text{dom } A} \text{atoms } A_i} \right\}$.

Reversely, let $K \in \text{GR} \sqcup \left\{ \frac{\prod^{\text{RLD}} a}{a \in \prod_{i \in \text{dom } A} \text{atoms } A_i} \right\}$.

Consequently $K \in \text{GR} \prod^{\text{RLD}} a$ for every $a \in \prod_{i \in \text{dom } A} \text{atoms } A_i$; $K \supseteq \prod X$ and thus $K \supseteq \bigcup_{a \in \prod_{i \in \text{dom } A} \text{atoms } A_i} \prod X_a$ for some $X_a \in \prod_{i \in \text{dom } A} \text{atoms } A_i$.

But $\bigcup_{a \in \prod_{i \in \text{dom } A} \text{atoms } A_i} \prod X_a = \prod_{i \in \text{dom } A} \bigcup_{a \in \text{atoms } A_i} \langle \text{Pr}_i \rangle^* X_a \supseteq \prod_{j \in \text{dom } A} Z_j$ for some $Z_j \in \text{up } A_j$ because $\langle \text{Pr}_i \rangle^* X \in \text{up } a_i$ and our lattice is atomistic. So $K \in \text{GR} \prod^{\text{RLD}} A$. \square

THEOREM 1891. Let a, b be indexed families of filters on powersets of the same form \mathfrak{A} . Then

$$\prod^{\text{RLD}} a \sqcap \prod^{\text{RLD}} b = \prod^{\text{RLD}} (a_i \sqcap b_i).$$

PROOF.

$$\begin{aligned} & \text{up} \left(\prod^{\text{RLD}} a \sqcap \prod^{\text{RLD}} b \right) = \\ & \prod^{\text{RLD}(\mathfrak{A})} \left\{ \frac{P \sqcap Q}{P \in \text{GR} \prod^{\text{RLD}} a, Q \in \prod^{\text{RLD}} b} \right\} = \\ & \prod^{\text{RLD}(\mathfrak{A})} \left\{ \frac{\prod p \sqcap \prod q}{p \in \text{up} \prod a, q \in \text{up} \prod b} \right\} = \\ & \prod^{\text{RLD}(\mathfrak{A})} \left\{ \frac{\prod_{i \in \text{dom } \mathfrak{A}} (p_i \sqcap q_i)}{p \in \prod \text{up } a, q \in \prod \text{up } b} \right\} = \\ & \prod^{\text{RLD}(\mathfrak{A})} \left\{ \frac{\prod r}{r \in \text{up} \prod_{i \in \text{dom } \mathfrak{A}} (a_i \sqcap b_i)} \right\} = \\ & \text{up} \prod^{\text{RLD}} (a_i \sqcap b_i). \end{aligned}$$

\square

THEOREM 1892. If $S \in \mathcal{P} \prod_{i \in \text{dom } \mathfrak{Z}} \mathcal{F}(\mathfrak{Z}_i)$ where \mathfrak{Z} is an indexed family of sets, then

$$\prod_{a \in S} \prod^{\text{RLD}} a = \prod_{i \in \text{dom } \mathfrak{Z}} \prod^{\mathcal{F}(\mathfrak{Z}_i)} \text{Pr}_i S.$$

PROOF. If $S = \emptyset$ then $\prod_{a \in S} \prod^{\text{RLD}} a = \prod \emptyset = \top^{\text{RLD}(\mathfrak{Z})}$ and

$$\prod_{i \in \text{dom } \mathfrak{Z}} \prod^{\mathcal{F}(\mathfrak{Z}_i)} \text{Pr}_i S = \prod_{i \in \text{dom } \mathfrak{Z}} \prod^{\mathcal{F}(\mathfrak{Z}_i)} \emptyset = \prod_{i \in \text{dom } \mathfrak{Z}} \top^{\mathcal{F}(\mathfrak{Z}_i)} = \top^{\text{RLD}(\mathfrak{Z})},$$

thus $\prod_{a \in S} \prod^{\text{RLD}} a = \prod_{i \in \text{dom } \mathfrak{z}} \prod^{\mathfrak{z}(i)} \text{Pr}_i S$.

Let $S \neq \emptyset$.

$\prod^{\mathfrak{z}(i)} \text{Pr}_i S \subseteq \prod^{\mathfrak{z}(i)} \{a_i\} = a_i$ for every $a \in S$ because $a_i \in \text{Pr}_i S$. Thus $\prod_{i \in \text{dom } \mathfrak{z}} \prod^{\mathfrak{z}(i)} \text{Pr}_i S \subseteq \prod^{\text{RLD}} a$;

$$\prod_{a \in S} \prod^{\text{RLD}} a \supseteq \prod_{i \in \text{dom } \mathfrak{z}} \prod^{\mathfrak{z}(i)} \text{Pr}_i S.$$

Now suppose $F \in \text{GR} \prod_{i \in \text{dom } \mathfrak{z}} \prod^{\mathfrak{z}(i)} \text{Pr}_i S$. Then there exists $X \in \text{up} \prod_{i \in \text{dom } \mathfrak{z}} \prod^{\mathfrak{z}(i)} \text{Pr}_i S$ such that $F \supseteq \prod X$. It is enough to prove that there exist $a \in S$ such that $F \in \text{GR} \prod^{\text{RLD}} a$. For this it is enough $\prod X \in \text{GR} \prod^{\text{RLD}} a$.

Really, $X_i \in \text{up} \prod^{\mathfrak{z}(i)} \text{Pr}_i S$ thus $X_i \in \text{up} a_i$ for every $A \in S$ because $\text{Pr}_i S \supseteq \{a_i\}$.

Thus $\prod X \in \text{GR} \prod^{\text{RLD}} a$. \square

DEFINITION 1893. I call a multireloid *principal* iff its graph is a principal filter.

DEFINITION 1894. I call a multireloid *convex* iff it is a join of reloidal products.

THEOREM 1895. $\text{StarComp}(a \sqcup b, f) = \text{StarComp}(a, f) \sqcup \text{StarComp}(b, f)$ for multireloids a, b and an indexed family f of reloids with $\text{Src } f_i = (\text{form } a)_i = (\text{form } b)_i$.

PROOF.

$$\begin{aligned} & \text{GR}(\text{StarComp}(a, f) \sqcup \text{StarComp}(b, f)) = \\ & \prod \left\{ \frac{\uparrow^{\text{RLD}(\text{form } a)} \text{StarComp}(A, F)}{A \in \text{GR } a, F \in \prod_{i \in n} \text{GR } f_i} \right\} \sqcup \prod \left\{ \frac{\uparrow^{\text{RLD}(\text{form } b)} \text{StarComp}(B, F)}{B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{RLD}(\text{form } a)} \text{StarComp}(A, F) \sqcup \uparrow^{\text{RLD}(\text{form } b)} \text{StarComp}(B, F)}{A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{RLD}(\text{form } a)} (\text{StarComp}(A, F) \cup \text{StarComp}(B, F))}{A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{RLD}(\text{form } a)} \text{StarComp}(A \cup B, F)}{A \in \text{GR } a, B \in \text{GR } b, F \in \prod_{i \in n} \text{GR } f_i} \right\} = \\ & \prod \left\{ \frac{\uparrow^{\text{RLD}(\text{form } a)} \text{StarComp}(C, F)}{C \in \text{GR}(a \sqcup b), F \in \prod_{i \in n} \text{GR } f_i} \right\} = \\ & \text{GR } \text{StarComp}(a \sqcup b, f). \end{aligned}$$

\square

23.11.1. Starred reloidal product. Tychonoff product of topological spaces inspired me the following definition, which seems possibly useful just like Tychonoff product:

DEFINITION 1896. Let a be an n -indexed (n is an arbitrary index set) family of filters on sets. $\prod^{\text{RLD}^*} a$ (*starred reloidal product*) is the reloid of the form $\prod_{i \in n} \text{Base}(a_i)$ induced by the filter base

$$\left\{ \frac{\prod_{i \in n} \left(\begin{cases} A_i & \text{if } i \in m \\ \text{Base}(a_i) & \text{if } i \in n \setminus m \end{cases} \right)}{m \text{ is a finite subset of } n, A \in \prod(a|_m)} \right\}.$$

OBVIOUS 1897. It is really a filter base.

OBVIOUS 1898. $\prod^{\text{RLD}^*} a \supseteq \prod^{\text{RLD}} a$.

PROPOSITION 1899. $\prod^{\text{RLD}^*} a = \prod^{\text{RLD}} a$ if n is finite.

PROOF. Take $m = n$ to show that $\prod^{\text{RLD}^*} a \subseteq \prod^{\text{RLD}} a$. \square

PROPOSITION 1900. $\prod^{\text{RLD}^*} a = \perp^{\text{RLD}(\lambda i \in n: \text{Base}(a_i))}$ if a_i is the non-proper filter for some $i \in n$.

PROOF. Take $A_i = \perp$ and $m = \{i\}$. Then $\prod_{i \in n} \left(\begin{cases} A_i & \text{if } i \in m \\ \text{Base}(a_i) & \text{if } i \in n \setminus m \end{cases} \right) = \perp$. \square

EXAMPLE 1901. There exists an indexed family a of principal filters such that $\prod^{\text{RLD}^*} a$ is non-principal.

PROOF. Let n be infinite and $\text{Base}(a_i)$ is a set of at least two elements. Let each a_i be a trivial ultrafilter.

Every $\prod_{i \in n} \left(\begin{cases} A_i & \text{if } i \in m \\ \text{Base}(a_i) & \text{if } i \in n \setminus m \end{cases} \right)$ has at least 2^n elements.

There are elements up $\prod^{\text{RLD}} a$ with cardinality 1. They can't be elements of up $\prod^{\text{RLD}^*} a$ because of cardinality issues. \square

COROLLARY 1902. There exists an indexed family a of principal filters such that $\prod^{\text{RLD}^*} a \neq \prod^{\text{RLD}} a$.

PROOF. Because $\prod^{\text{RLD}} a$ is principal. \square

PROPOSITION 1903. $\text{Pr}_k^{\text{RLD}} \prod^{\text{RLD}^*} x = x_k$ for every indexed family x of proper filters.

PROOF. $\text{Pr}_k^{\text{RLD}} \prod^{\text{RLD}^*} x = \langle \text{Pr}_k \rangle^* \text{GR} \prod^{\text{RLD}^*} x = x_k$. \square

THEOREM 1904. $\text{Pr}_i^{\text{RLD}} f \subseteq \mathcal{A}_i$ for all $i \in n$ iff $f \subseteq \prod^{\text{RLD}^*} \mathcal{A}$ (for every reloid f of arity n and n -indexed family \mathcal{A} of filters on sets).

PROOF. $f \subseteq \prod^{\text{RLD}^*} \mathcal{A} \Rightarrow \text{Pr}_i^{\text{RLD}} f \subseteq \text{Pr}_i^{\text{RLD}} \prod^{\text{RLD}^*} \mathcal{A} \subseteq \mathcal{A}_i$.

Let now $\text{Pr}_i^{\text{RLD}} f \subseteq \mathcal{A}_i$.

$f \subseteq \prod \left(\begin{cases} \text{Pr}_i^{\text{RLD}} f & \text{if } i \in m \\ \text{Base}(\text{form } f)_i & \text{if } i \notin m \end{cases} \right)$ for finite $m \subseteq n$, as it can be easily be proved by induction.

It follows $f \subseteq \prod^{\text{RLD}^*} \mathcal{A}$. \square

23.12. Subatomic product of funcoids

DEFINITION 1905. Let f be an indexed family of funcoids. Then $\prod^{(A)} f$ (*subatomic product*) is a funcoid $\prod_{i \in \text{dom } f} \text{Src } f_i \rightarrow \prod_{i \in \text{dom } f} \text{Dst } f_i$ such that for every $a \in \text{atoms}^{\text{RLD}(\lambda i \in \text{dom } f: \text{Src } f_i)}$, $b \in \text{atoms}^{\text{RLD}(\lambda i \in \text{dom } f: \text{Dst } f_i)}$

$$a \left[\prod^{(A)} f \right] b \Leftrightarrow \forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} a [f_i] \text{Pr}_i^{\text{RLD}} b.$$

PROPOSITION 1906. The funcoid $\prod^{(A)} f$ exists.

PROOF. To prove that $\prod^{(A)} f$ exists we need to prove (for every $a \in \text{atoms}^{\text{RLD}(\lambda i \in \text{dom } f : \text{Src } f_i)}$, $b \in \text{atoms}^{\text{RLD}(\lambda i \in \text{dom } f : \text{Dst } f_i)}$)

$$\forall X \in \text{GR } a, Y \in \text{GR } b$$

$$\exists x \in \text{atoms}^{\uparrow \text{RLD}(\lambda i \in \text{dom } f : \text{Src } f_i)} X, y \in \text{atoms}^{\uparrow \text{RLD}(\lambda i \in \text{dom } f : \text{Dst } f_i)} Y : x \left[\prod^{(A)} f \right] y \Rightarrow a \left[\prod^{(A)} f \right] b.$$

Let

$$\forall X \in \text{GR } a, Y \in \text{GR } b$$

$$\exists x \in \text{atoms}^{\uparrow \text{RLD}(\lambda i \in \text{dom } f : \text{Src } f_i)} X, y \in \text{atoms}^{\uparrow \text{RLD}(\lambda i \in \text{dom } f : \text{Dst } f_i)} Y : x \left[\prod^{(A)} f \right] y.$$

Then

$$\forall X \in \text{GR } a, Y \in \text{GR } b$$

$$\exists x \in \text{atoms}^{\uparrow \text{RLD}(\lambda i \in \text{dom } f : \text{Src } f_i)} X, y \in \text{atoms}^{\uparrow \text{RLD}(\lambda i \in \text{dom } f : \text{Dst } f_i)} Y$$

$$\forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} x [f_i] \text{Pr}_i^{\text{RLD}} y.$$

Then because $\text{Pr}_i^{\text{RLD}} x \in \text{atoms}^{\uparrow \text{Src } f_i} \text{Pr}_i X$ and likewise for y :

$$\forall X \in \text{GR } a, Y \in \text{GR } b \forall i \in \text{dom } f$$

$$\exists x \in \text{atoms}^{\uparrow \text{Src } f_i} \text{Pr}_i X, y \in \text{atoms}^{\uparrow \text{Dst } f_i} \text{Pr}_i Y : x [f_i] y.$$

Thus $\forall X \in \text{GR } a, Y \in \text{GR } b \forall i \in \text{dom } f : \uparrow^{\text{Src } f_i} \text{Pr}_i X [f_i] \uparrow^{\text{Dst } f_i} \text{Pr}_i Y$;

$$\forall X \in \text{GR } a, Y \in \text{GR } b \forall i \in \text{dom } f : \text{Pr}_i X [f_i]^* \text{Pr}_i Y.$$

$$\text{Then } \forall X \in \langle \text{Pr}_i \rangle^* \text{GR } a, Y \in \langle \text{Pr}_i \rangle^* \text{GR } b : X [f_i]^* Y.$$

$$\text{Thus } \text{Pr}_i^{\text{RLD}} a [f_i] \text{Pr}_i^{\text{RLD}} b. \text{ So}$$

$$\forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} a [f_i] \text{Pr}_i^{\text{RLD}} b$$

and thus $a \left[\prod^{(A)} f \right] b.$ □

REMARK 1907. It seems that the proof of the above theorem can be simplified using cross-composition product.

THEOREM 1908. $\prod_{i \in n}^{(A)} (g_i \circ f_i) = \prod^{(A)} g \circ \prod^{(A)} f$ for indexed (by an index set n) families f and g of funcoids such that $\forall i \in n : \text{Dst } f_i = \text{Src } g_i$.

PROOF. Let a, b be ultrafilters on $\prod_{i \in n} \text{Src } f_i$ and $\prod_{i \in n} \text{Dst } g_i$ correspondingly,

$$\begin{aligned}
& a \left[\prod_{i \in n}^{(A)} (g_i \circ f_i) \right] b \Leftrightarrow \\
& \forall i \in \text{dom } f : \langle \text{Pr}_i \rangle^* a [g_i \circ f_i] \langle \text{Pr}_i \rangle^* b \Leftrightarrow \\
& \forall i \in \text{dom } f \exists C \in \text{atoms}^{\mathcal{F}(\text{Dst } f_i)} : \left(\langle \text{Pr}_i \rangle^* a [f_i] C \wedge C [g_i] \langle \text{Pr}_i \rangle^* b \right) \Leftrightarrow \\
& \forall i \in \text{dom } f \exists c \in \text{atoms}^{\text{RLD}(\lambda i \in n : \text{Dst } f)} : \left(\langle \text{Pr}_i \rangle^* a [f_i] \langle \text{Pr}_i \rangle^* c \wedge \langle \text{Pr}_i \rangle^* c [g_i] \langle \text{Pr}_i \rangle^* b \right) \Leftrightarrow \\
& \exists c \in \text{atoms}^{\text{RLD}(\lambda i \in n : \text{Dst } f)} \forall i \in \text{dom } f : \left(\langle \text{Pr}_i \rangle^* a [f_i] \langle \text{Pr}_i \rangle^* c \wedge \langle \text{Pr}_i \rangle^* c [g_i] \langle \text{Pr}_i \rangle^* b \right) \Leftrightarrow \\
& \exists c \in \text{atoms}^{\text{RLD}(\lambda i \in n : \text{Dst } f)} : \left(a \left[\prod_{i \in n}^{(A)} f \right] c \wedge c \left[\prod_{i \in n}^{(A)} g \right] b \right) \Leftrightarrow \\
& a \left[\prod_{i \in n}^{(A)} g \circ \prod_{i \in n}^{(A)} f \right] b.
\end{aligned}$$

But

$$\forall i \in \text{dom } f \exists C \in \text{atoms}^{\mathcal{F}(\text{Dst } f_i)} : \left(\langle \text{Pr}_i \rangle^* a [f_i] C \wedge C [g_i] \langle \text{Pr}_i \rangle^* b \right)$$

implies

$$\exists C \in \prod_{i \in n} \text{atoms}^{\mathcal{F}(\text{Dst } f_i)} \forall i \in \text{dom } f : \left(\langle \text{Pr}_i \rangle^* a [f_i] C_i \wedge C_i [g_i] \langle \text{Pr}_i \rangle^* b \right).$$

Take $c \in \text{atoms} \prod^{\text{RLD}} C$. Then

$$\forall i \in \text{dom } f : \left(\langle \text{Pr}_i \rangle^* a [f_i] \text{Pr}_i c \wedge \text{Pr}_i c [g_i] \langle \text{Pr}_i \rangle^* b \right)$$

that is

$$\forall i \in \text{dom } f : \left(\langle \text{Pr}_i \rangle^* a [f_i] \langle \text{Pr}_i \rangle^* c \wedge \langle \text{Pr}_i \rangle^* c [g_i] \langle \text{Pr}_i \rangle^* b \right)$$

$$\text{We have } a \left[\prod_{i \in n}^{(A)} (g_i \circ f_i) \right] b \Leftrightarrow a \left[\prod^{(A)} g \circ \prod^{(A)} f \right] b. \quad \square$$

COROLLARY 1909. $\left(\prod^{(A)} f_{k-1} \right) \circ \dots \circ \left(\prod^{(A)} f_0 \right) = \prod_{i \in n}^{(A)} (f_{k-1} \circ \dots \circ f_0)$ for every n -indexed families f_0, \dots, f_{n-1} of composable functors.

PROPOSITION 1910. $\prod^{\text{RLD}} a \left[\prod^{(A)} f \right] \prod^{\text{RLD}} b \Leftrightarrow \forall i \in \text{dom } f : a_i [f_i] b_i$ for an indexed family f of functors and indexed families a and b of filters where $a_i \in \mathcal{F}(\text{Src } f_i)$, $b_i \in \mathcal{F}(\text{Dst } f_i)$ for every $i \in \text{dom } f$.

PROOF. If $a_i = \perp$ or $b_i = \perp$ for some i our theorem is obvious. We will take $a_i \neq \perp$ and $b_i \neq \perp$, thus there exist

$$x \in \text{atoms} \prod^{\text{RLD}} a, \quad y \in \text{atoms} \prod^{\text{RLD}} b.$$

$$\begin{aligned}
& \prod_{\text{RLD}} a \left[\prod_{(A)} f \right] \prod_{\text{RLD}} b \Leftrightarrow \\
& \exists x \in \text{atoms} \prod_{\text{RLD}} a, y \in \text{atoms} \prod_{\text{RLD}} b : x \left[\prod_{(A)} f \right] y \Leftrightarrow \\
& \exists x \in \text{atoms} \prod_{\text{RLD}} a, y \in \text{atoms} \prod_{\text{RLD}} b \forall i \in \text{dom } f : \langle \text{Pr}_i \rangle^* x [f_i] \langle \text{Pr}_i \rangle^* y \Leftrightarrow \\
& \forall i \in \text{dom } f \exists x \in \text{atoms } a_i, y \in \text{atoms } b_i : x [f_i] y \Leftrightarrow \\
& \forall i \in \text{dom } f : a_i [f_i] b_i.
\end{aligned}$$

□

THEOREM 1911. $\langle \prod_{(A)} f \rangle x = \prod_{i \in \text{dom } f}^{\text{RLD}} \langle f_i \rangle \text{Pr}_i^{\text{RLD}} x$ for an indexed family f of funcoids and $x \in \text{atoms}^{\text{RLD}(\lambda i \in \text{dom } f : \text{Src } f_i)}$ for every $n \in \text{dom } f$.

PROOF. For every ultrafilter $y \in \mathcal{F} \left(\prod_{i \in \text{dom } f} \text{Dst } f_i \right)$ we have:

$$\begin{aligned}
& y \not\prec \prod_{i \in \text{dom } f}^{\text{RLD}} \langle f_i \rangle \text{Pr}_i^{\text{RLD}} x \Leftrightarrow \\
& \forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} y \not\prec \langle f_i \rangle \text{Pr}_i^{\text{RLD}} x \Leftrightarrow \\
& \forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} x [f_i] \text{Pr}_i^{\text{RLD}} y \Leftrightarrow \\
& x \left[\prod_{(A)} f \right] y \Leftrightarrow \\
& y \not\prec \left\langle \prod_{(A)} f \right\rangle x.
\end{aligned}$$

Thus $\langle \prod_{(A)} f \rangle x = \prod_{i \in \text{dom } f}^{\text{RLD}} \langle f_i \rangle \text{Pr}_i^{\text{RLD}} x$. □

COROLLARY 1912. $\langle f \times^{(A)} g \rangle x = \langle f \rangle (\text{dom } x) \times^{\text{RLD}} \langle g \rangle (\text{im } x)$ for atomic x .

23.13. On products and projections

CONJECTURE 1913. For principal funcoids $\prod^{(C)}$ and $\prod^{(A)}$ coincide with the conventional product of binary relations.

23.13.1. Staroidal product. Let f be a staroid, whose form components are boolean lattices.

DEFINITION 1914. *Staroidal projection* of a staroid f is the filter $\text{Pr}_k^{\text{Strd}} f$ corresponding to the free star

$$(\text{val } f)_k (\lambda i \in (\text{arity } f) \setminus \{k\} : \top^{(\text{form } f)_i}).$$

PROPOSITION 1915. $\text{Pr}_k \text{GR} \prod^{\text{Strd}} x = \star x_k$ if x is an indexed family of proper filters, and $k \in \text{dom } x$.

PROOF.

$$\begin{aligned} & \Pr_k \text{GR} \prod_k^{\text{Strd}} x = \\ & \Pr_k \left\{ \frac{L \in \text{form } x}{\forall i \in \text{dom } x : x_i \neq L_i} \right\} = \\ & \text{(used the fact that } x_i \text{ are proper filters)} \\ & \left\{ \frac{l \in (\text{form } x)_k}{x_k \neq l} \right\} = \star x_k. \end{aligned}$$

□

PROPOSITION 1916. $\Pr_k^{\text{Strd}} \prod^{\text{Strd}} x = x_k$ if x is an indexed family of proper filters, and $k \in \text{dom } x$.

PROOF.

$$\begin{aligned} & \partial \Pr_k^{\text{Strd}} \prod^{\text{Strd}} x = \\ & \left(\text{val} \prod_k^{\text{Strd}} x \right) (\lambda i \in (\text{dom } x) \setminus \{k\} : \top^{(\text{form } x)_i}) = \\ & \left\{ \frac{X \in (\text{form } \prod_k^{\text{Strd}} x)_k}{(\lambda i \in (\text{dom } x) \setminus \{k\} : \top^{(\text{form } x)_i}) \cup \{(k, X)\} \in \text{GR} \prod^{\text{Strd}} x} \right\} = \\ & \left\{ \frac{X \in \text{Base } x_k}{(\forall i \in (\text{dom } x) \setminus \{k\} : \top^{(\text{form } x)_i} \neq x_i) \wedge X \neq x_k} \right\} = \\ & \left\{ \frac{X \in \text{Base } x_k}{X \neq x_k} \right\} = \partial x_k. \end{aligned}$$

Consequently $\Pr_k^{\text{Strd}} \prod^{\text{Strd}} x = x_k$.

□

23.13.2. Cross-composition product of pointfree functors.

DEFINITION 1917. Zero pointfree functor $\perp^{\text{pFCD}(\mathfrak{A}, \mathfrak{B})}$ from a poset \mathfrak{A} to a poset \mathfrak{B} is the least pointfree functor in the set $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROPOSITION 1918. A pointfree functor f is zero iff $[f] = \emptyset$.

PROOF. Direct implication is obvious.

Let now $[f] = \emptyset$. Then $\langle f \rangle x \asymp y$ for every $x \in \text{Src } f$, $y \in \text{Dst } f$ and thus $\langle f \rangle x \asymp \langle f \rangle x$. It is possible only when $\langle f \rangle x = \perp^{\text{Dst } f}$. □

COROLLARY 1919. A pointfree functor is zero iff its reverse is zero.

PROPOSITION 1920. Values x_i (for every $i \in \text{dom } x$) can be restored from the value of $\prod^{(C)} x$ provided that x is an indexed family of non-zero pointfree functors, $\text{Src } f_i$ (for every $i \in n$) is an atomic lattice and every $\text{Dst } f_i$ is an atomic poset with greatest element.

PROOF. $\langle \prod^{(C)} x \rangle \prod^{\text{Strd}} p = \prod_{i \in n}^{\text{Strd}} \langle x_i \rangle p_i$ by theorem 1872.

Since x_i is non-zero there exist p such that $\langle x_i \rangle p_i$ is non-least. Take $k \in n$, $p'_i = p_i$ for $i \neq k$ and $p'_k = q$ for an arbitrary value q ; then (using the staroidal projections from the previous subsection)

$$\langle x_k \rangle q = \Pr_k^{\text{Strd}} \prod_{i \in n}^{\text{Strd}} \langle x_i \rangle p'_i = \Pr_k^{\text{Strd}} \left\langle \prod^{(C)} x \right\rangle \prod^{\text{Strd}} p'.$$

So the value of x can be restored from $\prod^{(C)} x$ by this formula. \square

23.13.3. Subatomic product.

PROPOSITION 1921. Values x_i (for every $i \in \text{dom } x$) can be restored from the value of $\prod^{(A)} x$ provided that x is an indexed family of non-zero functors.

PROOF. Fix $k \in \text{dom } f$. Let for some filters \mathcal{X} and \mathcal{Y}

$$a = \begin{cases} \top^{\mathcal{F}(\text{Base}(x))} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{cases} \quad \text{and} \quad b = \begin{cases} \top^{\mathcal{F}(\text{Base}(y))} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k. \end{cases}$$

Then $\mathcal{X} [x_k] \mathcal{Y} \Leftrightarrow a_k [x_k] b_k \Leftrightarrow \forall i \in \text{dom } f : a_i [x_i] b_i \Leftrightarrow \prod^{\text{RLD}} a \left[\prod^{(A)} x \right] \prod^{\text{RLD}} b$.
So we have restored x_k from $\prod^{(A)} x$. \square

DEFINITION 1922. For every functor $f : \prod A \rightarrow \prod B$ (where A and B are indexed families of typed sets) consider the functor $\text{Pr}_k^{(A)} f$ defined by the formula

$$X \left[\text{Pr}_k^{(A)} f \right]^* Y \Leftrightarrow \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} X & \text{if } i = k \end{cases} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases} \right).$$

PROPOSITION 1923. $\text{Pr}_k^{(A)} f$ is really a functor.

PROOF. $\neg(\perp \left[\text{Pr}_k^{(A)} f \right]^* Y)$ is obvious.

$$\begin{aligned} I \sqcup J \left[\text{Pr}_k^{(A)} f \right]^* Y &\Leftrightarrow \\ \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} (I \sqcup J) & \text{if } i = k \end{cases} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases} \right) &\Leftrightarrow \\ \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} I \sqcup \uparrow^{A_i} J & \text{if } i = k \end{cases} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases} \right) &\Leftrightarrow \\ \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} I & \text{if } i = k \end{cases} \right) \sqcup \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} J & \text{if } i = k \end{cases} \right) [f] &\Leftrightarrow \\ \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases} \right) &\Leftrightarrow \\ \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} I & \text{if } i = k \end{cases} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases} \right) \vee &\Leftrightarrow \\ \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} J & \text{if } i = k \end{cases} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases} \right) &\Leftrightarrow \\ I \left[\text{Pr}_k^{(A)} f \right]^* Y \vee J \left[\text{Pr}_k^{(A)} f \right]^* Y. & \end{aligned}$$

The rest follows from symmetry. \square

PROPOSITION 1924. For every functor $f : \prod A \rightarrow \prod B$ (where A and B are indexed families of typed sets) the functor $\text{Pr}_k^{(A)} f$ conforms to the formula

$$\mathcal{X} \left[\text{Pr}_k^{(A)} f \right]^* \mathcal{Y} \Leftrightarrow \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{cases} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\begin{cases} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k \end{cases} \right).$$

PROOF.

$$\begin{aligned}
& \mathcal{X} \left[\begin{array}{c} (A) \\ \text{Pr } f \\ k \end{array} \right] \mathcal{Y} \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X \left[\begin{array}{c} (A) \\ \text{Pr } f \\ k \end{array} \right]^* Y \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : \\
& \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\left\{ \begin{array}{ll} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} X & \text{if } i = k \end{array} \right\} [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\left\{ \begin{array}{ll} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{array} \right\} \right) \Leftrightarrow \\
& \forall X \in \text{up } \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\left\{ \begin{array}{ll} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{array} \right\} \right), Y \in \text{up } \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\left\{ \begin{array}{ll} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k \end{array} \right\} \right) : \\
& X [f]^* Y \Leftrightarrow \\
& \prod_{i \in \text{dom } A}^{\text{RLD}} \left(\left\{ \begin{array}{ll} \top^{\mathcal{F}(A_i)} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{array} \right\} \right) [f] \prod_{i \in \text{dom } B}^{\text{RLD}} \left(\left\{ \begin{array}{ll} \top^{\mathcal{F}(B_i)} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k \end{array} \right\} \right).
\end{aligned}$$

□

REMARK 1925. Reloidal product above can be replaced with starred reloidal product, because of finite number of non-maximal multipliers in the products.

OBVIOUS 1926. $\text{Pr}_k^{(A)} \prod^{(A)} x = x_k$ provided that x is an indexed family of non-zero functors.

23.13.4. Other.

DEFINITION 1927. *Displaced product* $\prod^{(DP)} f = \Downarrow \prod^{(C)} f$ for every indexed family of pointfree functors, where downgrading is defined for the filtrator

$$\left(\text{FCD}(\text{StarHom}(\text{Src } \circ f), \text{StarHom}(\text{Dst } \circ f)), \mathbf{Rel} \left(\prod(\text{Src } \circ f), \prod(\text{Dst } \circ f) \right) \right).$$

REMARK 1928. Displaced product is a functor (not just a pointfree functor).

CONJECTURE 1929. Values x_i (for every $i \in \text{dom } x$) can be restored from the value of $\prod^{(DP)} x$ provided that x is an indexed family of non-zero functors.

DEFINITION 1930. Let $f \in \mathcal{S} \left(Z \prod^Y \right)$ where Z is a set and Y is a function.

$$\text{Pr}_k^{(D)} f = \text{Pr}_k \left\{ \frac{\text{curry } z}{z \in f} \right\}.$$

PROPOSITION 1931. $\text{Pr}_k^{(D)} \prod^{(D)} F = F_k$ for every indexed family F of non-empty relations.

PROOF. Obvious. □

COROLLARY 1932. $\text{GR Pr}_k^{(D)} \prod^{(D)} F = \text{GR } F_k$ and form $\text{Pr}_k^{(D)} \prod^{(D)} F = \text{form } F_k$ for every indexed family F of non-empty anchored relations.

23.14. Relationships between cross-composition and subatomic products

PROPOSITION 1933. $a [f \times^{(C)} g] b \Leftrightarrow \text{dom } a [f] \text{ dom } b \wedge \text{im } a [g] \text{ im } b$ for functors f and g and atomic functors $a \in \text{FCD}(\text{Src } f, \text{Src } g)$ and $b \in \text{FCD}(\text{Dst } f, \text{Dst } g)$.

PROOF.

$$\begin{aligned}
& a \left[f \times^{(C)} g \right] b \Leftrightarrow \\
& a \circ f^{-1} \not\neq g^{-1} \circ b \Leftrightarrow \\
& (\text{dom } a \times^{\text{FCD}} \text{im } a) \circ f^{-1} \not\neq g^{-1} \circ (\text{dom } b \times^{\text{FCD}} \text{im } b) \Leftrightarrow \\
& \langle f \rangle \text{dom } a \times^{\text{FCD}} \text{im } a \not\neq \text{dom } b \times^{\text{FCD}} \langle g^{-1} \rangle \text{im } b \Leftrightarrow \\
& \langle f \rangle \text{dom } a \not\neq \text{dom } b \wedge \text{im } a \not\neq \langle g^{-1} \rangle \text{im } b \Leftrightarrow \\
& \text{dom } a [f] \text{dom } b \wedge \text{im } a [g] \text{im } b.
\end{aligned}$$

□

PROPOSITION 1934. $\mathcal{X} \left[\prod^{(A)} f \right] \mathcal{Y} \Leftrightarrow \forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} \mathcal{X} [f_i] \text{Pr}_i^{\text{RLD}} \mathcal{Y}$ for every indexed family f of funcoids and $\mathcal{X} \in \text{RLD}(\text{Src } \circ f)$, $\mathcal{Y} \in \text{RLD}(\text{Dst } \circ f)$.

PROOF.

$$\begin{aligned}
& \mathcal{X} \left[\prod^{(A)} f \right] \mathcal{Y} \Leftrightarrow \\
& \exists a \in \text{atoms } \mathcal{X}, b \in \text{atoms } \mathcal{Y} : a \left[\prod^{(A)} f \right] b \Leftrightarrow \\
& \exists a \in \text{atoms } \mathcal{X}, b \in \text{atoms } \mathcal{Y} \forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} a [f_i] \text{Pr}_i^{\text{RLD}} b \Leftrightarrow \\
& \forall i \in \text{dom } f \exists x \in \text{atoms } \text{Pr}_i^{\text{RLD}} \mathcal{X}, y \in \text{atoms } \text{Pr}_i^{\text{RLD}} \mathcal{Y} : x_i [f_i] y_i \Leftrightarrow \\
& \forall i \in \text{dom } f : \text{Pr}_i^{\text{RLD}} \mathcal{X} [f_i] \text{Pr}_i^{\text{RLD}} \mathcal{Y}.
\end{aligned}$$

□

COROLLARY 1935. $\mathcal{X} [f \times^{(A)} g] \mathcal{Y} \Leftrightarrow \text{dom } \mathcal{X} [f] \text{dom } \mathcal{Y} \wedge \text{im } \mathcal{X} [g] \text{im } \mathcal{Y}$ for funcoids f, g and reloids $\mathcal{X} \in \text{RLD}(\text{Src } f, \text{Src } g)$, and $\mathcal{Y} \in \text{RLD}(\text{Dst } f, \text{Dst } g)$.

LEMMA 1936. For every $A \in \mathbf{Rel}(X, Y)$ (for every sets X, Y) we have:

$$\left\{ \frac{(\text{dom } a, \text{im } a)}{a \in \text{atoms } \uparrow^{\text{FCD}} A} \right\} = \left\{ \frac{(\text{dom } a, \text{im } a)}{a \in \text{atoms } \uparrow^{\text{RLD}} A} \right\}.$$

PROOF. Let $x \in \left\{ \frac{(\text{dom } a, \text{im } a)}{a \in \text{atoms } \uparrow^{\text{RLD}} A} \right\}$. Take $x_0 = \text{dom } a$ and $x_1 = \text{im } a$ where $a \in \text{atoms } \uparrow^{\text{RLD}} A$.

Then $x_0 = \text{dom}(\text{FCD})a$ and $x_1 = \text{im}(\text{FCD})a$ and obviously $(\text{FCD})a \in \text{atoms } \uparrow^{\text{FCD}} A$. So $x \in \left\{ \frac{(\text{dom } a, \text{im } a)}{a \in \text{atoms } \uparrow^{\text{FCD}} A} \right\}$.

Let now $x \in \left\{ \frac{(\text{dom } a, \text{im } a)}{a \in \text{atoms } \uparrow^{\text{FCD}} A} \right\}$. Take $x_0 = \text{dom } a$ and $x_1 = \text{im } a$ where $a \in \text{atoms } \uparrow^{\text{FCD}} A$.

$x_0 [\uparrow^{\text{FCD}} A] x_1 \Leftrightarrow x_0 [(\text{FCD}) \uparrow^{\text{RLD}} A] x_1 \Leftrightarrow x_0 \times^{\text{RLD}} x_1 \not\neq \uparrow^{\text{RLD}} A$. Thus there exists atomic reloid x' such that $x' \in \text{atoms } \uparrow^{\text{RLD}} A$ and $\text{dom } x' = x_0$, $\text{im } x' = x_1$.

$$\text{So } x \in \left\{ \frac{(\text{dom } a', \text{im } a')}{a' \in \text{atoms } \uparrow^{\text{RLD}} A} \right\}. \quad \square$$

THEOREM 1937. $\uparrow^{\text{FCD}} A [f \times^{(C)} g] \uparrow^{\text{FCD}} B \Leftrightarrow \uparrow^{\text{RLD}} A [f \times^{(A)} g] \uparrow^{\text{RLD}} B$ for funcoids f, g , and **Rld**-morphisms $A : \text{Src } f \rightarrow \text{Src } g$, and $B : \text{Dst } f \rightarrow \text{Dst } g$.

PROOF.

$$\begin{aligned} \uparrow^{\text{FCD}} A \left[f \times^{(C)} g \right] \uparrow^{\text{FCD}} B &\Leftrightarrow \\ \exists a \in \text{atoms } \uparrow^{\text{FCD}} A, b \in \text{atoms } \uparrow^{\text{FCD}} B : a \left[f \times^{(C)} g \right] b &\Leftrightarrow \\ \exists a \in \text{atoms } \uparrow^{\text{FCD}} A, b \in \text{atoms } \uparrow^{\text{FCD}} B : (\text{dom } a [f] \text{ dom } b \wedge \text{im } a [g] \text{ im } b) &\Rightarrow \\ \exists a_0 \in \text{atoms dom } \uparrow^{\text{FCD}} A, a_1 \in \text{atoms im } \uparrow^{\text{FCD}} A, & \\ b_0 \in \text{atoms dom } \uparrow^{\text{FCD}} B, b_1 \in \text{atoms im } \uparrow^{\text{FCD}} B : (a_0 [f] b_0 \wedge a_1 [g] b_1). & \end{aligned}$$

On the other hand:

$$\begin{aligned} \exists a_0 \in \text{atoms dom } \uparrow^{\text{FCD}} A, a_1 \in \text{atoms im } \uparrow^{\text{FCD}} A, & \\ b_0 \in \text{atoms dom } \uparrow^{\text{FCD}} B, b_1 \in \text{atoms im } \uparrow^{\text{FCD}} B : (a_0 [f] b_0 \wedge a_1 [g] b_1) &\Rightarrow \\ \exists a_0 \in \text{atoms dom } \uparrow^{\text{FCD}} A, a_1 \in \text{atoms im } \uparrow^{\text{FCD}} A, & \\ b_0 \in \text{atoms dom } \uparrow^{\text{FCD}} B, b_1 \in \text{atoms im } \uparrow^{\text{FCD}} B : (a_0 \times^{\text{FCD}} b_0 \not\neq f \wedge a_1 \times^{\text{FCD}} b_1 \not\neq g) &\Rightarrow \\ \exists a \in \text{atoms } \uparrow^{\text{FCD}} A, b \in \text{atoms } \uparrow^{\text{FCD}} B : (\text{dom } a [f] \text{ dom } b \wedge \text{im } a [g] \text{ im } b). & \end{aligned}$$

Also using the lemma we have

$$\begin{aligned} \exists a \in \text{atoms } \uparrow^{\text{FCD}} A, b \in \text{atoms } \uparrow^{\text{FCD}} B : (\text{dom } a [f] \text{ dom } b \wedge \text{im } a [g] \text{ im } b) &\Leftrightarrow \\ \exists a \in \text{atoms } \uparrow^{\text{RLD}} A, b \in \text{atoms } \uparrow^{\text{RLD}} B : (\text{dom } a [f] \text{ dom } b \wedge \text{im } a [g] \text{ im } b). & \end{aligned}$$

So

$$\begin{aligned} \uparrow^{\text{FCD}} A \left[f \times^{(C)} g \right] \uparrow^{\text{FCD}} B &\Leftrightarrow \\ \exists a \in \text{atoms } \uparrow^{\text{RLD}} A, b \in \text{atoms } \uparrow^{\text{RLD}} B : (\text{dom } a [f] \text{ dom } b \wedge \text{im } a [g] \text{ im } b) &\Leftrightarrow \\ \exists a \in \text{atoms } \uparrow^{\text{RLD}} A, b \in \text{atoms } \uparrow^{\text{RLD}} B : a \left[f \times^{(A)} g \right] b &\Leftrightarrow \\ \uparrow^{\text{RLD}} A \left[f \times^{(A)} g \right] \uparrow^{\text{RLD}} B. & \end{aligned}$$

□

COROLLARY 1938. $f \times^{(A)} g = \uparrow\downarrow (f \times^{(C)} g)$ where downgrading is taken on the filtrator

$$\left(\text{pFCD}(\text{FCD}(\text{Src } \circ f), \text{FCD}(\text{Dst } \circ f)), \text{FCD} \left(\mathscr{P} \prod (\text{Src } \circ f), \mathscr{P} \prod (\text{Dst } \circ f) \right) \right)$$

and upgrading is taken on the filtrator

$$\left(\text{pFCD}(\text{RLD}(\text{Src } \circ f), \text{RLD}(\text{Dst } \circ f)), \text{FCD} \left(\mathscr{P} \prod (\text{Src } \circ f), \mathscr{P} \prod (\text{Dst } \circ f) \right) \right).$$

where we equate n -ary relations with corresponding principal multifuncoids and principal multireloids, when appropriate.

PROOF. Leave as an exercise for the reader. □

CONJECTURE 1939. $\uparrow^{\text{FCD}} A \left[\prod^{(C)} f \right] \uparrow^{\text{FCD}} B \Leftrightarrow \uparrow^{\text{RLD}} A \left[\prod^{(A)} f \right] \uparrow^{\text{RLD}} B$ for every indexed family f of functors and $A \in \mathscr{P} \prod_{i \in \text{dom } f} \text{Src } f_i$, $B \in \mathscr{P} \prod_{i \in \text{dom } f} \text{Dst } f_i$.

THEOREM 1940. For every filters a_0, a_1, b_0, b_1 we have

$$a_0 \times^{\text{FCD}} b_0 \left[f \times^{(C)} g \right] a_1 \times^{\text{FCD}} b_1 \Leftrightarrow a_0 \times^{\text{RLD}} b_0 \left[f \times^{(A)} g \right] a_1 \times^{\text{RLD}} b_1.$$

PROOF.

$$a_0 \times^{\text{RLD}} b_0 \left[f \times^{(A)} g \right] a_1 \times^{\text{RLD}} b_1 \Leftrightarrow$$

$$\forall A_0 \in a_0, B_0 \in b_0, A_1 \in a_1, B_1 \in b_1 : A_0 \times B_0 \left[f \times^{(A)} g \right]^* A_1 \times B_1.$$

$$A_0 \times B_0 \left[f \times^{(A)} g \right]^* A_1 \times B_1 \Leftrightarrow A_0 \times B_0 \left[f \times^{(C)} g \right]^* A_1 \times B_1 \Leftrightarrow A_0 [f]^* A_1 \wedge B_0 [g]^* B_1.$$

(Here by $A_0 \times B_0 \left[f \times^{(C)} g \right]^* A_1 \times B_1$ I mean $\uparrow^{\text{FCD}(\text{Base } a, \text{Base } b)} (A_0 \times B_0) \left[f \times^{(C)} g \right]^* \uparrow^{\text{FCD}(\text{Base } a, \text{Base } b)} (A_1 \times B_1)$.)

Thus it is equivalent to $a_0 [f] a_1 \wedge b_0 [g] b_1$ that is $a_0 \times^{\text{FCD}} b_0 \left[f \times^{(C)} g \right]^* a_1 \times^{\text{FCD}} b_1$.

(It was used the corollary 1749.) \square

Can the above theorem be generalized for the infinitary case?

23.15. Cross-inner and cross-outer product

Let f be an indexed family of funcoids.

DEFINITION 1941. $\prod_{i \in \text{dom } f}^{\text{in}} f = \prod_{i \in \text{dom } f}^{(C)} (\text{RLD})_{\text{in}} f_i$ (*cross-inner product*).

DEFINITION 1942. $\prod_{i \in \text{dom } f}^{\text{out}} f = \prod_{i \in \text{dom } f}^{(C)} (\text{RLD})_{\text{out}} f_i$ (*cross-outer product*).

PROPOSITION 1943. $\prod_{i \in \text{dom } f}^{\text{in}} f$ and $\prod_{i \in \text{dom } f}^{\text{out}} f$ are funcoids (not just pointfree funcoids).

PROOF. They are both morphisms $\text{StarHom}(\lambda i \in \text{dom } f : \text{Src } f_i) \rightarrow \text{StarHom}(\lambda i \in \text{dom } f : \text{Src } f_i)$ for the category of multireloids with star-morphisms, that is $\text{StarHom}(\lambda i \in \text{dom } f : \text{Src } f_i)$ is the set of filters on the cartesian product $\prod_{i \in \text{dom } f} \text{Src } f_i$ and likewise for $\text{StarHom}(\lambda i \in \text{dom } f : \text{Src } f_i)$. \square

OBVIOUS 1944. For every funcoids f and g

- 1°. $f \times^{\text{in}} g = (\text{RLD})_{\text{in}} f \times^{(C)} (\text{RLD})_{\text{in}} g$;
- 2°. $f \times^{\text{out}} g = (\text{RLD})_{\text{out}} f \times^{(C)} (\text{RLD})_{\text{out}} g$.

COROLLARY 1945.

- 1°. $\langle f \times^{\text{in}} g \rangle a = (\text{RLD})_{\text{in}} g \circ a \circ (\text{RLD})_{\text{in}} f^{-1}$;
- 2°. $\langle f \times^{\text{out}} g \rangle a = (\text{RLD})_{\text{out}} g \circ a \circ (\text{RLD})_{\text{out}} f^{-1}$

COROLLARY 1946. For every funcoids f and g and filters a and b on suitable sets:

- 1°. $a [f \times^{\text{in}} g] b \Leftrightarrow b \not\prec (\text{RLD})_{\text{in}} g \circ a \circ (\text{RLD})_{\text{in}} f^{-1} \Leftrightarrow b \circ (\text{RLD})_{\text{in}} f \not\prec (\text{RLD})_{\text{in}} g \circ a$;
- 2°. $a [f \times^{\text{out}} g] b \Leftrightarrow b \not\prec (\text{RLD})_{\text{out}} g \circ a \circ (\text{RLD})_{\text{out}} f^{-1} \Leftrightarrow b \circ (\text{RLD})_{\text{out}} f \not\prec (\text{RLD})_{\text{out}} g \circ a$.

PROPOSITION 1947. Knowing that every f_i is nonzero, we can restore the values of f_i from the value of $\prod_{i \in \text{dom } f}^{\text{in}} f$.

PROOF. It follows that every $(\text{RLD})_{\text{in}} f_i$ is nonzero, thus we can restore each $(\text{RLD})_{\text{in}} f_i$ from $\prod_{i \in \text{dom } f}^{(C)} (\text{RLD})_{\text{in}} f_i = \prod_{i \in \text{dom } f}^{\text{in}} f$ and then we know $f_i = (\text{FCD})(\text{RLD})_{\text{in}} f_i$. \square

EXAMPLE 1948. The values of f and g cannot be restored from $f \times^{\text{out}} g$ for some nonzero funcoids f and g .

PROOF. Obviously $\text{id}_{\Omega(\mathbb{N})}^{\text{FCD}} \neq \text{id}_{\Omega(\mathbb{R})}^{\text{FCD}}$, but $\text{id}_{\Omega(\mathbb{N})}^{\text{FCD}} \times^{\text{out}} \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}} = (\text{RLD})_{\text{out}} \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}} \times^{(C)} (\text{RLD})_{\text{out}} \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}} = \perp \times^{(C)} \perp = (\text{RLD})_{\text{out}} \text{id}_{\Omega(\mathbb{R})}^{\text{FCD}} \times^{(C)} (\text{RLD})_{\text{out}} \text{id}_{\Omega(\mathbb{R})}^{\text{FCD}} = \text{id}_{\Omega(\mathbb{R})}^{\text{FCD}} \times^{\text{out}} \text{id}_{\Omega(\mathbb{R})}^{\text{FCD}}$.

That is the product $f \times^{\text{out}} g$ is the same if we take $f = g = \text{id}_{\Omega(\mathbb{N})}^{\text{FCD}}$ and if we take $f = g = \text{id}_{\Omega(\mathbb{R})}^{\text{FCD}}$. \square

QUESTION 1949. Which of the following are pairwise equal (for a. two functors, b. any (possibly infinite) number of functors)?

- 1°. subatomic product;
- 2°. displaced product;
- 3°. cross-inner product.

23.16. Coordinate-wise continuity

THEOREM 1950. Let μ and ν be indexed (by some index set n) families of endomorphisms for a quasi-invertible dagger category with star-morphisms, and $f_i \in \text{Hom}(\text{Ob } \mu_i, \text{Ob } \nu_i)$ for every $i \in n$. Then:

- 1°. $\forall i \in n : f_i \in C(\mu_i, \nu_i) \Rightarrow \prod^{(C)} f \in C\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right)$;
- 2°. $\forall i \in n : f_i \in C'(\mu_i, \nu_i) \Rightarrow \prod^{(C)} f \in C'\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right)$;
- 3°. $\forall i \in n : f_i \in C''(\mu_i, \nu_i) \Rightarrow \prod^{(C)} f \in C''\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right)$.

PROOF. Using the corollary 1850:

$$\begin{aligned} \forall i \in n : f_i \in C(\mu_i, \nu_i) &\Leftrightarrow \forall i \in n : f_i \circ \mu_i \sqsubseteq \nu_i \circ f_i \Rightarrow \prod_{i \in n}^{(C)} (f_i \circ \mu_i) \sqsubseteq \prod_{i \in n}^{(C)} (\nu_i \circ f_i) \Leftrightarrow \\ &\left(\prod^{(C)} f\right) \circ \left(\prod^{(C)} \mu\right) \sqsubseteq \left(\prod^{(C)} \nu\right) \circ \left(\prod^{(C)} f\right) \Leftrightarrow \prod^{(C)} f \in C\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right). \end{aligned}$$

$$\begin{aligned} \forall i \in n : f_i \in C'(\mu_i, \nu_i) &\Leftrightarrow \forall i \in n : \mu_i \sqsubseteq f_i^\dagger \circ \nu_i \circ f_i \Rightarrow \prod_{i \in n}^{(C)} \mu \sqsubseteq \prod_{i \in n}^{(C)} (f_i^\dagger \circ \nu_i \circ f_i) \Leftrightarrow \\ &\prod_{i \in n}^{(C)} \mu \sqsubseteq \left(\prod_{i \in n}^{(C)} f_i^\dagger\right) \circ \left(\prod_{i \in n}^{(C)} \nu_i\right) \circ \left(\prod_{i \in n}^{(C)} f_i\right) \Leftrightarrow \\ &\prod_{i \in n}^{(C)} \mu \sqsubseteq \left(\prod_{i \in n}^{(C)} f_i\right)^\dagger \circ \left(\prod_{i \in n}^{(C)} \nu_i\right) \circ \left(\prod_{i \in n}^{(C)} f_i\right) \Leftrightarrow \prod^{(C)} f \in C'\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right). \end{aligned}$$

$$\begin{aligned} \forall i \in n : f_i \in C''(\mu_i, \nu_i) &\Leftrightarrow \forall i \in n : f_i \circ \mu_i \circ f_i^\dagger \sqsubseteq \nu_i \Rightarrow \\ &\prod_{i \in n}^{(C)} (f_i \circ \mu_i \circ f_i^\dagger) \sqsubseteq \prod_{i \in n}^{(C)} \nu_i \Leftrightarrow \prod_{i \in n}^{(C)} f_i \circ \prod_{i \in n}^{(C)} \mu_i \circ \prod_{i \in n}^{(C)} f_i^\dagger \sqsubseteq \prod_{i \in n}^{(C)} \nu_i \Leftrightarrow \\ &\prod_{i \in n}^{(C)} f_i \circ \prod_{i \in n}^{(C)} \mu_i \circ \left(\prod_{i \in n}^{(C)} f_i\right)^\dagger \sqsubseteq \prod_{i \in n}^{(C)} \nu_i \Leftrightarrow \prod_{i \in n}^{(C)} f_i \in C''\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right). \end{aligned}$$

\square

THEOREM 1951. Let μ and ν be indexed (by some index set n) families of endofunctors, and $f_i \in \text{FCD}(\text{Ob } \mu_i, \text{Ob } \nu_i)$ for every $i \in n$. Then:

- 1°. $\forall i \in n : f_i \in \text{C}(\mu_i, \nu_i) \Rightarrow \prod^{(A)} f \in \text{C}\left(\prod^{(A)} \mu, \prod^{(A)} \nu\right)$;
- 2°. $\forall i \in n : f_i \in \text{C}'(\mu_i, \nu_i) \Rightarrow \prod^{(A)} f \in \text{C}'\left(\prod^{(A)} \mu, \prod^{(A)} \nu\right)$;
- 3°. $\forall i \in n : f_i \in \text{C}''(\mu_i, \nu_i) \Rightarrow \prod^{(A)} f \in \text{C}''\left(\prod^{(A)} \mu, \prod^{(A)} \nu\right)$.

PROOF. Similar to the previous theorem. \square

THEOREM 1952. Let μ and ν be indexed (by some index set n) families of point-free endofunctors between posets with least elements, and $f_i \in \text{pFCD}(\text{Ob } \mu_i, \text{Ob } \nu_i)$ for every $i \in n$. Then:

- 1°. $\forall i \in n : f_i \in \text{C}(\mu_i, \nu_i) \Rightarrow \prod^{(S)} f \in \text{C}\left(\prod^{(S)} \mu, \prod^{(S)} \nu\right)$;
- 2°. $\forall i \in n : f_i \in \text{C}'(\mu_i, \nu_i) \Rightarrow \prod^{(S)} f \in \text{C}'\left(\prod^{(S)} \mu, \prod^{(S)} \nu\right)$;
- 3°. $\forall i \in n : f_i \in \text{C}''(\mu_i, \nu_i) \Rightarrow \prod^{(S)} f \in \text{C}''\left(\prod^{(S)} \mu, \prod^{(S)} \nu\right)$.

PROOF. Similar to the previous theorem. \square

23.17. Upgrading and downgrading multifunctors

LEMMA 1953. $\left\{ \frac{\langle f \rangle_k^* X}{X \in \text{up} \prod_{i \in n \setminus \{k\}} \mathfrak{Z}_i \mathcal{X}} \right\}$ is a filter base on \mathfrak{A}_k for every family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of primary filtrators where $i \in n$ for some index set n (provided that f is a multifunctor of the form \mathfrak{Z} and $k \in n$ and $\mathcal{X} \in \prod_{i \in n \setminus \{k\}} \mathfrak{A}_i$).

PROOF. Let $\mathcal{K}, \mathcal{L} \in \left\{ \frac{\langle f \rangle_k^* X}{X \in \text{up} \mathcal{X}} \right\}$. Then there exist $X, Y \in \text{up} \mathcal{X}$ such that $\mathcal{K} = \langle f \rangle_k^* X$, $\mathcal{L} = \langle f \rangle_k^* Y$. We can take $Z \in \text{up} \mathcal{X}$ such that $Z \sqsubseteq X, Y$. Then evidently $\langle f \rangle_k^* Z \sqsubseteq \mathcal{K}$ and $\langle f \rangle_k^* Z \sqsubseteq \mathcal{L}$ and $\langle f \rangle_k^* Z \in \left\{ \frac{\langle f \rangle_k^* X}{X \in \text{up} \mathcal{X}} \right\}$. \square

DEFINITION 1954. *Square mult* is a mult whose base and core are the same.

DEFINITION 1955. $\mathcal{L} \in [f] \Leftrightarrow \forall L \in \text{up } \mathcal{L} : L \in [f]^*$ for every mult f .

DEFINITION 1956. $\langle f \rangle \mathcal{X} = \prod_{X \in \text{up } \mathcal{X}} \langle f \rangle^* X$ for every mult f whose base is a complete lattice.

DEFINITION 1957. Let f be a mult whose base is a complete lattice. *Upgrading* of this mult is square mult $\uparrow\uparrow f$ with base $\uparrow\uparrow f = \text{core } \uparrow\uparrow f = \text{base } f$ and $\langle \uparrow\uparrow f \rangle^* \mathcal{X} = \langle f \rangle \mathcal{X}$ for every $\mathcal{X} \in \prod \text{base } f$.

LEMMA 1958. $\mathcal{L}_i \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \forall L \in \text{up } \mathcal{L} : L_i \not\prec \langle f \rangle^* L|_{(\text{dom } \mathcal{L}) \setminus \{i\}}$, if every $((\text{base } f)_i, (\text{core } f)_i)$ is a primary filtrator over a meet-semilattice with least element.

PROOF.

$$\begin{aligned}
& \mathcal{L}_i \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \\
& \quad \mathcal{L}_i \not\prec \langle f \rangle \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \\
& \mathcal{L}_i \not\prec \prod_{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \langle f \rangle^* X \Leftrightarrow \\
& \mathcal{L}_i \sqcap \prod_{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \langle f \rangle^* X \neq \perp \Leftrightarrow \\
& \prod_{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \langle \mathcal{L}_i \sqcap \rangle^* \langle f \rangle^* X \neq \perp \Leftrightarrow \\
& \prod \left\{ \frac{\mathcal{L}_i \sqcap \langle f \rangle^* X}{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \right\} \neq \perp \Leftrightarrow (*) \\
& \quad \perp \notin \left\{ \frac{\mathcal{L}_i \sqcap \langle f \rangle^* X}{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \right\} \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} : \mathcal{L}_i \sqcap \langle f \rangle^* X \neq \perp \Leftrightarrow (**) \\
& \quad \forall L \in \text{up } \mathcal{L} : \langle f \rangle^* L|_{\text{dom } \mathcal{L}} \sqcap \mathcal{L}_i \neq \perp \Leftrightarrow \\
& \quad \forall L \in \text{up } \mathcal{L} : \mathcal{L}_i \not\prec \langle f \rangle^* L|_{\text{dom } \mathcal{L}}.
\end{aligned}$$

(*) because $\left\{ \frac{\mathcal{L}_i \sqcap \langle f \rangle^* X}{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \right\}$ is a filter base (by lemma 1953) of the filter $\prod \left\{ \frac{\mathcal{L}_i \sqcap \langle f \rangle^* X}{X \in \text{up } \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}} \right\}$.
(**) by theorem 537. □

PROPOSITION 1959. $\uparrow\uparrow f$ is a square multifunctor, if every $((\text{base } f)_i, (\text{core } f)_i)$ is a primary filtrator over a bounded meet-semilattice.

PROOF. Our filtrators are with complete base by corollary 518.

$\mathcal{L}_i \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \forall L \in \text{up } \mathcal{L} : \mathcal{L}_i \not\prec \langle f \rangle^* L|_{(\text{dom } \mathcal{L}) \setminus \{i\}}$ by the lemma.

Similarly $\mathcal{L}_j \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{j\}} \Leftrightarrow \forall L \in \text{up } \mathcal{L} : \mathcal{L}_j \not\prec \langle f \rangle^* L|_{(\text{dom } \mathcal{L}) \setminus \{j\}}$.
So $\mathcal{L}_i \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \mathcal{L}_j \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{j\}}$ because $\mathcal{L}_i \not\prec \langle f \rangle^* L|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \mathcal{L}_j \not\prec \langle f \rangle^* L|_{(\text{dom } \mathcal{L}) \setminus \{j\}}$. □

PROPOSITION 1960. $[\uparrow\uparrow f]^* = [f]$ if every $((\text{base } f)_i, (\text{core } f)_i)$ is a primary filtrator over a bounded meet-semilattice.

PROOF. Our filtrators are with complete base by corollary 518.

$$\begin{aligned}
& \mathcal{L} \in [\uparrow\uparrow f]^* \Leftrightarrow \\
& \quad \mathcal{L}_i \not\prec \langle \uparrow\uparrow f \rangle^* \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \text{(by the lemma)} \\
& \forall L \in \text{up } \mathcal{L} : \mathcal{L}_i \not\prec \langle f \rangle^* L|_{(\text{dom } \mathcal{L}) \setminus \{i\}} \Leftrightarrow \\
& \quad \forall L \in \text{up } \mathcal{L} : L \in [f]^* \Leftrightarrow \\
& \quad \mathcal{L} \in [f].
\end{aligned}$$
□

PROPOSITION 1961. $\mathcal{L} \in [f] \Leftrightarrow \mathcal{L}_i \not\prec \langle f \rangle \mathcal{L}|_{(\text{dom } \mathcal{L}) \setminus \{i\}}$ if every $((\text{base } f)_i, (\text{core } f)_i)$ is a primary filtrator over a bounded meet-semilattice.

PROOF. Our filtrators are with complete base by corollary 518.

The theorem holds because $\uparrow\uparrow f$ is a multifunctor and $[f] = [\uparrow\uparrow f]^*$ and $\langle f \rangle = \langle \uparrow\uparrow f \rangle^*$. □

PROPOSITION 1962. $\Lambda \uparrow\uparrow g = \uparrow\uparrow \Lambda g$ for every prearoid g on boolean lattices.

PROOF. Our filtrators are with separable core by theorem 537.

$$\begin{aligned}
& Y \in \langle \Lambda \uparrow\uparrow g \rangle_i^* \mathcal{L} \Leftrightarrow \\
& \mathcal{L} \cup \{(i, Y)\} \in \text{GR } \uparrow\uparrow g \Leftrightarrow \\
& \text{up}(\mathcal{L} \cup \{(i, Y)\}) \subseteq \text{GR } g \Leftrightarrow \\
& \forall K \in \text{up}(\mathcal{L} \cup \{(i, Y)\}) : K \in \text{GR } g \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L}, P \in \text{up } Y : X \cup \{(i, P)\} \in \text{GR } g \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L}, P \in \text{up } Y : P \neq (\text{val } g)_i X \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L} : Y \neq (\text{val } g)_i X \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L} : Y \in (\text{val } g)_i X \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L} : X \cup \{(i, Y)\} \in \text{GR } g \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L} : Y \in \langle \Lambda g \rangle^* X \Leftrightarrow \\
& \forall X \in \text{up } \mathcal{L} : Y \sqcap \langle \Lambda g \rangle^* X \neq \perp \Leftrightarrow \\
& \perp \notin \left\{ \frac{Y \sqcap \langle \Lambda g \rangle^* X}{X \in \text{up } \mathcal{L}} \right\} \Leftrightarrow (*) \\
& \prod \left\{ \frac{Y \sqcap \langle \Lambda g \rangle^* X}{X \in \text{up } \mathcal{L}} \right\} \neq \perp \Leftrightarrow \\
& \prod_{X \in \text{up } \mathcal{L}} \langle Y \sqcap \rangle^* \langle \Lambda g \rangle^* X \neq \perp \Leftrightarrow \\
& Y \neq \prod_{X \in \text{up } \mathcal{L}} \langle \Lambda g \rangle^* X \Leftrightarrow \\
& Y \in \prod_{X \in \text{up } \mathcal{L}} \langle \Lambda g \rangle^* X \Leftrightarrow \\
& Y \in \langle \Lambda g \rangle_i \mathcal{L} \Leftrightarrow \\
& Y \in \langle \uparrow\uparrow \Lambda g \rangle_i^* \mathcal{L}.
\end{aligned}$$

(*) because $\left\{ \frac{Y \sqcap \langle \Lambda g \rangle^* X}{X \in \text{up } \mathcal{L}} \right\}$ is a filter base (by the lemma 1953) of $\prod \left\{ \frac{Y \sqcap \langle \Lambda g \rangle^* X}{X \in \text{up } \mathcal{L}} \right\}$. \square

DEFINITION 1963. Fix an indexed family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of filtrators. *Downgrading* of a square mult f of the form $(\mathfrak{A}_i, \mathfrak{A}_i)$ is the mult $\downarrow\downarrow f$ of the form $(\mathfrak{A}_i, \mathfrak{Z}_i)$ defined by the formula $\langle \downarrow\downarrow f \rangle_i^* = \langle f \rangle_i^* |_{\mathfrak{Z}_i}$ for every i .

OBVIOUS 1964. Downgrading of a square multifunctor is a multifunctor.

OBVIOUS 1965. $\downarrow\uparrow f = f$ for every mult f of the form $(\mathfrak{A}_i, \mathfrak{Z}_i)$.

PROPOSITION 1966. Let f be a square mult whose base is a complete lattice. Then $\uparrow\uparrow\downarrow f = f$.

PROOF. $\langle \uparrow\uparrow\downarrow f \rangle^* \mathcal{X} = \prod_{X \in \text{up } \mathcal{X}} \langle \downarrow\downarrow f \rangle^* X = \prod_{X \in \text{up } \mathcal{X}} \langle f \rangle^* X = \langle f \rangle^* \mathcal{X}$ for every $\mathcal{X} \in \prod_{i \in \text{arity } f} (\text{base } f)_i$. \square

23.18. On pseudofunctors

DEFINITION 1967. *Pseudofunctor* from a set A to a set B is a relation f between filters on A and B such that:

$$\begin{aligned}
\neg(\mathcal{I} f \perp), \quad \mathcal{I} \sqcup \mathcal{J} f \mathcal{K} &\Leftrightarrow \mathcal{I} f \mathcal{K} \vee \mathcal{J} f \mathcal{K} & (\text{for every } \mathcal{I}, \mathcal{J} \in \mathcal{F}(A), \mathcal{K} \in \mathcal{F}(B)), \\
\neg(\perp f \mathcal{I}), \quad \mathcal{K} f \mathcal{I} \sqcup \mathcal{J} &\Leftrightarrow \mathcal{K} f \mathcal{I} \vee \mathcal{K} f \mathcal{J} & (\text{for every } \mathcal{I}, \mathcal{J} \in \mathcal{F}(B), \mathcal{K} \in \mathcal{F}(A)).
\end{aligned}$$

OBVIOUS 1968. Pseudofuncoid is just a staroid of the form $(\mathcal{F}(A), \mathcal{F}(B))$.

OBVIOUS 1969. $[f]$ is a pseudofuncoid for every funcoid f .

EXAMPLE 1970. If A and B are infinite sets, then there exist two different pseudofuncoids f and g from A to B such that $f \cap (\mathcal{T}A \times \mathcal{T}B) = g \cap (\mathcal{T}A \times \mathcal{T}B) = [c] \cap (\mathcal{T}A \times \mathcal{T}B)$ for some funcoid c .

REMARK 1971. Considering a pseudofuncoid f as a staroid, we get $f \cap (\mathcal{T}A \times \mathcal{T}B) = \Downarrow f$.

PROOF. Take

$$f = \left\{ \frac{(\mathcal{X}, \mathcal{Y})}{\mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B), \bigcap \mathcal{X} \text{ and } \bigcap \mathcal{Y} \text{ are infinite}} \right\}$$

and

$$g = f \cup \left\{ \frac{(\mathcal{X}, \mathcal{Y})}{\mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B), \mathcal{X} \supseteq a, \mathcal{Y} \supseteq b} \right\}$$

where a and b are nontrivial ultrafilters on A and B correspondingly, c is the funcoid defined by the relation

$$[c]^* = \delta = \left\{ \frac{(X, Y)}{X \in \mathcal{P}A, Y \in \mathcal{P}B, X \text{ and } Y \text{ are infinite}} \right\}.$$

First prove that f is a pseudofuncoid. The formulas $\neg(\mathcal{I} f \perp)$ and $\neg(\perp f \mathcal{I})$ are obvious. We have

$$\begin{aligned} \mathcal{I} \sqcup \mathcal{J} f \mathcal{K} &\Leftrightarrow \bigcap(\mathcal{I} \sqcup \mathcal{J}) \text{ and } \bigcap \mathcal{Y} \text{ are infinite} \Leftrightarrow \\ \bigcap \mathcal{I} \cup \bigcap \mathcal{J} \text{ and } \bigcap \mathcal{Y} \text{ are infinite} &\Leftrightarrow \left(\bigcap \mathcal{I} \text{ or } \bigcap \mathcal{J} \text{ is infinite} \right) \wedge \bigcap \mathcal{Y} \text{ is infinite} \Leftrightarrow \\ \left(\bigcap \mathcal{I} \text{ and } \bigcap \mathcal{Y} \text{ are infinite} \right) \vee &\left(\bigcap \mathcal{J} \text{ and } \bigcap \mathcal{Y} \text{ are infinite} \right) \Leftrightarrow \\ &\mathcal{I} f \mathcal{K} \vee \mathcal{J} f \mathcal{K}. \end{aligned}$$

Similarly $\mathcal{K} f \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} f \mathcal{I} \vee \mathcal{K} f \mathcal{J}$. So f is a pseudofuncoid.

Let now prove that g is a pseudofuncoid. The formulas $\neg(\mathcal{I} g \perp)$ and $\neg(\perp g \mathcal{I})$ are obvious. Let $\mathcal{I} \sqcup \mathcal{J} g \mathcal{K}$. Then either $\mathcal{I} \sqcup \mathcal{J} f \mathcal{K}$ and then $\mathcal{I} \sqcup \mathcal{J} g \mathcal{K}$ or $\mathcal{I} \sqcup \mathcal{J} \supseteq a$ and then $\mathcal{I} \supseteq a \vee \mathcal{J} \supseteq a$ thus having $\mathcal{I} g \mathcal{K} \vee \mathcal{J} g \mathcal{K}$. So $\mathcal{I} \sqcup \mathcal{J} g \mathcal{K} \Rightarrow \mathcal{I} g \mathcal{K} \vee \mathcal{J} g \mathcal{K}$. The reverse implication is obvious. We have $\mathcal{I} \sqcup \mathcal{J} g \mathcal{K} \Leftrightarrow \mathcal{I} g \mathcal{K} \vee \mathcal{J} g \mathcal{K}$ and similarly $\mathcal{K} g \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} g \mathcal{I} \vee \mathcal{K} g \mathcal{J}$. So g is a pseudofuncoid.

Obviously $f \neq g$ ($a g b$ but not $a f b$).

It remains to prove $f \cap (\mathcal{T}A \times \mathcal{T}B) = g \cap (\mathcal{T}A \times \mathcal{T}B) = [c] \cap (\mathcal{T}A \times \mathcal{T}B)$. Really, $f \cap (\mathcal{T}A \times \mathcal{T}B) = [c] \cap (\mathcal{T}A \times \mathcal{T}B)$ is obvious. If $(\uparrow^A X, \uparrow^B Y) \in g \cap (\mathcal{T}A \times \mathcal{T}B)$ then either $(\uparrow^A X, \uparrow^B Y) \in f \cap (\mathcal{T}A \times \mathcal{T}B)$ or $X \in \text{up } a, Y \in \text{up } b$, so X and Y are infinite and thus $(\uparrow^A X, \uparrow^B Y) \in f \cap (\mathcal{T}A \times \mathcal{T}B)$. So $g \cap (\mathcal{T}A \times \mathcal{T}B) = f \cap (\mathcal{T}A \times \mathcal{T}B)$. \square

REMARK 1972. The above counter-example shows that pseudofuncoids (and more generally, any staroids on filters) are “second class” objects, they are not full-fledged because they don’t bijectively correspond to funcoids and the elegant funcoids theory does not apply to them.

From the above it follows that staroids on filters do not correspond (by restriction) to staroids on principal filters (or staroids on sets).

23.18.1. More on free stars and principal free stars.

PROPOSITION 1973. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is a filtrator.
- 4°. $\partial\mathcal{F} = \Downarrow \star\mathcal{F}$ for every $\mathcal{F} \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Obvious.

3° \Rightarrow 4°. $X \in \partial\mathcal{F} \Leftrightarrow X \not\prec^{\mathfrak{A}} \mathcal{F} \Leftrightarrow X \in \star\mathcal{F} \Leftrightarrow X \in \Downarrow \star\mathcal{F}$ for every $X \in \mathfrak{J}$.

□

PROPOSITION 1974. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a meet-semilattice with least element.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is a filtrator with separable core.
- 4°. $\star\mathcal{F} = \Uparrow \partial\mathcal{F}$ for every $\mathcal{F} \in \mathfrak{A}$.

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. Theorem 537.

3° \Rightarrow 4°. $\mathcal{X} \in \Uparrow \partial\mathcal{F} \Leftrightarrow \text{up } \mathcal{X} \subseteq \partial\mathcal{F} \Leftrightarrow \forall X \in \text{up } \mathcal{X} : X \not\prec \mathcal{F} \Leftrightarrow \mathcal{X} \not\prec \mathcal{F} \Leftrightarrow \mathcal{X} \in \star\mathcal{F}$.

□

PROPOSITION 1975. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a complete boolean lattice.
- 3°. $(\mathfrak{A}, \mathfrak{J})$ is a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.
- 4°. The following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:
 - (a) $\mathcal{F} \in \mathfrak{J}$.
 - (b) $\partial\mathcal{F}$ is a principal free star on \mathfrak{J} .
 - (c) $\star\mathcal{F}$ is a principal free star on \mathfrak{A} .

PROOF.

1° \Rightarrow 2°. Obvious.

2° \Rightarrow 3°. The filtrator $(\mathfrak{A}, \mathfrak{J})$ is with with join-closed core by theorem 534, binarily meet-closed core by corollary 536, with separable core by theorem 537.

3° \Rightarrow 4°.

4°a \Rightarrow 4°b. That $\partial\mathcal{F}$ does not contain the least element is obvious. That $\partial\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 583.

4°b \Rightarrow 4°c. That $\star\mathcal{F}$ does not contain the least element is obvious. That $\star\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 583.

4°c \Rightarrow 4°a. Apply theorem 583.

□

PROPOSITION 1976. The following is an implications tuple:

- 1°. $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2°. $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a join-semilattice.
- 3°. The filtrator $(\mathfrak{A}, \mathfrak{J})$ is weakly down-aligned and with binarily join-closed core and \mathfrak{J} is a join-semilattice.
- 4°. If S is a free star on \mathfrak{A} then $\Downarrow S$ is a free star on \mathfrak{J} .

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. It is weakly down-aligned by obvious 511 and with join-closed core by theorem 534.

$3^\circ \Rightarrow 4^\circ$. For every $X, Y \in \mathfrak{J}$ we have

$$\begin{aligned} X \sqcup^{\mathfrak{J}} Y \in \Downarrow S &\Leftrightarrow X \sqcup^{\mathfrak{J}} Y \in S \Leftrightarrow X \sqcup^{\mathfrak{A}} Y \in S \Leftrightarrow \\ &X \in S \vee Y \in S \Leftrightarrow X \in \Downarrow S \vee Y \in \Downarrow S; \end{aligned}$$

Suppose there is least element $\perp^{\mathfrak{J}} \in \Downarrow S$. Then $\perp^{\mathfrak{A}} = \perp^{\mathfrak{J}} \in S$ what is impossible. \square

PROPOSITION 1977. The following is an implications tuple:

- 1° . $(\mathfrak{A}, \mathfrak{J})$ is a powerset filtrator.
- 2° . $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator over a boolean lattice.
- 3° . If S is a free star on \mathfrak{J} then $\Uparrow S$ is a free star on \mathfrak{A} .

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. There exists a filter \mathcal{F} such that $S = \partial\mathcal{F}$. For every filters $\mathcal{X}, \mathcal{Y} \in \mathfrak{A}$

$$\begin{aligned} \mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \in \Uparrow S &\Leftrightarrow \text{up}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) \subseteq S \Leftrightarrow \forall K \in \text{up}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) : K \in \partial\mathcal{F} \Leftrightarrow \\ \forall K \in \text{up}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) : K \not\in \mathcal{F} &\Leftrightarrow \mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \not\in \mathcal{F} \Leftrightarrow \mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \in \star\mathcal{F} \Leftrightarrow \mathcal{X} \in \star\mathcal{F} \vee \mathcal{Y} \in \star\mathcal{F} \Leftrightarrow \\ \mathcal{X} \not\in \mathcal{F} \vee \mathcal{Y} \not\in \mathcal{F} &\Leftrightarrow \forall X \in \text{up} \mathcal{X} : X \not\in \mathcal{F} \vee \forall Y \in \text{up} \mathcal{Y} : Y \not\in \mathcal{F} \Leftrightarrow \\ \forall X \in \text{up} \mathcal{X} : X \in \partial\mathcal{F} &\vee \forall Y \in \text{up} \mathcal{Y} : Y \in \partial\mathcal{F} \Leftrightarrow \\ \text{up} \mathcal{X} \subseteq S \vee \text{up} \mathcal{Y} \subseteq S &\Leftrightarrow \mathcal{X} \in \Uparrow S \vee \mathcal{Y} \in \Uparrow S; \end{aligned}$$

$$\perp \in \Uparrow S \Leftrightarrow \text{up} \perp \subseteq S \Leftrightarrow \perp \in S \text{ what is false.}$$

\square

PROPOSITION 1978. The following is an implications tuple:

- 1° . $(\mathfrak{A}, \mathfrak{J})$ is primary filtrator over a complete lattice.
- 2° . $(\mathfrak{A}, \mathfrak{J})$ is down-aligned filtrator with join-closed core over a complete lattice.
- 3° . If S is a principal free star on \mathfrak{A} then $\Downarrow S$ is a principal free star on \mathfrak{J} .

PROOF.

$1^\circ \Rightarrow 2^\circ$. It is down-aligned by obvious 506 and with join-closed core by theorem 534.

$2^\circ \Rightarrow 3^\circ$. $\sqcup^{\mathfrak{J}} T \in \Downarrow S \Leftrightarrow \sqcup^{\mathfrak{J}} T \in S \Leftrightarrow \sqcup^{\mathfrak{A}} T \in S \Leftrightarrow T \cap S \neq \emptyset \Leftrightarrow T \cap \Downarrow S \neq \emptyset$ for every $T \in \mathcal{P}\mathfrak{J}$; $\perp \notin \Downarrow S$ is obvious. \square

PROPOSITION 1979. The following is an implications tuple:

- 1° . $(\mathfrak{A}, \mathfrak{J})$ is powerset filtrator.
- 2° . $(\mathfrak{A}, \mathfrak{J})$ is primary filtrator over a boolean lattice.
- 3° . If S is a principal free star on \mathfrak{J} then $\Uparrow S$ is a principal free star on \mathfrak{A} .

PROOF.

$1^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 3^\circ$. There exists a principal filter \mathcal{F} such that $S = \partial\mathcal{F}$.

$$\begin{aligned} \bigsqcup^{\mathfrak{A}} T \in \uparrow\uparrow S &\Leftrightarrow \text{up} \bigsqcup^{\mathfrak{A}} T \subseteq S \Leftrightarrow \forall K \in \text{up} \bigsqcup^{\mathfrak{A}} T : K \in \partial\mathcal{F} \Leftrightarrow \\ &\forall K \in \text{up} \bigsqcup^{\mathfrak{A}} T : K \not\prec \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \not\prec \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in \star\mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T : \mathcal{K} \in \star\mathcal{F} \Leftrightarrow \\ \exists \mathcal{K} \in T : \mathcal{K} \not\prec \mathcal{F} &\Leftrightarrow \exists \mathcal{K} \in T \forall K \in \text{up} \mathcal{K} : K \not\prec \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \text{up} \mathcal{K} : K \in \partial\mathcal{F} \Leftrightarrow \\ &\exists \mathcal{K} \in T : \text{up} \mathcal{K} \subseteq S \Leftrightarrow \exists \mathcal{K} \in T : \mathcal{K} \in \uparrow\uparrow S \Leftrightarrow T \cap \uparrow\uparrow S \neq \emptyset. \\ \perp \in \uparrow\uparrow S &\Leftrightarrow \text{up} \perp \subseteq S \Leftrightarrow \perp \in S \text{ what is false.} \end{aligned}$$

□

23.18.2. Complete staroids and multifuncoids.

DEFINITION 1980. Consider an indexed family \mathfrak{Z} of posets. A pre-staroid f of the form \mathfrak{Z} is *complete* in argument $k \in \text{arity } f$ when $(\text{val } f)_k L$ is a principal free star for every $L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} \mathfrak{Z}_i$.

DEFINITION 1981. Consider an indexed family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of filtrators and multifuncoid f is of the form $(\mathfrak{A}, \mathfrak{Z})$. Then f is *complete* in argument $k \in \text{arity } f$ iff $\langle f \rangle_k^* L \in \mathfrak{Z}_k$ for every family $L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} \mathfrak{Z}_i$.

PROPOSITION 1982. Consider an indexed family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of primary filtrators over complete boolean lattices. Let f be a multifuncoid of the form $(\mathfrak{A}, \mathfrak{Z})$ and $k \in \text{arity } f$. The following are equivalent:

- 1°. Multifuncoid f is complete in argument k .
- 2°. Pre-staroid $\Downarrow [f]^*$ is complete in argument k .

PROOF. Let $L \in \prod \mathfrak{Z}$. We have $L \in \text{GR} [f]^* \Leftrightarrow L_i \not\prec \langle f \rangle_i^* L|_{(\text{dom } L) \setminus \{i\}}$;

$(\text{val } [f]^*)_k L = \partial \langle f \rangle_k^* L$ by the definition.

So $(\text{val } [f]^*)_k L$ is a principal free star iff $\langle f \rangle_k^* L \in \mathfrak{Z}_k$ (proposition 1975) for every $L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} \mathfrak{Z}_i$. □

EXAMPLE 1983. Consider funcoid $f = 1_{\mathcal{U}}^{\text{CD}}$. It is obviously complete in each its two arguments. Then $[f]^*$ is not complete in each of its two arguments because $(\mathcal{X}, \mathcal{Y}) \in [f]^* \Leftrightarrow \mathcal{X} \not\prec \mathcal{Y}$ what does not generate a principal free star if one of the arguments (say \mathcal{X}) is a fixed nonprincipal filter.

THEOREM 1984. Consider an indexed family $(\mathfrak{A}, \mathfrak{Z})$ of filtrators which are down-aligned, separable, with join-closed, binarily meet-closed and with separable core which is a complete boolean lattice.

Let f be a multifuncoid of the aforementioned form. Let $k, l \in \text{arity } f$ and $k \neq l$. The following are equivalent:

- 1°. f is complete in the argument k .
- 2°. $\langle f \rangle_l^* (L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l^* (L \cup \{(k, x)\})$ for every $X \in \mathcal{P} \mathfrak{Z}_k$, $L \in \prod_{i \in (\text{arity } f) \setminus \{k, l\}} \mathfrak{Z}_i$.
- 3°. $\langle f \rangle_l^* (L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l^* (L \cup \{(k, x)\})$ for every $X \in \mathcal{P} \mathfrak{A}_k$, $L \in \prod_{i \in (\text{arity } f) \setminus \{k, l\}} \mathfrak{Z}_i$.

PROOF.

$3^\circ \Rightarrow 2^\circ$. Obvious.

$2^\circ \Rightarrow 1^\circ$. Let $Y \in \mathfrak{J}$.

$$\begin{aligned} \bigsqcup X \not\prec \langle f \rangle_k^*(L \cup \{(l, Y)\}) &\Leftrightarrow Y \not\prec \langle f \rangle_l^*(L \cup \{(k, \bigsqcup X)\}) \Leftrightarrow \\ &Y \not\prec \bigsqcup_{x \in X} \langle f \rangle_l^*(L \cup \{(k, x)\}) \Leftrightarrow (\text{proposition 583}) \Leftrightarrow \\ &\exists x \in X : Y \not\prec \langle f \rangle_l^*(L \cup \{(k, x)\}) \Leftrightarrow \exists x \in X : x \not\prec \langle f \rangle_k^*(L \cup \{(l, Y)\}). \end{aligned}$$

It is equivalent (proposition 1975 and the fact that $[f]^*$ is an upper set) to $\langle f \rangle_k^*(L \cup \{(l, Y)\})$ being a principal filter and thus $(\text{val } f)_k L$ being a principal free star.

$1^\circ \Rightarrow 3^\circ$.

$$\begin{aligned} Y \not\prec \langle f \rangle_l^*(L \cup \{(k, \bigsqcup X)\}) &\Leftrightarrow \bigsqcup X \not\prec \langle f \rangle_k(L \cup \{(l, Y)\}) \Leftrightarrow \\ \exists x \in X : x \not\prec \langle f \rangle_k^*(L \cup \{(l, Y)\}) &\Leftrightarrow \exists x \in X : Y \not\prec \langle f \rangle_l^*(L \cup \{(k, x)\}) \Leftrightarrow \\ &Y \not\prec \bigsqcup_{x \in X} \langle f \rangle_l^*(L \cup \{(k, x)\}) \end{aligned}$$

for every principal Y . Thus $\langle f \rangle_l^*(L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l^*(L \cup \{(k, x)\})$ by separability. \square

23.19. Identity staroids and multifuncoids

23.19.1. Identity relations. Denote $\text{id}_{A[n]} = \left\{ \frac{\lambda i \in n : x}{x \in A} \right\} = \left\{ \frac{n \times \{x\}}{x \in A} \right\}$ the n -ary identity relation on a set A (for each index set n).

PROPOSITION 1985. $\prod X \not\prec \text{id}_{A[n]} \Leftrightarrow \bigcap_{i \in n} X_i \cap A \neq \emptyset$ for every indexed family X of sets.

PROOF.

$$\prod X \not\prec \text{id}_{A[n]} \Leftrightarrow \exists t \in A : n \times \{t\} \in \prod X \Leftrightarrow \exists t \in A \forall i \in n : t \in X_i \Leftrightarrow \bigcap_{i \in n} X_i \cap A \neq \emptyset.$$

\square

23.19.2. General definitions of identity staroids. Consider a filtrator $(\mathfrak{A}, \mathfrak{J})$ and $\mathcal{A} \in \mathfrak{A}$.

I will define below *small identity staroids* $\text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ and *big identity staroids* $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$. That they are really staroids and even complementary staroids (under certain conditions) is proved below.

DEFINITION 1986. Consider a filtrator $(\mathfrak{A}, \mathfrak{J})$. Let \mathfrak{J} be a complete lattice. Let $\mathcal{A} \in \mathfrak{A}$, let n be an index set.

$$\text{form id}_{\mathcal{A}[n]}^{\text{Strd}} = \mathfrak{J}^n; \quad L \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \prod_{i \in n}^{\mathfrak{J}} L_i \in \partial \mathcal{A}.$$

OBVIOUS 1987. $X \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \forall A \in \text{up } \mathcal{A} : \prod_{i \in n}^{\mathfrak{J}} X_i \cap A \neq \emptyset$ if our filtrator is with separable core.

DEFINITION 1988. The subset X of a poset \mathfrak{A} has a nontrivial lower bound (I denote this predicate as $\text{MEET}(X)$) iff there is nonleast $a \in \mathfrak{A}$ such that $\forall x \in X : a \sqsubseteq x$.

DEFINITION 1989. Staroid $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ (for any $\mathcal{A} \in \mathfrak{A}$ where \mathfrak{A} is a poset) is defined by the formulas:

$$\text{form ID}_{\mathcal{A}[n]}^{\text{Strd}} = \mathfrak{A}^n; \quad \mathcal{L} \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \text{MEET} \left(\left\{ \frac{\mathcal{L}_i}{i \in n} \right\} \cup \{\mathcal{A}\} \right).$$

OBVIOUS 1990. If \mathfrak{A} is complete lattice, then $\mathcal{L} \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \prod \mathcal{L} \not\leq \mathcal{A}$.

OBVIOUS 1991. If \mathfrak{A} is complete lattice and a is an atom, then $\mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \prod \mathcal{L} \supseteq a$.

OBVIOUS 1992. If \mathfrak{A} is a complete lattice then there exists a multifunctor $\Lambda \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ such that $\langle \Lambda \text{ID}_{\mathcal{A}[n]}^{\text{Strd}} \rangle_k L = \prod_{i \in n} L_i \sqcap \mathcal{A}$ for every $k \in n$, $L \in \mathfrak{A}^{n \setminus \{k\}}$.

PROPOSITION 1993. Let $(\mathfrak{A}, \mathfrak{B})$ be a meet-closed filtrator and \mathfrak{B} be a complete lattice and \mathfrak{A} be a meet-semilattice. There exists a multifunctor $\Lambda \text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ such that $\langle \Lambda \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \rangle_k L = \prod_{i \in n}^{\mathfrak{B}} L_i \sqcap^{\mathfrak{A}} \mathcal{A}$ for every $k \in n$, $L \in \mathfrak{B}^{n \setminus \{k\}}$.

PROOF. We need to prove that $L \cup \{(k, X)\} \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \prod_{i \in n}^{\mathfrak{B}} L_i \sqcap^{\mathfrak{A}} \mathcal{A} \not\leq^{\mathfrak{A}} X$.
But

$$\begin{aligned} \prod_{i \in n}^{\mathfrak{B}} L_i \sqcap^{\mathfrak{A}} \mathcal{A} \not\leq^{\mathfrak{A}} X &\Leftrightarrow \prod_{i \in n}^{\mathfrak{B}} L_i \sqcap^{\mathfrak{A}} X \not\leq^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \\ &\prod_{i \in n}^{\mathfrak{B}} (L \cup \{(k, X)\})_i \not\leq^{\mathfrak{A}} \mathcal{A} \Leftrightarrow L \cup \{(k, X)\} \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}}. \end{aligned}$$

□

23.19.3. Identities are staroids.

PROPOSITION 1994. Let \mathfrak{A} be a complete meet infinite distributive lattice and $\mathcal{A} \in \mathfrak{A}$. Then $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ is a staroid.

PROOF. That $L \notin \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}}$ if $L_k = \perp$ for some $k \in n$ is obvious. It remains to prove

$$L \cup \{(k, X \sqcup Y)\} \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow L \cup \{(k, X)\} \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}} \vee L \cup \{(k, Y)\} \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}}.$$

It is equivalent to

$$\prod_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\leq \mathcal{A} \Leftrightarrow \prod_{i \in n \setminus \{k\}} L_i \sqcap X \not\leq \mathcal{A} \vee \prod_{i \in n \setminus \{k\}} L_i \sqcap Y \not\leq \mathcal{A}.$$

Really,

$$\begin{aligned} \prod_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\leq \mathcal{A} &\Leftrightarrow \prod_{i \in n \setminus \{k\}} ((L_i \sqcap X) \sqcup (L_i \sqcap Y)) \not\leq \mathcal{A} \Leftrightarrow \\ &\left(\prod_{i \in n \setminus \{k\}} L_i \sqcap X \right) \sqcup \left(\prod_{i \in n \setminus \{k\}} L_i \sqcap Y \right) \not\leq \mathcal{A} \Leftrightarrow \\ &\prod_{i \in n \setminus \{k\}} L_i \sqcap X \not\leq \mathcal{A} \vee \prod_{i \in n \setminus \{k\}} L_i \sqcap Y \not\leq \mathcal{A}. \end{aligned}$$

□

PROPOSITION 1995. Let $(\mathfrak{A}, \mathfrak{B})$ be a starrish filtrator over a complete meet infinite distributive lattice and $\mathcal{A} \in \mathfrak{A}$. Then $\text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ is a staroid.

PROOF. That $L \notin \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}}$ if $L_k = \perp$ for some $k \in n$ is obvious. It remains to prove

$$L \cup \{(k, X \sqcup Y)\} \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow L \cup \{(k, X)\} \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \vee L \cup \{(k, Y)\} \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}}.$$

It is equivalent to

$$\prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap (X \sqcup Y) \not\leq \mathcal{A} \Leftrightarrow \prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap X \not\leq \mathcal{A} \vee \prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap Y \not\leq \mathcal{A}.$$

Really,

$$\begin{aligned} \prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap (X \sqcup Y) \not\leq \mathcal{A} &\Leftrightarrow \prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} ((L_i \sqcap X) \sqcup (L_i \sqcap Y)) \not\leq \mathcal{A} \Leftrightarrow \\ &\left(\prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap X \right) \sqcup \left(\prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap Y \right) \not\leq \mathcal{A} \Leftrightarrow \\ &\prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap X \not\leq \mathcal{A} \vee \prod_{i \in n \setminus \{k\}}^{\mathfrak{Z}} L_i \sqcap Y \not\leq \mathcal{A}. \end{aligned}$$

□

PROPOSITION 1996. Let $(\mathfrak{A}, \mathfrak{Z})$ be a primary filtrator over a boolean lattice. $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ is a completary staroid for every $\mathcal{A} \in \mathfrak{A}$.

PROOF. $\star\mathcal{A}$ is a free star by theorem 614.

$$\begin{aligned} L_0 \sqcup L_1 \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}} &\Leftrightarrow \forall i \in n : (L_0 \sqcup L_1)i \in \star\mathcal{A} \Leftrightarrow \forall i \in n : L_0i \sqcup L_1i \in \star\mathcal{A} \Leftrightarrow \\ &\forall i \in n : (L_0i \in \star\mathcal{A} \vee L_1i \in \star\mathcal{A}) \Leftrightarrow \exists c \in \{0, 1\}^n \forall i \in n : L_{c(i)}i \in \star\mathcal{A} \Leftrightarrow \\ &\exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)}i) \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}}. \end{aligned}$$

□

LEMMA 1997. $X \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \text{Cor}' \prod_{i \in n}^{\mathfrak{A}} X_i \not\leq \mathcal{A}$ for a join-closed filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that both \mathfrak{A} and \mathfrak{Z} are complete lattices, provided that $\mathcal{A} \in \mathfrak{A}$.

PROOF. $X \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \prod_{i \in n}^{\mathfrak{Z}} X_i \not\leq \mathcal{A} \Leftrightarrow \text{Cor}' \prod_{i \in n}^{\mathfrak{A}} X_i \not\leq \mathcal{A}$ (theorem 602). □

CONJECTURE 1998. $\text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ is a completary staroid for every set-theoretic filter \mathcal{A} .

CONJECTURE 1999. $\uparrow\uparrow \text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ is a completary staroid if \mathcal{A} is a filter on a set and n is an index set.

23.19.4. Special case of sets and filters.

PROPOSITION 2000. $\uparrow^{3^n} X \in \text{GR id}_{a[n]}^{\text{Strd}} \Leftrightarrow \forall A \in a : \prod X \not\leq \text{id}_{A[n]}$ for every filter a on a powerset and index set n .

PROOF.

$$\begin{aligned} \forall A \in a : \prod X \not\leq \text{id}_{A[n]} &\Leftrightarrow \forall A \in a : \bigcap_{i \in n} X_i \cap A \neq \emptyset \Leftrightarrow \forall A \in a : \prod_{i \in n}^{\mathfrak{Z}} X_i \not\leq A \Leftrightarrow \\ \forall A \in a : \prod_{i \in n}^{\mathfrak{Z}} X_i \not\leq^{\mathfrak{A}} A &\Leftrightarrow \prod_{i \in n}^{\mathfrak{Z}} \uparrow^{\mathfrak{Z}} X_i \not\leq^{\mathfrak{A}} a \Leftrightarrow \prod_{i \in n}^{\mathfrak{Z}} (\uparrow^{3^n} X)_i \not\leq^{\mathfrak{A}} a \Leftrightarrow \uparrow^{3^n} X \in \text{GR id}_{a[n]}^{\text{Strd}}. \end{aligned}$$

□

PROPOSITION 2001. $Y \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \forall A \in \text{up } \mathcal{A} : Y \in \text{GR } \uparrow^{\text{Strd}} \text{ id}_{A[n]}$ for every filter \mathcal{A} on a powerset and $Y \in \mathfrak{3}^n$.

PROOF. Take $Y = \uparrow^{\mathfrak{3}^n} X$.

$$\begin{aligned} \forall A \in \text{up } \mathcal{A} : Y \in \text{GR } \uparrow^{\text{Strd}} \text{ id}_{A[n]} &\Leftrightarrow \forall A \in \text{up } \mathcal{A} : \uparrow^{\mathfrak{3}^n} X \in \text{GR } \uparrow^{\text{Strd}} \text{ id}_{A[n]} \Leftrightarrow \\ \forall A \in \text{up } \mathcal{A} : \prod X \not\prec \text{id}_{A[n]} &\Leftrightarrow \uparrow^{\mathfrak{3}^n} X \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow Y \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}}. \end{aligned}$$

□

PROPOSITION 2002. $\uparrow^{\mathfrak{3}^n} X \in \text{GR id}_{a[n]}^{\text{Strd}} \Leftrightarrow \forall A \in a \exists t \in A \forall i \in n : t \in X_i$.

PROOF.

$$\uparrow^{\mathfrak{3}^n} X \in \text{GR id}_{a[n]}^{\text{Strd}} \Leftrightarrow \exists A \in a \exists t \in A : n \times \{t\} \in \prod X \Leftrightarrow \forall A \in a \exists t \in A \forall i \in n : t \in X_i.$$

□

23.19.5. Relationships between big and small identity staroids.

DEFINITION 2003. $a_{\text{Strd}}^n = \prod_{i \in n}^{\text{Strd}} a$ for every element a of a poset and an index set n .

LEMMA 2004. $\mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}}$ iff $\bigcup_{i \in n} \text{up } \mathcal{L}_i \cup \{a\}$ has finite intersection property (for primary filtrators over meet semilattices with greatest element).

PROOF. The lattice \mathfrak{A} is complete by corollary 518. $\mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \prod_{i \in n} \text{up } \mathcal{L} \sqcap a \neq \perp^{\mathcal{F}} \Leftrightarrow \forall X \in \prod_{i \in n} \text{up } \mathcal{L} \sqcap a : X \neq \perp$ what is equivalent of $\bigcup_{i \in n} \mathcal{L}_i \cup \{a\}$ having finite intersection property. □

PROPOSITION 2005. $\uparrow \uparrow \text{id}_{a[n]}^{\text{Strd}} \sqsubseteq \text{ID}_{a[n]}^{\text{Strd}} \sqsubseteq a_{\text{Strd}}^n$ for every filter a (on any distributive lattice with least element) and an index set n .

PROOF.

$$\text{GR } \uparrow \uparrow \text{id}_{a[n]}^{\text{Strd}} \sqsubseteq \text{GR ID}_{a[n]}^{\text{Strd}}.$$

$$\mathcal{L} \in \text{GR } \uparrow \uparrow \text{id}_{a[n]}^{\text{Strd}} \Leftrightarrow \text{up } \mathcal{L} \subseteq \text{GR id}_{a[n]}^{\text{Strd}} \Leftrightarrow \forall L \in \text{up } \mathcal{L} : L \in \text{GR id}_{a[n]}^{\text{Strd}} \Leftrightarrow$$

$$\text{(theorem 537)} \Leftrightarrow \forall L \in \text{up } \mathcal{L} \forall A \in \text{up } a : \prod_{i \in n}^3 L_i \not\prec A \Leftrightarrow$$

$$\forall L \in \text{up } \mathcal{L} \forall A \in \text{up } a : \prod_{i \in n}^3 L_i \sqcap A \neq \perp \Rightarrow$$

$$\bigcup_{i \in n} \text{up } \mathcal{L}_i \cup \{a\} \text{ has finite intersection property} \Leftrightarrow \mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}}.$$

$$\text{GR ID}_{a[n]}^{\text{Strd}} \sqsubseteq \text{GR } a_{\text{Strd}}^n. \mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \text{MEET}(\{\frac{\mathcal{L}_i}{i \in n}\} \cup \{a\}) \Rightarrow \forall i \in a : \mathcal{L}_i \not\prec a \Leftrightarrow \mathcal{L} \in \text{GR } a_{\text{Strd}}^a.$$

□

PROPOSITION 2006. $\uparrow \uparrow \text{id}_{a[a]}^{\text{Strd}} \sqsubset \text{ID}_{a[a]}^{\text{Strd}} = a_{\text{Strd}}^a$ for every nontrivial ultrafilter a on a set.

PROOF.

□

GR $\uparrow \uparrow \text{id}_{a[a]}^{\text{Strd}} \neq \text{GR ID}_{a[a]}^{\text{Strd}}$. Let $\mathcal{L}_i = \uparrow^{\text{Base}(a)} i$. Then trivially $\mathcal{L} \in \text{GR ID}_{a[a]}^{\text{Strd}}$. But to disprove $\mathcal{L} \in \text{GR } \uparrow \uparrow \text{id}_{a[a]}^{\text{Strd}}$ it's enough to show $L \notin \text{GR id}_{a[a]}^{\text{Strd}}$ for some $L \in \text{up } \mathcal{L}$. Really, take $L_i = \mathcal{L}_i = \uparrow^{\text{Base}(a)} i$. Then $L \in \text{GR id}_{a[a]}^{\text{Strd}} \Leftrightarrow \forall A \in a \exists t \in A \forall i \in a : t \in i$ what is clearly false (we can always take $i \in a$ such that $t \notin i$ for any point t).

$$\text{GR ID}_{a[a]}^{\text{Strd}} = \text{GR } a_{\text{Strd}}^a. \quad \mathcal{L} \in \text{GR ID}_{a[a]}^{\text{Strd}} \Leftrightarrow \forall i \in a : \mathcal{L}_i \sqsupseteq a \Leftrightarrow \forall i \in a : \mathcal{L}_i \not\neq a \Leftrightarrow \mathcal{L} \in \text{GR } a_{\text{Strd}}^a.$$

COROLLARY 2007. a_{Strd}^a isn't an atom when a is a nontrivial ultrafilter.

COROLLARY 2008. Staroidal product of an infinite indexed family of ultrafilters may be non-atomic.

PROPOSITION 2009. $\text{id}_{a[n]}^{\text{Strd}}$ is determined by the value of $\uparrow\uparrow \text{id}_{a[n]}^{\text{Strd}}$ (for every element a of a filtrator $(\mathfrak{A}, \mathfrak{I})$ over a complete lattice \mathfrak{J}). Moreover $\text{id}_{a[n]}^{\text{Strd}} = \downarrow\downarrow \uparrow\uparrow \text{id}_{a[n]}^{\text{Strd}}$.

PROOF. Use general properties of upgrading and downgrading (proposition 1773). \square

PROPOSITION 2010. $\text{ID}_{a[n]}^{\text{Strd}}$ is determined by the value of $\downarrow\downarrow \text{ID}_{a[n]}^{\text{Strd}}$, moreover $\text{ID}_{a[n]}^{\text{Strd}} = \uparrow\uparrow \downarrow\downarrow \text{ID}_{a[n]}^{\text{Strd}}$ (for filter a on a primary filtrator over a meet semilattice with greatest element).

PROOF.

$$\begin{aligned} \mathcal{L} \in \uparrow\uparrow \downarrow\downarrow \text{ID}_{a[n]}^{\text{Strd}} &\Leftrightarrow \text{up } \mathcal{L} \subseteq \downarrow\downarrow \text{ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \text{up } \mathcal{L} \subseteq \text{ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \\ &\forall L \in \text{up } \mathcal{L} : L \in \text{ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \forall L \in \text{up } \mathcal{L} : \prod_{i \in n} L_i \sqcap a \neq \perp^{\mathfrak{F}} \Leftrightarrow \end{aligned}$$

$$\bigcup_{i \in n} \text{up } \mathcal{L}_i \cup \{a\} \text{ has finite intersection property} \Leftrightarrow (\text{lemma}) \Leftrightarrow \mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}}.$$

\square

PROPOSITION 2011. $\text{id}_{a[n]}^{\text{Strd}} \sqsubseteq \downarrow\downarrow \text{ID}_{a[n]}^{\text{Strd}}$ for every filter a and an index set n .

PROOF. $\text{id}_{a[n]}^{\text{Strd}} = \downarrow\downarrow \uparrow\uparrow \text{id}_{a[n]}^{\text{Strd}} \sqsubseteq \downarrow\downarrow \text{ID}_{a[n]}^{\text{Strd}}$. \square

PROPOSITION 2012. $\text{id}_{a[a]}^{\text{Strd}} \sqsubset \downarrow\downarrow \text{ID}_{a[a]}^{\text{Strd}}$ for every nontrivial ultrafilter a .

PROOF. Suppose $\text{id}_{a[a]}^{\text{Strd}} = \downarrow\downarrow \text{ID}_{a[a]}^{\text{Strd}}$. Then $\text{ID}_{a[a]}^{\text{Strd}} = \uparrow\uparrow \downarrow\downarrow \text{ID}_{a[a]}^{\text{Strd}} = \uparrow\uparrow \text{id}_{a[a]}^{\text{Strd}}$ what contradicts to the above. \square

OBVIOUS 2013. $\mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}} \Leftrightarrow a \sqcap \prod_{i \in n} \mathcal{L}_i \neq \perp$ if a is an element of a complete lattice.

OBVIOUS 2014. $\mathcal{L} \in \text{GR ID}_{a[n]}^{\text{Strd}} \Leftrightarrow \forall i \in n : \mathcal{L}_i \sqsupseteq a \Leftrightarrow \forall i \in n : \mathcal{L}_i \not\neq a$ if a is an ultrafilter on \mathfrak{A} .

23.19.6. Identity staroids on principal filters. For principal filter $\uparrow A$ (where A is a set) the above definitions coincide with n -ary identity relation, as formulated in the following propositions:

PROPOSITION 2015. $\uparrow^{\text{Strd}} \text{id}_{A[n]} = \text{id}_{\uparrow A[n]}^{\text{Strd}}$.

PROOF.

$$\begin{aligned} L \in \text{GR } \uparrow^{\text{Strd}} \text{id}_{A[n]} &\Leftrightarrow \prod L \not\neq \text{id}_{A[n]} \Leftrightarrow \exists t \in A \forall i \in n : t \in L_i \Leftrightarrow \\ &\bigcap_{i \in n} L_i \cap A \neq \emptyset \Leftrightarrow L \in \text{GR } \text{id}_{\uparrow A[n]}^{\text{Strd}}. \end{aligned}$$

Thus $\uparrow^{\text{Strd}} \text{id}_{A[n]} = \text{id}_{\uparrow A[n]}^{\text{Strd}}$. \square

COROLLARY 2016. $\text{id}_{\uparrow A[n]}^{\text{Strd}}$ is a principal staroid.

QUESTION 2017. Is $\text{ID}_{A[n]}^{\text{Strd}}$ principal for every principal filter A on a set and index set n ?

PROPOSITION 2018. $\uparrow^{\text{Strd}} \text{id}_{A[n]} \sqsubseteq \Downarrow \text{ID}_{\uparrow A[n]}^{\text{Strd}}$ for every set A .

PROOF.

$$\begin{aligned} L \in \text{GR } \uparrow^{\text{Strd}} \text{id}_{A[n]} &\Leftrightarrow L \in \text{GR } \text{id}_{\uparrow A[n]}^{\text{Strd}} \Leftrightarrow \uparrow A \not\neq \prod_{i \in n}^3 L_i \Rightarrow \\ &\uparrow A \not\neq \prod_{i \in n}^{\aleph} L_i \Leftrightarrow L \in \Downarrow \text{GR } \text{ID}_{\uparrow A[n]}^{\text{Strd}}. \end{aligned}$$

□

PROPOSITION 2019. $\uparrow^{\text{Strd}} \text{id}_{A[n]} \sqsubset \Downarrow \text{ID}_{\uparrow A[n]}^{\text{Strd}}$ for some set A and index set n .

PROOF. $L \in \text{GR } \uparrow^{\text{Strd}} \text{id}_{A[n]} \Leftrightarrow \prod_{i \in n}^3 L_i \not\neq \uparrow A$ what is not implied by $\prod_{i \in n}^{\aleph} L_i \not\neq \uparrow A$ that is $L \in \Downarrow \text{GR } \text{ID}_{\uparrow A[n]}^{\text{Strd}}$. (For a counter example take $n = \mathbb{N}$, $L_i =]0; 1/i[$, $A = \mathbb{R}$.) □

PROPOSITION 2020. $\uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} = \uparrow\uparrow \text{id}_{\uparrow A[n]}^{\text{Strd}}$.

PROOF. $\uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} = \uparrow\uparrow \text{id}_{\uparrow A[n]}^{\text{Strd}}$ is obvious from the above. □

PROPOSITION 2021. $\uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} \sqsubseteq \text{ID}_{\uparrow A[n]}^{\text{Strd}}$.

PROOF.

$$\begin{aligned} \mathcal{X} \in \text{GR } \uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} &\Leftrightarrow \text{up } \mathcal{X} \subseteq \text{GR } \uparrow^{\text{Strd}} \text{id}_{A[n]} \Leftrightarrow \\ &\forall Y \in \text{up } \mathcal{X} : Y \in \text{GR } \uparrow^{\text{Strd}} \text{id}_{A[n]} \Leftrightarrow \forall Y \in \text{up } \mathcal{X} : Y \in \text{GR } \text{id}_{\uparrow A[n]}^{\text{Strd}} \Leftrightarrow \\ &\forall Y \in \text{up } \mathcal{X} : \prod_{i \in n}^3 Y_i \cap \uparrow A \neq \perp \Rightarrow \prod_{i \in n}^{\aleph} \mathcal{X}_i \cap \uparrow A \neq \perp \Leftrightarrow \mathcal{X} \in \text{GR } \text{ID}_{\uparrow A[n]}^{\text{Strd}}. \end{aligned}$$

□

PROPOSITION 2022. $\uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} \sqsubset \text{ID}_{\uparrow A[n]}^{\text{Strd}}$ for some set A .

PROOF. We need to prove $\uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} \neq \text{ID}_{\uparrow A[n]}^{\text{Strd}}$ that is it's enough to prove (see the above proof) that $\forall Y \in \text{up } \mathcal{X} : \prod_{i \in n}^3 Y_i \cap \uparrow A \neq \perp \not\Leftarrow \prod_{i \in n}^{\aleph} \mathcal{X}_i \cap \uparrow A \neq \perp$. A counter-example follows:

$\forall Y \in \text{up } \mathcal{X} : \prod_{i \in n}^3 Y_i \cap \uparrow A \neq \perp$ does not hold for $n = \mathbb{N}$, $\mathcal{X}_i =] - 1/i; 0[$ for $i \in n$, $A =] - \infty; 0[$. To show this, it's enough to prove $\prod_{i \in n}^3 Y_i \cap \uparrow A = \perp$ for $Y_i =] - 1/i; 0[$ but this is obvious since $\prod_{i \in n}^3 Y_i = \perp$.

On the other hand, $\prod_{i \in n}^{\aleph} \mathcal{X}_i \cap \uparrow A \neq \perp$ for the same \mathcal{X} and A . □

The above theorems are summarized in the diagram at figure 13:

$$\begin{array}{ccc} \Downarrow \text{ID}_{\uparrow A[n]}^{\text{Strd}} & \sqsubseteq & \uparrow^{\text{Strd}} \text{id}_{A[n]} = \text{id}_{\uparrow A[n]}^{\text{Strd}} \\ \downarrow \uparrow & & \downarrow \uparrow \\ \text{ID}_{\uparrow A[n]}^{\text{Strd}} & \sqsubseteq & \uparrow\uparrow^{\text{Strd}} \text{id}_{A[n]} = \uparrow\uparrow \text{id}_{\uparrow A[n]}^{\text{Strd}} \end{array}$$

FIGURE 13. Relationships of identity staroids for principal filters.

REMARK 2023. \sqsubseteq on the diagram means inequality which can become strict for some A and n .

23.19.7. Identity staroids represented as meets and joins.

PROPOSITION 2024. $\text{id}_{a[n]}^{\text{Strd}} = \prod_{A \in \text{up } a}^{\text{Anch}} \text{id}_{A[n]} = \prod_{A \in \text{up } a}^{\text{Strd}} \text{id}_{A[n]}$ for every filter a on a powerset.

PROOF. Since $\text{id}_{a[n]}^{\text{Strd}}$ is a staroid (proposition 1995), it's enough to prove that $\text{id}_{a[n]}^{\text{Strd}}$ is the greatest lower bound of $\left\{ \prod_{A \in \text{up } a}^{\text{Strd}} \text{id}_{A[n]} \right\}$.

That $\text{id}_{a[n]}^{\text{Strd}} \sqsubseteq \prod_{A \in \text{up } a}^{\text{Strd}} \text{id}_{A[n]}$ for every $A \in \text{up } a$ is obvious.

Let $f \sqsubseteq \prod_{A \in \text{up } a}^{\text{Strd}} \text{id}_{A[n]}$ for every $A \in \text{up } a$.

$$L \in \text{GR } f \Rightarrow \forall A \in \text{up } a : L \in \text{GR } \prod_{A \in \text{up } a}^{\text{Strd}} \text{id}_{A[n]} \Leftrightarrow$$

$$\forall A \in \text{up } a : \prod_{i \in n} L \not\neq \text{id}_{A[n]} \Leftrightarrow \forall A \in \text{up } a : \prod_{i \in n}^{\exists} L_i \not\neq A \Rightarrow$$

$$\forall A \in \text{up } a : \prod_{i \in n}^{\exists} L_i \not\neq A \Rightarrow \prod_{i \in n}^{\exists} L_i \not\neq a \Rightarrow L \in \text{GR } \text{id}_{a[n]}^{\text{Strd}}.$$

Thus $f \sqsubseteq \text{id}_{a[n]}^{\text{Strd}}$. \square

PROPOSITION 2025. $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}} = \bigsqcup_{a \in \text{atoms } \mathcal{A}} \text{ID}_{a[n]}^{\text{Strd}} = \bigsqcup_{a \in \text{atoms } \mathcal{A}} a_{\text{Strd}}^n$ where the join may be taken on every of the following posets: anchored relations, staroids, complementary staroids, provided that \mathcal{A} is a filter on a set.

PROOF. $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ is a complementary staroid (proposition 1996). Thus, it's enough to prove that $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ is the lowest upper bound of $\left\{ \frac{\text{ID}_{a[n]}^{\text{Strd}}}{a \in \text{atoms } \mathcal{A}} \right\}$ (also use the fact that $\text{ID}_{a[n]}^{\text{Strd}} = a_{\text{Strd}}^n$).

$\text{ID}_{\mathcal{A}[n]}^{\text{Strd}} \supseteq \text{ID}_{a[n]}^{\text{Strd}}$ for every $a \in \text{atoms } \mathcal{A}$ is obvious.

Let $f \supseteq \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ for every $a \in \text{atoms } \mathcal{A}$. Then $\forall L \in \text{GR } \text{ID}_{\mathcal{A}[n]}^{\text{Strd}} : L \in \text{GR } f$ that is

$$\forall L \in \text{form } f : \left(\text{MEET} \left(\left\{ \frac{L_i}{i \in n} \right\} \cup \{a\} \right) \Rightarrow L \in \text{GR } f \right).$$

But

$$\begin{aligned} \exists a \in \text{atoms } \mathcal{A} : \text{MEET} \left(\left\{ \frac{L_i}{i \in n} \right\} \cup \{a\} \right) \Leftrightarrow \exists a \in \text{atoms } \mathcal{A} : \prod_{i \in n}^{\exists} L_i \not\neq a \Leftrightarrow \\ \prod_{i \in n}^{\exists} L_i \not\neq \mathcal{A} \Leftrightarrow L \in \text{GR } \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}. \end{aligned}$$

So $L \in \text{GR } \text{ID}_{\mathcal{A}[n]}^{\text{Strd}} \Rightarrow L \in \text{GR } f$. Thus $f \supseteq \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$. \square

PROPOSITION 2026. $\text{id}_{\mathcal{A}[n]}^{\text{Strd}} = \bigsqcup_{a \in \text{atoms } \mathcal{A}} \text{id}_{a[n]}^{\text{Strd}}$ where the meet may be taken on every of the following posets: anchored relations, staroids, provided that \mathcal{A} is a filter on a set.

PROOF. Since $\text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ is a staroid (proposition 1995), it's enough to prove the result for join on anchored relations.

$\text{id}_{\mathcal{A}[n]}^{\text{Strd}} \supseteq \text{id}_{a[n]}^{\text{Strd}}$ for every $a \in \text{atoms } \mathcal{A}$ is obvious.

Let $f \sqsupseteq \text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ for every $a \in \text{atoms } \mathcal{A}$. Then $\forall L \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} : L \in \text{GR } f$ that is

$$\forall L \in \text{form } f : \left(\prod_{i \in n}^{\mathfrak{Z}} L_i \not\neq a \Rightarrow L \in \text{GR } f \right).$$

But $\exists a \in \text{atoms } \mathcal{A} : \prod_{i \in n}^{\mathfrak{Z}} L_i \not\neq a \Leftarrow \prod_{i \in n}^{\mathfrak{Z}} L_i \not\neq \mathcal{A} \Leftrightarrow L \in \text{id}_{\mathcal{A}[n]}^{\text{Strd}}$.

So $L \in \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \Rightarrow L \in \text{GR } f$. Thus $f \sqsupseteq \text{id}_{\mathcal{A}[n]}^{\text{Strd}}$. \square

23.19.8. Finite case.

THEOREM 2027. Let n be a finite set.

- 1°. $\text{id}_{\mathcal{A}[n]}^{\text{Strd}} = \Downarrow \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ if \mathfrak{A} and \mathfrak{Z} are meet-semilattices and $(\mathfrak{A}, \mathfrak{Z})$ is a binarily meet-closed filtrator.
- 2°. $\text{ID}_{\mathcal{A}[n]}^{\text{Strd}} = \Uparrow \text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ if $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.

PROOF.

1°.

$$\begin{aligned} L \in \text{GR } \Downarrow \text{ID}_{\mathcal{A}[n]}^{\text{Strd}} &\Leftrightarrow L \in \text{GR ID}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \text{MEET} \left(\left\{ \frac{L_i}{i \in n} \right\} \cup \{\mathcal{A}\} \right) \Leftrightarrow \\ &\prod_{i \in n}^{\mathfrak{A}} L_i \sqcap \mathcal{A} \neq 0 \Leftrightarrow (\text{by finiteness}) \Leftrightarrow \prod_{i \in n}^{\mathfrak{Z}} L_i \sqcap \mathcal{A} \neq 0 \Leftrightarrow L \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \end{aligned}$$

for every $L \in \prod \mathfrak{Z}$.

2°.

$$\begin{aligned} L \in \text{GR } \Uparrow \text{id}_{\mathcal{A}[n]}^{\text{Strd}} &\Leftrightarrow \text{up } L \subseteq \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \forall K \in \text{up } L : K \in \text{GR id}_{\mathcal{A}[n]}^{\text{Strd}} \Leftrightarrow \\ &\forall K \in \text{up } L : \prod_{i \in n}^{\mathfrak{Z}} K_i \in \partial \mathcal{A} \Leftrightarrow \forall K \in \text{up } L : \prod_{i \in n}^{\mathfrak{Z}} K_i \not\neq \mathcal{A} \Leftrightarrow \\ &(\text{by finiteness and theorem 535}) \Leftrightarrow \\ &\forall K \in \text{up } L : \prod_{i \in n}^{\mathfrak{A}} K_i \not\neq \mathcal{A} \Leftrightarrow \mathcal{A} \in \bigcap \langle \star \rangle^* \left\{ \frac{\prod_{i \in n}^{\mathfrak{A}} K_i}{K \in \text{up } L} \right\} \Leftrightarrow \\ &(\text{by the formula for finite meet of filters, theorem 523}) \Leftrightarrow \\ &\mathcal{A} \in \bigcap \langle \star \rangle^* \text{up } \prod_{i \in n}^{\mathfrak{A}} L_i \Leftrightarrow \forall K \in \text{up } \prod_{i \in n}^{\mathfrak{A}} L_i : \mathcal{A} \in \star K \Leftrightarrow \forall K \in \text{up } \prod_{i \in n}^{\mathfrak{A}} L_i : \mathcal{A} \not\neq K \Leftrightarrow \\ &(\text{by separability of core, theorem 537}) \Leftrightarrow \\ &\prod_{i \in n}^{\mathfrak{A}} L_i \not\neq \mathcal{A} \Leftrightarrow L \in \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}. \end{aligned}$$

\square

PROPOSITION 2028. Let $(\mathfrak{A}, \mathfrak{Z})$ be a binarily meet closed filtrator whose core is a meet-semilattice. $\Downarrow \text{ID}_{\mathcal{A}[n]}^{\text{Strd}}$ and $\text{id}_{\mathcal{A}[n]}^{\text{Strd}}$ are the same for finite n .

PROOF. Because $\prod_{i \in \text{dom } L}^{\mathfrak{Z}} L_i = \prod_{i \in \text{dom } L}^{\mathfrak{A}} L_i$ for finitary L . \square

23.20. Counter-examples

EXAMPLE 2029. $\Uparrow \Downarrow f \neq f$ for some staroid f whose form is an indexed family of filters on a set.

PROOF. Let $f = \left\{ \frac{A \in \mathcal{F}(\mathcal{U})}{\uparrow \text{Cor } A \neq \Delta} \right\}$ for some infinite set \mathcal{U} where Δ is some non-principal filter on \mathcal{U} .

$$\begin{aligned} A \sqcup B \in f &\Leftrightarrow \uparrow^{\mathcal{U}} \text{Cor}(A \sqcup B) \neq \Delta \Leftrightarrow \uparrow^{\mathcal{U}} \text{Cor } A \sqcup \uparrow^{\mathcal{U}} \text{Cor } B \neq \Delta \Leftrightarrow \\ &\uparrow^{\mathcal{U}} \text{Cor } A \sqcap \Delta \neq \perp^{\mathcal{F}(\mathcal{U})} \vee \uparrow^{\mathcal{U}} \text{Cor } B \sqcap \Delta \neq \perp^{\mathcal{F}(\mathcal{U})} \Leftrightarrow A \in f \vee B \in f. \end{aligned}$$

Obviously $\perp^{\mathcal{F}(\mathcal{U})} \notin f$. So f is a free star. But free stars are essentially the same as 1-staroids.

$$\Downarrow f = \partial\Delta. \quad \Downarrow\Downarrow f = \left\{ \frac{Z \in \mathcal{F}}{\text{up } Z \subseteq \partial\Delta} \right\} = \left\{ \frac{Z \in \mathcal{F}}{\forall K \in \text{up } Z: K \neq \Delta} \right\} = \left\{ \frac{Z \in \mathcal{F}}{Z \neq \Delta} \right\} = \star\Delta \neq f. \quad \square$$

For the below counter-examples we will define a staroid ϑ with arity $\vartheta = \mathbb{N}$ and $\text{GR } \vartheta \in \mathcal{P}(\mathbb{N}^{\mathbb{N}})$ (based on a suggestion by ANDREAS BLASS):

$$A \in \text{GR } \vartheta \Leftrightarrow \sup_{i \in \mathbb{N}} \text{card}(A_i \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_i \neq \emptyset.$$

PROPOSITION 2030. ϑ is a staroid.

PROOF. $(\text{val } \vartheta)_i L = \mathcal{P}\mathbb{N} \setminus \{\emptyset\}$ for every $L \in (\mathcal{P}\mathbb{N})^{\mathbb{N} \setminus \{i\}}$ if

$$\sup_{j \in \mathbb{N} \setminus \{i\}} \text{card}(A_j \cap j) = \mathbb{N} \wedge \forall j \in \mathbb{N} \setminus \{i\} : L_j \neq \emptyset.$$

Otherwise $(\text{val } \vartheta)_i L = \emptyset$. Thus $(\text{val } \vartheta)_i L$ is a free star. So ϑ is a staroid. (That ϑ is an upper set, is obvious.) \square

PROPOSITION 2031. ϑ is a complementary staroid.

PROOF.

$$\begin{aligned} A_0 \sqcup A_1 \in \text{GR } \vartheta &\Leftrightarrow A_0 \cup A_1 \in \text{GR } \vartheta \Leftrightarrow \\ &\sup_{i \in \mathbb{N}} \text{card}((A_0 i \cup A_1 i) \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_0 i \cup A_1 i \neq \emptyset \Leftrightarrow \\ &\sup_{i \in \mathbb{N}} \text{card}((A_0 i \cap i) \cup (A_1 i \cap i)) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_0 i \cup A_1 i \neq \emptyset. \end{aligned}$$

If $A_0 i = \emptyset$ then $A_0 i \cap i = \emptyset$ and thus $A_1 i \cap i \supseteq A_0 i \cap i$. Thus we can select $c(i) \in \{0, 1\}$ in such a way that $\forall d \in \{0, 1\} : \text{card}(A_{c(i)} i \cap i) \supseteq \text{card}(A_d i \cap i)$ and $A_{c(i)} i \neq \emptyset$. (Consider the case $A_0 i, A_1 i \neq \emptyset$ and the similar cases $A_0 i = \emptyset$ and $A_1 i = \emptyset$.)

So

$$\begin{aligned} A_0 \sqcup A_1 \in \text{GR } \vartheta &\Leftrightarrow \sup_{i \in \mathbb{N}} \text{card}(A_{c(i)} i \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_{c(i)} i \neq \emptyset \Leftrightarrow \\ &(\lambda i \in \mathbb{N} : A_{c(i)} i) \in \text{GR } \vartheta. \end{aligned}$$

Thus ϑ is complementary. \square

OBVIOUS 2032. ϑ is non-zero.

EXAMPLE 2033. There is such a nonzero staroid f on powersets that $f \not\sqsupseteq \prod^{\text{Strd}} a$ for every family $a = a_{i \in \mathbb{N}}$.

PROOF. It's enough to prove $\vartheta \not\sqsupseteq \prod^{\text{Strd}} a$.

Let $\uparrow^{\mathbb{N}} R_i = a_i$ if a_i is principal and $R_i = \mathbb{N} \setminus i$ if a_i is non-principal.

We have $\forall i \in \mathbb{N} : R_i \in a_i$.

We have $R \notin \text{GR } \vartheta$ because $\sup_{i \in \mathbb{N}} \text{card}(R_i \cap i) \neq \mathbb{N}$.

$R \in \prod^{\text{Strd}} a$ because $\forall X \in a_i : X \cap R_i \neq \emptyset$.

So $\vartheta \not\sqsupseteq \prod^{\text{Strd}} a$. \square

REMARK 2034. At <http://mathoverflow.net/questions/60925/special-infinitary-relations-and-ultrafilters> there is a proof for arbitrary infinite form, not just for \mathbb{N} .

CONJECTURE 2035. For every family $a = a_{i \in \mathbb{N}}$ of ultrafilters $\prod^{\text{Strd}} a$ is not an atom nor of the poset of staroids neither of the poset of cometary staroids of the form $\lambda i \in \mathbb{N} : \text{Base}(a_i)$.

CONJECTURE 2036. There exists a non-cometary staroid on powersets.

CONJECTURE 2037. There exists a prestaroid which is not a staroid.

CONJECTURE 2038. The set of staroids of the form A^B where A and B are sets is atomic.

CONJECTURE 2039. The set of staroids of the form A^B where A and B are sets is atomistic.

CONJECTURE 2040. The set of cometary staroids of the form A^B where A and B are sets is atomic.

CONJECTURE 2041. The set of cometary staroids of the form A^B where A and B are sets is atomistic.

EXAMPLE 2042. $\text{StarComp}(a, f \sqcup g) \neq \text{StarComp}(a, f) \sqcup \text{StarComp}(a, g)$ in the category of binary relations with star-morphisms for some n -ary relation a and an n -indexed families f and g of functions.

PROOF. Let $n = \{0, 1\}$. Let $\text{GR } a = \{(0, 1), (1, 0)\}$ and $f = [\{(0, 1)\}, \{(1, 0)\}]$, $g = [\{(1, 0)\}, \{(0, 1)\}]$.

For every $\{0, 1\}$ -indexed family of μ of functions:

$$L \in \text{StarComp}(a, \mu) \Leftrightarrow \exists y \in a : (y_0 \mu_0 L_0 \wedge y_1 \mu_1 L_1) \Leftrightarrow \exists y_0 \in \text{dom } \mu_0, y_1 \in \text{dom } \mu_1 : (y_0 \mu_0 L_0 \wedge y_1 \mu_1 L_1)$$

for every n -ary relation μ .

Consequently

$$L \in \text{StarComp}(a, f) \Leftrightarrow L_0 = 1 \wedge L_1 = 0 \Leftrightarrow L = (1, 0)$$

that is $\text{StarComp}(a, f) = \{(1, 0)\}$. Similarly

$$\text{StarComp}(a, g) = \{(0, 1)\}.$$

Also

$$L \in \text{StarComp}(a, f \sqcup g) \Leftrightarrow \exists y_0, y_1 \in \{0, 1\} : ((y_0 f_0 L_0 \vee y_0 g_0 L_0) \wedge (y_1 f_1 L_1 \vee y_1 g_1 L_1)).$$

Thus

$$\text{StarComp}(a, f \sqcup g) = \{(0, 1), (1, 0), (0, 0), (1, 1)\}.$$

□

COROLLARY 2043. The above inequality is possible also for star-morphisms of funcoids and star-morphisms of reloids.

PROOF. Because finitary funcoids and reloids between finite sets are essentially the same as finitary relations and our proof above works for binary relations. □

The following example shows that the theorem 1984 can't be strengthened:

EXAMPLE 2044. For some multifuncoid f on powersets complete in argument k the following formula is false:

$$\langle f \rangle_i(L \cup \{(k, \sqcup X)\}) = \sqcup_{x \in X} \langle f \rangle_i(L \cup \{(k, x)\}) \text{ for every } X \in \mathcal{P}\mathfrak{Z}_k, L \in \prod_{i \in (\text{arity } f) \setminus \{k, l\}} \mathcal{F}_i.$$

PROOF. Consider multifunctor $f = \Lambda \text{id}_{\uparrow U[3]}^{\text{Strd}}$ where U is an infinite set (of the form \mathfrak{Z}^3) and $L = (Y)$ where Y is a nonprincipal filter on U .

$$\langle f \rangle_0(L \cup \{(k, \sqcup X)\}) = Y \sqcap \sqcup X;$$

$$\bigsqcup_{x \in X} \langle f \rangle_0(L \cup \{(k, x)\}) = \bigsqcup_{x \in X} (Y \sqcap x).$$

It can be $Y \sqcap \sqcup X = \bigsqcup_{x \in X} (Y \sqcap x)$ only if Y is principal: Really: $Y \sqcap \sqcup X = \bigsqcup_{x \in X} (Y \sqcap x)$ implies $Y \not\sqsubset \sqcup X \Rightarrow \bigsqcup_{x \in X} (Y \sqcap x) \neq \perp \Rightarrow \exists x \in X : Y \not\sqsubset x$ and thus Y is principal. But we claimed above that it is nonprincipal. \square

EXAMPLE 2045. There exists a staroid f and an indexed family X of principal filters (with arity $f = \text{dom } X$ and $(\text{form } f)_i = \text{Base}(X_i)$ for every $i \in \text{arity } f$), such that $f \sqsubseteq \prod^{\text{Strd}} X$ and $Y \sqcap X \notin \text{GR } f$ for some $Y \in \text{GR } f$.

REMARK 2046. Such examples obviously do not exist if both f is a principal staroid and X and Y are indexed families of principal filters (because for powerset algebras staroidal product is equivalent to Cartesian product). This makes the above example inspired.

PROOF. (MONROE ESKEW) Let a be any (trivial or nontrivial) ultrafilter on an infinite set U . Let $A, B \in a$ be such that $A \cap B \subset A, B$. In other words, A, B are arbitrary nonempty sets such that $\emptyset \neq A \cap B \subset A, B$ and a be an ultrafilter on $A \cap B$.

Let f be the staroid whose graph consists of functions $p : U \rightarrow a$ such that either $p(n) \supseteq A$ for all but finitely many n or $p(n) \supseteq B$ for all but finitely many n . Let's prove f is really a staroid.

It's obvious $px \neq \emptyset$ for every $x \in U$. Let $k \in U$, $L \in a^{U \setminus \{k\}}$. It is enough (taking symmetry into account) to prove that

$$L \cup \{(k, x \sqcup y)\} \in \text{GR } f \Leftrightarrow L \cup \{(k, x)\} \in \text{GR } f \vee L \cup \{(k, y)\} \in \text{GR } f. \quad (36)$$

Really, $L \cup \{(k, x \sqcup y)\} \in \text{GR } f$ iff $x \sqcup y \in a$ and $L(n) \supseteq A$ for all but finitely many n or $L(n) \supseteq B$ for all but finitely many n ; $L \cup \{(k, x)\} \in \text{GR } f$ iff $x \in a$ and $L(n) \supseteq A$ for all but finitely many n or $L(n) \supseteq B$; and similarly for y .

But $x \sqcup y \in a \Leftrightarrow x \in a \vee y \in a$ because a is an ultrafilter. So, the formula (36) holds, and we have proved that f is really a staroid.

Take X be the constant function with value A and Y be the constant function with value B .

$\forall p \in \text{GR } f : p \not\sqsubset X$ because $p_i \cap X_i \in a$; so $\text{GR } f \subseteq \text{GR } \prod^{\text{Strd}} X$ that is $f \sqsubseteq \prod^{\text{Strd}} X$.

Finally, $Y \sqcap X \notin \text{GR } f$ because $X \sqcap Y = \lambda i \in U : A \cap B$. \square

23.21. Conjectures

REMARK 2047. Below I present special cases of possible theorems. The theorems may be generalized after the below special cases are proved.

CONJECTURE 2048. For every two functors f and g we have:

- 1°. $(\text{RLD})_{\text{in}} a [f \times^{(DP)} g] (\text{RLD})_{\text{in}} b \Leftrightarrow a [f \times^{(C)} g] b$ for every functors $a \in \text{FCD}(\text{Src } f, \text{Src } g)$, $b \in \text{FCD}(\text{Dst } f, \text{Dst } g)$;
- 2°. $(\text{RLD})_{\text{out}} a [f \times^{(DP)} g] (\text{RLD})_{\text{out}} b \Leftrightarrow a [f \times^{(C)} g] b$ for every functors $a \in \text{FCD}(\text{Src } f, \text{Src } g)$, $b \in \text{FCD}(\text{Dst } f, \text{Dst } g)$;
- 3°. $(\text{FCD}) a [f \times^{(C)} g] (\text{FCD}) b \Leftrightarrow a [f \times^{(DP)} g] b$ for every relicts $a \in \text{RLD}(\text{Src } f, \text{Src } g)$, $b \in \text{RLD}(\text{Dst } f, \text{Dst } g)$.

CONJECTURE 2049. For every two functors f and g we have:

- 1°. $(\text{RLD})_{\text{in}} a [f \times^{(A)} g] (\text{RLD})_{\text{in}} b \Leftrightarrow a [f \times^{(C)} g] b$ for every functors $a \in \text{FCD}(\text{Src } f, \text{Src } g)$, $b \in \text{FCD}(\text{Dst } f, \text{Dst } g)$;

- 2°. $(\text{RLD})_{\text{out}} a [f \times^{(A)} g] (\text{RLD})_{\text{out}} b \Leftrightarrow a [f \times^{(C)} g] b$ for every funcoids $a \in \text{FCD}(\text{Src } f, \text{Src } g)$, $b \in \text{FCD}(\text{Dst } f, \text{Dst } g)$;
 3°. $(\text{FCD})_a [f \times^{(C)} g] (\text{FCD})_b \Leftrightarrow a [f \times^{(A)} g] b$ for every reloids $a \in \text{RLD}(\text{Src } f, \text{Src } g)$, $b \in \text{RLD}(\text{Dst } f, \text{Dst } g)$.

CONJECTURE 2050. $\prod^{\text{Strd}} a \neq \prod^{\text{Strd}} b \Leftrightarrow b \in \prod^{\text{Strd}} a \Leftrightarrow a \in \prod^{\text{Strd}} b \Leftrightarrow \forall i \in n : a_i \neq b_i$ for every n -indexed families a and b of filters on powersets.

CONJECTURE 2051. Let f be a staroid on powersets and $a \in \prod_{i \in \text{arity } f} \text{Src } f_i$, $b \in \prod_{i \in \text{arity } f} \text{Dst } f_i$. Then

$$\prod^{\text{Strd}} a \left[\prod^{(C)} f \right] \prod^{\text{Strd}} b \Leftrightarrow \forall i \in n : a_i [f_i] b_i.$$

PROPOSITION 2052. The conjecture 2051 is a consequence of the conjecture 2050.

PROOF.

$$\begin{aligned} \prod^{\text{Strd}} a \left[\prod^{(C)} f \right] \prod^{\text{Strd}} b \Leftrightarrow \prod^{\text{Strd}} b \neq \left\langle \prod^{(C)} f \right\rangle \prod^{\text{Strd}} a \Leftrightarrow \prod^{\text{Strd}} b \neq \prod_{i \in n} \langle f_i \rangle a_i \Leftrightarrow \\ \forall i \in n : b_i \neq \langle f_i \rangle a_i \Leftrightarrow \forall i \in n : a_i [f_i] b_i. \end{aligned}$$

□

CONJECTURE 2053. For every indexed families a and b of filters and an indexed family f of pointfree funcoids we have

$$\prod^{\text{Strd}} a \left[\prod^{(C)} f \right] \prod^{\text{Strd}} b \Leftrightarrow \prod^{\text{RLD}} a \left[\prod^{(DP)} f \right] \prod^{\text{RLD}} b.$$

CONJECTURE 2054. For every indexed families a and b of filters and an indexed family f of pointfree funcoids we have

$$\prod^{\text{Strd}} a \left[\prod^{(C)} f \right] \prod^{\text{Strd}} b \Leftrightarrow \prod^{\text{RLD}} a \left[\prod^{(A)} f \right] \prod^{\text{RLD}} b.$$

Strengthening of an above result:

CONJECTURE 2055. If a is a complementary staroid and $\text{Dst } f_i$ is a starrish poset for every $i \in n$ then $\text{StarComp}(a, f)$ is a complementary staroid.

Strengthening of above results:

CONJECTURE 2056.

- 1°. $\prod^{(D)} F$ is a prestaroid if every F_i is a prestaroid.
 2°. $\prod^{(D)} F$ is a complementary staroid if every F_i is a complementary staroid.

CONJECTURE 2057. If f_1 and f_2 are funcoids, then there exists a pointfree funcoid $f_1 \times f_2$ such that

$$\langle f_1 \times f_2 \rangle x = \bigsqcup \left\{ \frac{\langle f_1 \rangle X \times^{\text{FCD}} \langle f_2 \rangle X}{X \in \text{atoms } x} \right\}$$

for every ultrafilter x .

CONJECTURE 2058. Let $(\mathfrak{A}, \mathfrak{Z}) = (\mathfrak{A}, \mathfrak{Z})_{i \in n}$ be a family of filtrators on boolean lattices.

A relation $\delta \in \mathcal{P} \prod \text{atoms}^{\mathfrak{A}_i}$ such that for every $a \in \prod \text{atoms}^{\mathfrak{A}_i}$

$$\forall A \in a : \delta \cap \prod_{i \in n} \text{atoms} \uparrow^{\mathfrak{Z}_i} A_i \neq \emptyset \Rightarrow a \in \delta \quad (37)$$

can be continued till the function $\uparrow\uparrow f$ for a unique staroid f of the form $\lambda i \in n : \mathfrak{A}_i$. The funcoïd f is completary.

CONJECTURE 2059. For every $\mathcal{X} \in \prod_{i \in n} \mathcal{F}(\mathfrak{A}_i)$

$$\mathcal{X} \in \text{GR} \uparrow\uparrow f \Leftrightarrow \delta \cap \prod_{i \in n} \text{atoms} \mathcal{X}_i \neq \emptyset. \quad (38)$$

CONJECTURE 2060. Let R be a set of staroids of the form $\lambda i \in n : \mathcal{F}(\mathfrak{A}_i)$ where every \mathfrak{A}_i is a boolean lattice. If $x \in \prod_{i \in n} \text{atoms}^{\mathcal{F}(\mathfrak{A}_i)}$ then $x \in \text{GR} \uparrow\uparrow \prod R \Leftrightarrow \forall f \in R : x \in \uparrow\uparrow f$.

There exists a completary staroid f and an indexed family X of principal filters (with arity $f = \text{dom } X$ and $(\text{form } f)_i = \text{Base}(X_i)$ for every $i \in \text{arity } f$), such that $f \sqsubseteq \prod^{\text{Strd}} X$ and $Y \sqcap X \notin \text{GR } f$ for some $Y \in \text{GR } f$.

CONJECTURE 2061. There exists a staroid f and an indexed family x of ultrafilters (with arity $f = \text{dom } x$ and $(\text{form } f)_i = \text{Base}(x_i)$ for every $i \in \text{arity } f$), such that $f \sqsubseteq \prod^{\text{Strd}} x$ and $Y \sqcap x \notin \text{GR } f$ for some $Y \in \text{GR } f$.

Other conjectures:

CONJECTURE 2062. If staroid $\perp \neq f \sqsubseteq a_{\text{Strd}}^n$ for an ultrafilter a and an index set n , then $n \times \{a\} \in \text{GR } f$. (Can it be generalized for arbitrary staroidal products?)

CONJECTURE 2063. The following posets are atomic:

- 1°. anchored relations on powersets;
- 2°. staroids on powersets;
- 3°. completary staroids on powersets.

CONJECTURE 2064. The following posets are atomistic:

- 1°. anchored relations on powersets;
- 2°. staroids on powersets;
- 3°. completary staroids on powersets.

The above conjectures seem difficult, because we know almost nothing about structure of atomic staroids.

CONJECTURE 2065. A staroid on powersets is principal iff it is complete in every argument.

CONJECTURE 2066. If a is an ultrafilter, then $\text{id}_{a[n]}^{\text{Strd}}$ is an atom of the lattice of:

- 1°. anchored relations of the form $(\mathcal{P} \text{Base}(a))^n$;
- 2°. staroids of the form $(\mathcal{P} \text{Base}(a))^n$;
- 3°. completary staroids of the form $(\mathcal{P} \text{Base}(a))^n$.

CONJECTURE 2067. If a is an ultrafilter, then $\uparrow\uparrow \text{id}_{a[n]}^{\text{Strd}}$ is an atom of the lattice of:

- 1°. anchored relations of the form $\mathcal{F}(\text{Base}(a))^n$;
- 2°. staroids of the form $\mathcal{F}(\text{Base}(a))^n$;
- 3°. completary staroids of the form $\mathcal{F}(\text{Base}(a))^n$.

23.21.1. On finite unions of infinite Cartesian products. Let \mathfrak{A} be an indexed family of sets.

Products are $\prod A$ for $A \in \prod \mathfrak{A}$.

Let the lattice Γ consists of all finite unions of products.

Let the lattice Γ^* be the lattice of complements of elements of the lattice Γ .

PROBLEM 2068. Is \prod^{FCD} a bijection from a. $\mathfrak{F}\Gamma$; b. $\mathfrak{F}\Gamma^*$ to:

- 1°. prestaroids on \mathfrak{A} ;
- 2°. staroids on \mathfrak{A} ;
- 3°. completary staroids on \mathfrak{A} ?

If yes, is up^Γ defining the inverse bijection?

If not, characterize the image of the function \prod^{FCD} defined on a. $\mathfrak{F}\Gamma$; b. $\mathfrak{F}\Gamma^*$.

23.21.2. Informal questions. Do products of funcoids and reloids coincide with Tychonoff topology?

Limit and generalized limit for multiple arguments.

Is product of connected spaces connected?

Product of T_0 -separable is T_0 , of T_1 is T_1 ?

Relationships between multireloids and staroids.

Generalize the section “Specifying funcoids by functions or relations on atomic filters” from [29].

Generalize “Relationships between funcoids and reloids”.

Explicitly describe the set of complemented funcoids.

Formulate and prove associativity of staroidal product.

What are necessary and sufficient conditions for $\text{up } f$ to be a filter (for a funcoid f)? (See also proposition 1124.)

Part 5

Postface

Postface

See this Web page for my research plans: <http://www.mathematics21.org/agt-plans.html>

I deem that now the most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- research pointfree reloids (see below);
- define and research compactness of funcoids;
- research categories related with funcoids and reloids;
- research multifuncoids and staroids in more details;
- research generalized limit of compositions of functions;
- research more on complete pointfree funcoids.

All my research of funcoids and reloids is presented at

<http://www.mathematics21.org/algebraic-general-topology.html>

Please write to porton@narod.ru, if you discover anything new related with my theory.

24.1. Pointfree reloids

Let us define something (let call it *pointfree reloids*) corresponding to pointfree funcoids in the same way as reloids correspond to funcoids.

First note that $\text{RLD}(A, B)$ are isomorphic to $\mathfrak{F}\mathcal{P}(\mathcal{P}X \times \mathcal{P}B)$. Then note that $\mathcal{P}(\mathcal{P}A \times \mathcal{P}B)$ are isomorphic both to $\text{pFCD}(\mathcal{P}A, \mathcal{P}B)$ and to $\text{atoms}^{\mathcal{P}A} \times \text{atoms}^{\mathcal{P}B}$.

But $\text{FCD}(A, B)$ is isomorphic to $\text{pFCD}(\mathfrak{F}(A), \mathfrak{F}(B))$.

Thus both $\mathfrak{F}\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{F}(\text{atoms}^{\mathfrak{A}} \times \text{atoms}^{\mathfrak{B}})$ correspond to $\text{pFCD}(\mathfrak{F}(\mathfrak{A}), \mathfrak{F}(\mathfrak{B}))$ in the same way (replace $\mathcal{P}A \rightarrow \mathfrak{A}$, $\mathcal{P}B \rightarrow \mathfrak{B}$) as $\text{RLD}(A, B)$ corresponds to $\text{FCD}(A, B)$.

So we can name either $\mathfrak{F}\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ or $\mathfrak{F}(\text{atoms}^{\mathfrak{A}} \times \text{atoms}^{\mathfrak{B}})$ as *pointfree reloids*.

Yes another possible way is to define pointfree reloids as the set of filters on the poset of Galois connections between two posets.

Note that there are three different definitions of pointfree reloids. They probably are not the same for arbitrary posets \mathfrak{A} and \mathfrak{B} .

I have defined pointfree reloids, but have not yet started to research their properties.

Research convergence for pointfree funcoids (should be easy).

24.2. Formalizing this theory

Despite of all measures taken, it is possible that there are errors in this book. While special cases, such as filters of powersets or funcoids, are most likely correct, general cases (such as filters on posets or pointfree funcoids) may possibly contain wrong theorem conditions.

Thus it would be good to formalize the theory presented in this book in a proof assistant¹ such as Coq.

If you want to work on formalizing this theory, please let me know.

See also https://coq.inria.fr/bugs/show_bug.cgi?id=2957

¹A *proof assistant* is a computer program which checks mathematical proofs written in a formal language understandable by computer.

CHAPTER 25

Please nominate me for prizes

Please nominate me for this book and other texts at my site <https://mathematics21.org> for math prizes.

In particular, this page gives instructions how to nominate me for Abel Prize: <https://mathematics21.org/nominate-me-for-abel-prize/>

I also need a recommendation letter from you to be nominated for the Break-through Prize. Please write and send me (porton@narod.ru) a letter.

Nominate me for other prizes, too.

Story of the discovery

I was a Protestant. (Now I have a new religion.¹).

I deemed that I should openly proclaim my faith: (Luk. 9:26) “For whoever shall be ashamed of me and of my words, of him shall the Son of man be ashamed, when he shall come in his own glory, and in his Father’s, and of the holy angels.” and Mrk. 8:38.

Moreover, I “reduced” my confession: “I am a sectarian”, “I am a religious fanatic.” I considered the word “sectarian” as one of the Christ’s words, because the Gospel, 2Cor. 6:17 contains the word “separate”, the root of which has the same meaning as the roots of the words “sectarian” and “holy”. I considered the word “fanatic” to be one of Christ’s words, because the Bible says (Rev. 3:19) “be zealous” and “jealous” and “fanatic” are words with the root of a similar meaning.

My so-called “confession of faith” caused a sharply negative reaction of people and led to religious discrimination, refusal to talk to me, insults, and often beatings. Moreover, realizing the hopelessness of my situation, I did not even try to improve my social status, since this was clearly impossible. In addition, with such my position, new opportunities would mean new problems for me.

When I was a first year student at Perm State University, I became interested in general topology and set a goal to discover algebraic general topology.

So I ended up on the street, without food. I began to eat grass and drink from a puddle and wait for death from hunger (as you know, I still survived).

From nothing to do, I continued my mathematical thoughts and came up with a definition of funcoid. The biggest math discovery in general topology since 1937 (when the filters were opened) was made by a hungry homeless on the street.

I wrote a term paper at my first year opening in the university.

Understanding that a religious fanatic cannot find a job and for me it is threatening soon again starvation and death, I decided to show humility: become economically weaker (abandon my economic goals and ambitions) in order to become richer. To become economically weaker, I decided to leave the university in the 5th year and filed a deduction.

My humility worked: I managed to get a second disability group that provided the conditions for my survival. Besides other things, I told psychiatrists that I have a strange object in my brain, a seraph (“genius” in Greek mythology). Consider both options: if I have a foreign object in my brain then I’m a disabled person in the psyche, if not then disabled in the psyche, too.

As you know, I wrote a doctoral dissertation in mathematics (you read it) and I was not awarded the title of Doctor of Science for religious reasons as a punishment for practicing my religion.

I sued, demanding compensation for the unpaid salary of a professor of mathematics and other things, as well as 4 trillion dollars as compensation for not made due to poverty scientific discoveries. (I valued this book along with amendments,

¹<https://www.smashwords.com/books/view/618525>

as well as my XML file processing method in 2 trillion dollars; well, how much is the limit of the discontinuous function?)

It was not that court, and after that I filed a lawsuit in the Tverskoy court of Moscow. This time without the requirement of 4 trillions and the title of hero of Russia.

But when she saw the word “sectarian”, the chairman of the court, Olga Nikolaevna Solopova, went crazy with laughter and shame and, deciding that humor took precedence over the law, did not respond to my lawsuit. It is clear that Solopova cannot answer, therefore I demanded that the qualification collegium of judges recognize her as incompetent and insane as a result of exposure to her brain with information about an abnormal sectarian and transfer the case to another judge. Qualification board has not yet responded. Such should be the reaction of a judge to a suit of a subhuman, in accordance with humor.

Note: I'm not going to actually bankrupt Russia.

About *mathematical* aspects of the story of my discoveries, see blog post:

<https://portonmath.wordpress.com/?p=2992>

Using logic of generalizations

A.1. Logic of generalization

In mathematics it is often encountered that a smaller set S naturally bijectively corresponds to a subset R of a larger set B . (In other words, there is specified an injection from S to B .) It is a widespread practice to equate S with R .

REMARK 2069. I denote the first set S from the first letter of the word “small” and the second set B from the first letter of the word “big”, because S is intuitively considered as smaller than B . (However we do not require $\text{card } S < \text{card } B$.)

The set B is considered as a generalization of the set S , for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as “without proof”.

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M : B \rightarrow B'$ such that $S \subseteq B'$. (I call this bijection M from the first letter of the word “move” which signifies the move from the old set B to a new set B').

The following is a formal but rather silly formalization of this situation in ZF. (A more natural formalization may be done in a smarter formalistic, such as dependent type theory.)

A.1.1. The formalistic. Let S and B be sets. Let E be an injection from S to B . Let $R = \text{im } E$.

Let $t = \mathcal{P} \cup \cup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t, x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t, x) = \{t, \{t, x\}\}$ by definition.

THEOREM 2070. $(t, x) \notin S$.

PROOF. Suppose $(t, x) \in S$. Then $\{t, \{t, x\}\} \in S$. Consequently $\{t\} \in \cup S$; $\{t\} \subseteq \cup \cup S$; $\{t\} \in \mathcal{P} \cup \cup S$; $\{t\} \in t$ what contradicts to the axiom of foundation (aka axiom of regularity). \square

DEFINITION 2071. Let $B' = \text{im } M$.

THEOREM 2072. $S \subseteq B'$.

PROOF. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$; $x \in \text{im } M = B'$. \square

OBVIOUS 2073. E is a bijection from S to R .

THEOREM 2074. M is a bijection from B to B' .

PROOF. Surjectivity of M is obvious. Let's prove injectivity. Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases: \square

$a, b \in R$. $M(a) = E^{-1}a$; $M(b) = E^{-1}b$; $E^{-1}a = E^{-1}b$; thus $a = b$ because E^{-1} is a bijection.

$a \in R, b \notin R$. $M(a) = E^{-1}a$; $M(b) = (t, b)$; $M(a) \in S$; $M(b) \notin S$. Thus $M(a) \neq M(b)$.

$a \notin R, b \in R$. Analogous.

$a, b \notin R$. $M(a) = (t, a)$; $M(b) = (t, b)$. Thus $M(a) = M(b)$ implies $a = b$.

THEOREM 2075. $M \circ E = \text{id}_S$.

PROOF. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$. □

OBVIOUS 2076. $E = M^{-1}|_S$.

A.1.2. Existence of primary filtrator.

THEOREM 2077. For every poset \mathfrak{J} there exists a poset $\mathfrak{A} \supseteq \mathfrak{J}$ such that $(\mathfrak{A}, \mathfrak{J})$ is a primary filtrator.

PROOF. Take $S = \mathfrak{J}$, $B = \mathfrak{F}$, $E = \uparrow$. By the above there exists an injection M defined on \mathfrak{F} such that $M \circ \uparrow = \text{id}_{\mathfrak{J}}$.

Take $\mathfrak{A} = \text{im } M$. Order (\sqsubseteq') elements of \mathfrak{A} in such a way that $M : \mathfrak{F}(\mathfrak{J}) \rightarrow \mathfrak{A}$ become order isomorphism. If $x \in \mathfrak{J}$ then $x = \text{id}_{\mathfrak{J}} x = M \uparrow x \in \text{im } M = \mathfrak{A}$. Thus $\mathfrak{A} \supseteq \mathfrak{J}$.

If $x \sqsubseteq y$ for elements x, y of \mathfrak{J} , then $\uparrow x \sqsubseteq \uparrow y$ and thus $M \uparrow x \sqsubseteq' M \uparrow y$ that is $x \sqsubseteq' y$, so \mathfrak{J} is a subposet of \mathfrak{A} , that is $(\mathfrak{A}, \mathfrak{J})$ is a filtrator.

It remains to prove that M is an isomorphism between filtrators $(\mathfrak{F}(\mathfrak{J}), \mathfrak{P})$ and $(\mathfrak{A}, \mathfrak{J})$. That M is an order isomorphism from $\mathfrak{F}(\mathfrak{J})$ to \mathfrak{A} is already known. It remains to prove that M maps \mathfrak{P} to \mathfrak{J} . We will instead prove that M^{-1} maps \mathfrak{J} to \mathfrak{P} . Really, $\uparrow x = M^{-1}x$ for every $x \in \mathfrak{J}$. □

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